Liquidity Flows in Interbank Networks*

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Abstract

This paper characterises the interbank deposit network as a flow network that is able to channel liquidity flows among banks. These flows are beneficial, allowing banks to cope with liquidity risk. First, we analyse the efficiency of three network structures–star-shaped, complete and incomplete–in transferring liquidity among banks. The star-shaped interbank network achieves the complete coverage of liquidity risk with the smallest amount of interbank deposits held by each bank. This result implies that the star-shaped network is most resilient to systemic risk. Second, we analyse the banks' decentralised interbank deposit decisions for a given network structure. We show that all network structures can generate an inefficiently low amount of interbank deposits. However, the star-shaped network induces banks to hold an amount of interbank deposits that is the closest to the efficient level. These results provide a rationale for consistent empirical evidence on sparse and centralized interbank networks.

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1. Introduction

Banks are characterised by a maturity mismatch between long-term assets and short-term liabilities. Liquidity transformation is indeed one of the main functions provided by the banks. A necessary consequence is that banks are exposed to a substantial amount of liquidity risk. As a form of coinsurance, banks share this risk by holding gross liquid positions, where each bank deposits a sum in other banks and receives deposits from other banks. These cross-holdings of interbank deposits form an interbank network.

This network of interbank deposits serves the purpose of reallocating liquidity from banks that have a liquidity surplus to banks that face liquidity deficits. The same interbank network, though, becomes a channel of contagion in case of defaults. Thus, there is a trade-off between the coverage of liquidity risk and exposure to contagion. On one hand, the larger the interbank deposits are, the larger the liquidity transfer may be (hence, the larger the insurance against liquidity risk is). On the other hand, the larger the interbank deposits are, the larger the exposure to systemic risk is. It is then relevant to identify the network structure that allows for the largest liquidity transfer with the smallest interbank exposures.

We address this issue in a novel way by applying flow network analysis to interbank networks.[1] We consider interbank networks as directed and weighted graphs, where the banks are represented as nodes and the interbank obligations are represented as links that connect the banks. We then transform these graphs into flow networks by characterizing two sets of banks depending on the liquidity shock they experience, either facing a liquidity surplus or dealing with a liquidity shortage. We model the *ex-post* re-allocation of liquidity across banks as a flow going from the banks in surplus to the banks in deficit. This modelling device allowed us to characterise the *carrying capacity* of an interbank network, which is the largest flow of liquidity that an interbank network can convey from banks in surplus to banks in deficit.

We take the structure of the interbank networks as given, and analyse the flow of liquidity in three interbank structures: complete, star-shaped (also known as money centre) and incomplete regular networks. In the complete network, every bank is connected to all other banks; in the star-shaped network, one bank is at the centre and is connected to all peripheral banks, while the peripheral banks are connected only with to bank at the

[1] See Ahuja, Magnanti and Orlin (1993) for an exhaustive textbook treatment of flow networks.

centre. In the incomplete regular network, each bank is connected to other banks in the network and all banks have the same number of neighbouring banks.

We first compare the size of the efficient interbank deposits determined by a social planner across the three interbank networks. An efficient interbank deposit is the minimum deposit that guarantees the complete coverage of liquidity risk, that is, the complete transfer of liquidity from surplus banks to deficit banks. The planner's objective is to prevent the costly early liquidation of long-term assets. The complete and star-shaped networks attain full insurance against liquidity risk with the same total amount of interbank deposits. However, the way this amount is split among banks is very different in the two networks. In the complete network, the total amount is split evenly among all the banks. In the star-shaped network, the peripheral banks equally share only half of the total amount. The advantage of the star-shaped network becomes particularly relevant when the number of the banks is large; a peripheral bank in the star-shaped network holds an amount of interbank deposits that is roughly half of the amount that a bank holds in the complete network. The incomplete regular network achieves the complete coverage of liquidity risk only if it is sufficiently connected. Even under this condition, this network requires banks to hold a higher amount of interbank deposits than the complete network.

The intuition for this result is as follows. In the complete network, and *a fortiori* in the incomplete one, the cross-holding of interbank deposits between pairs of banks that are both in need of liquidity, or both have an excess of liquidity, does not improve the ability of the network to transfer liquidity from surplus banks to deficit bank (i.e., the *carrying capacity* of the network). Such interbank deposits are somehow redundant with respect the coverage of liquidity risk. In the star-shaped interbank network, the centre bank acts as a hub that channels liquidity from banks in surplus towards banks in deficit with no redundant interbank exposures between peripheral banks. It is the crucial position of the bank in the centre of the network that allows the bank to channel most efficiently the liquidity flow through the network. Overall, the star-shaped interbank network achieves the largest possible liquidity transfer (i.e., it is characterised by the largest *carrying capacity*) for any given size of interbank deposits.

We compare the star-shaped and complete interbank networks, endowed with their efficient interbank deposits, in terms of their resiliency to systemic risk.[2] Following an

[2] The comparison in terms of exposure to systemic risk is not extended to incomplete regular networks because they provide an efficient transfer of liquidity only under restrictive conditions (see Section 2.2).

exogenous default of one or more banks in the system, we evaluate the exposure of the two networks to financial contagion. The star-shaped network guarantees the complete coverage of liquidity risk with the least exposure to systemic risk. The smallest shock capable of inducing the insolvency of all banks in a star-shaped network is larger than the corresponding shock in a complete network. The key determinant of the resiliency of a network is the ratio between customer deposits and interbank deposits in each bank, which determines the allocation of losses among the creditors of the defaulting banks. The higher this ratio is, the smaller the flow of losses that circulates within the interbank network is, and the more resilient the network is to solvency shocks. The star-shaped network, compared to the complete one, allows each bank to have the highest ratio between customer and interbank deposits since it requires banks to hold the lowest amount of interbank deposits.

We then analyse the performance of a given network when the decision regarding interbank deposits is decentralised and not made by the planner. Banks can obtain liquidity either by holding interbank deposits or by prematurely liquidating their long-term assets. The early liquidation of the long-term assets is assumed to be costly and occurs at a discount (or at cash-in-the-market prices). We show that banks, to prevent the costly early liquidation of long-term assets, have the right incentives to hold efficient interbank deposits both in the complete and the star-shaped networks. Therefore, the star-shaped network is the most efficient and most resilient network.

To make banks deviate from the efficient decision, we consider the existence of an intermediation cost due to the presence of information frictions in the interbank market. The intermediation activity is executed by a clearing house that collects the relative revenue. The presence of the intermediation cost induces banks to hold an amount of interbank deposits that is smaller than the efficient one in both networks. However, while the complete network is characterised by a unique equilibrium value of interbank deposits, the star-shaped network has a range of possible equilibrium values within which the peripheral banks and the bank at the centre exert their bargaining power. It turns out that the complete network and the star-shaped network are characterised by the same inefficiency when peripheral banks decide the size of the interbank deposits. Conversely, when the bank at the centre can determine the decentralised interbank deposit, the star-shaped network induces banks to hold an amount of interbank deposits that is the closest to the corresponding efficient level. This result is guaranteed under two conditions. The first is when the intermediation cost is sufficiently high. In this case, the bank at the centre is willing to hold an interbank deposit larger than that desired by a peripheral bank. The second is when the bank at the centre acts not only as a liquidity hub but also as a clearing house, collecting the relative revenue. In this case, the centre bank is willing to hold the efficient interbank deposit. It follows that, in both cases, the star-shaped network is less inefficient in providing full coverage from liquidity risk.

Finally, we compare the two decentralised networks in terms of exposure to systemic risk. It turns out that the star-shaped network is still more resilient than the decentralised complete network when the peripheral banks may choose the decentralised interbank deposits. Otherwise, if the bank at the centre has power in determining the interbank deposits, there is a range of parameters under which the complete network is more resilient. This occurs because the decentralised interbank deposit in the complete network can become relatively small compared to that chosen in the star-shaped network, making the complete network more resilient to shocks.

To motivate our analysis, we refer to recent empirical studies that documented the structures of existing interbank networks. This strand of empirical research was spurred by the crucial role that interbank markets played in the 2007-2008 financial crisis. The picture of the interbank network that emerged is consistent across different studies. Based on transaction data from the Fedwire system, Soromäki et al. (2007) and Beck and Atalay (2008) found that the actual interbank lending networks formed by commercial banks in the United States is quite sparse. It consists of a core of highly connected banks, while the remaining peripheral banks connect to the core banks. An almost identical feature is found in banking networks in the United Kingdom, Canada, Japan, Germany and Austria (see, respectively, Langfield, Liu and Ota, 2014; Embree and Roberts, 2009; Inaoka et al., 2004; Craig and von Peter, 2014; Boss et al., 2004). Our model provides the first rationale for these findings since it highlights how sparse and centralized interbank networks can indeed be optimal and more efficient than less sparse and more decentralised networks.

Several papers empirically analyzed the relationship between the interbank network structure and exposure to contagion. Degryse and Nguyen (2007) investigated the evolution of contagion risk in the Belgian banking system. They found that a change from a complete structure (where all banks have symmetric links) towards a money-center structure (where money centres are symmetrically linked to otherwise disconnected banks) decreased the risk and impact of contagion. Mistrulli (2011) focussed on the Italian interbank network and, analysing its evolution through time, found that complete connection among banks is not always less conducive to contagion than other structures. He showed that less connected networks could be more resilient to contagion. The evidence provided by these studies is supportive of our theoretical results.

The remainder of the paper is organized as follows. In the rest of the introduction, we discuss the related literature. Section 2 presents the model, the efficient interbank deposits in each network (Section 2.1) and the implications for systemic risk (section 2.2). Section 3 gives our analysis of the decentralised interbank deposit decision. Section 4 concludes the paper and the Appendix contains the proofs of our lemmae and propositions.

1.1 Related Literature

The motivation for our paper stemmed from the banking literature that investigated the relationship between interbank deposit structures and systemic risk. This literature focussed on simple network structures and *ad hoc* realizations of liquidity shocks in order to obtain analytical results. Allen and Gale (2000) showed that the banking system is more fragile when the interbank market is incomplete (cycle-shaped) than when the interbank market is complete. Brusco and Castiglionesi (2007) and Freixas, Parigi and Rochet (2000) instead showed that an incomplete cycle-shaped interbank market is more resilient than a complete interbank market.

Like the cited banking literature, the present paper considers the interbank network as a way to eliminate aggregate liquidity risk and analyzes how different network structures are able to cope with idiosyncratic risk, that is, how efficiently interbank networks channel liquidity from banks that have excessive liquidity holdings to banks that are in need of liquidity. Unlike the banking literature, we consider networks with arbitrary numbers of banks and a wide variety of network structures. Along with the complete and regular incomplete interbank networks, we also examine the star-shaped network.[3] More importantly, our approach allows us to contemplate generic realizations of liquidity shocks. While Allen and Gale (2000) assumed alternate liquidity shocks (i.e., adjacent banks have opposite liquidity shocks), we do not restrict the analysis to any particular realisation of

[3]Freixas, Parigi and Rochet (2000) also analyzed a three-banks example of a money-centre system, arguing that too-central-to-fail policies could be rationalized by avoiding contagious defaults to the peripheral banks. Unlike the present paper, they did not compare the money-centre system with other structures of interbank networks.

liquidity shocks (i.e., adjacent banks can have the same liquidity shock).

Our analysis was also inspired by the work by Eisenberg and Noe (2001). They represented the network formed by a payment system as a lattice and studied the flows of payment that clear this network of financial obligations. Given the operating cash flows of the agents in the system, and a generic network of obligations, they showed that the clearing payments vector is unique under a mildly restrictive condition.[4] Since the clearing payments vector cannot be characterised in analytical form for generic networks, Eisenberg and Noe (2001) provided a computational characterization of this vector. Our work also studies flows of payments in networks of obligations, but our method and focus are different. We use flow network theory (as opposed to lattice theory) to characterise the maximum flow of interbank payments. Even if there is no clear advantage of using either approach, the flow network theory seems to be more directly applicable in analysing issues related to the transfer of liquidity, such as those induced by short-term interbank obligations.

The present paper is related to the growing theoretical literature that models interbank relationships as networks (see the survey by Allen and Babus, 2009). Leitner (2005) showed how the threat of contagion may be part of an optimal network. The possibility that the failure of a bank can spread to the entire network makes it *ex-ante* optimal to establish links among banks to obtain mutual insurance and prevent the collapse of the network. Babus (2016) showed that this form of insurance between banks emerges endogenously in a network formation game. Castiglionesi and Navarro (2016) instead rationalized the formation of the interbank network structure as a trade-off between liquidity coinsurance and counterparty risk. Allen, Babus and Carletti (2012) analyzed the interaction between financial connections due to overlapping portfolio exposure and systemic risk. Castiglionesi and Wagner (2013) used a three-bank model to study weather liquidity cross-insurance among banks is socially efficient. Acemoglu, Ozdaglar and Tahbaz-Salehi (2015) and Eboli (2013) showed that the complete network has a robust-yet-fragile nature, in that it is resilient to relatively small shocks but becomes fragile if the shocks reach a given threshold.

A recent paper by Babus and Hu (2017) highlighted the benefit of less connected and more centralized networks. Unlike our paper, they considered informational networks that allowed for the observation of the past behaviours of the connected traders. With limited

[4] Eisenberg and Noe (2001) used Tarski's fixed-point theorem to establish that the clearing payments vector has a lower and an upper bound. To guarantee uniqueness, they introduced a regularity condition that requires that, in the set of agents involved in a contagion process, there is at least one agent with strictly positive operating cash flow.

commitment, transactions must take place through intermediaries in the network. They showed that a star-shaped network sustains trade more easily, and the centre trader must be compensated. It is also a stable network and, under certain conditions, a constrained efficient network. Similarly to Babus and Hu, we find that the star-shaped network is stable when the centre bank is rewarded for its intermediation. However, our analysis shows that the bank at the centre can also be willing to act as a liquidity hub without being compensated for its intermediation.

An alternative stream of literature resorted to numerical simulations to shed light on the dynamics of contagion processes in generic and complex financial networks. In this literature, the analysis relied on numerical simulations of default contagion either on randomly generated networks (see Alenton et al., 2007; Cifuentes, Ferrucci and Shin, 2005) or on national interbank systems (see Upper, 2010). More recently, Gofman (2017) used computational methods to investigate the effects of policies that aim to improve financial stability by imposing limits in terms of the number of connections in the interbank market. The author generates interbank networks with core-periphery structures. In particular, he calibrated one network on the topology of the Fed funds market and compared its performance with seven networks obtained by imposing a progressively lower cap on the number of connections that a bank can establish. Gofman (2017) found that a trade-off between efficiency (i.e., the capability to transfer liquidity in the network) and stability (i.e., exposure to systemic risk) exists. On one hand, highly interconnected banks improve efficiency, which is consistent with our analysis. In particular, Gofman showed, using an example, that the star-shaped network is the most efficient one. On the other hand, Gofman found that as progressively lower caps are set to the connections of core banks, the stability of a core–periphery network first improves and then rapidly deteriorates.

Flow network theory has been shown to be a useful tool for the analysis of economic issues. Che, Kim and Mierendor (2013) successfully applied it to the theory of auctions. To the best of our knowledge, the present paper is the first to apply flow network theory to study the efficiency and decentralised behaviour of interbank networks in transferring liquidity among banks.

2. The Model

Let $N := (\Omega, \Lambda)$ be an *interbank network*, i.e., a connected, directed and weighted graph. Each node ω_i, $(i = 1, 2, ..., n)$, in Ω represents a bank, and the links in $\Lambda \subseteq \Omega^2$ represent the interbank deposits that connect the banks in Ω. The capacity (i.e., the weight) of a link $c_{ij} \in \Lambda$ is the amount of money that bank ω_j deposited in bank ω_i. The direction of the link goes from the debtor node, ω_i, to the creditor node, ω_j. We assume that all interbank deposits are reciprocal, i.e. $c_{ij} = c_{ji}$ for all $c_{ij} \in \Lambda$. The liabilities of bank ω_i comprise customer (household) deposits, h_i, interbank deposits, d_i, and equity, e_i. On the asset side, bank ω_i holds long-term assets, a_i, which are liabilities of agents that do not belong to Ω, and short-term interbank deposits, c_i, which are deposits made by bank ω_i in other banks of the network. The budget identity of bank ω_i is as follows: $a_i + c_i = h_i + d_i + e_i$.

To analyse the flows of liquidity that can be carried by an interbank network, N, we need to model a liquidity shock. We consider a liquidity shock that consists of a reallocation of customer deposits across banks, while the aggregate liquidity in the network remains constant. We assume that, upon the occurrence of the shock, a bank can either experience an increase in customer deposits (i.e., a liquidity surplus) equal to δ or face a decrease in customer deposits (i.e., a liquidity deficit) of the same amount, $-\delta$, under the following constraint: $\sum_{\Omega} \delta_i = 0$.[5] Herein, we refer to a bank that experiences negative (positive) liquidity shock as a deficit (surplus) bank. If a deficit bank is not able to collect sufficient liquidity through the interbank network, then such a bank has to liquidate its long-term assets at a loss (or at cash-in-the-market prices).

The efficiency of an interbank network in providing coverage of liquidity risk depends on the banks' *ex-ante* choices about which neighbouring banks to place interbank deposits, and how much to deposit in these banks. These choices determine the shape of the network N and the capacity of its links, respectively. While we take the network structure as given, we characterise the efficient amount of interbank deposits.

The efficient interbank deposit is derived under the assumption that the social planner

[5] Assuming symmetric liquidity shock is conventional in the banking literature to represent liquidity risk. For example, in Allen and Gale (2000) banks use the interbank network to insure against customer deposits fluctuations from the expected liquidity shock $\gamma = (\omega_H + \omega_L)/2$, where ω_H (ω_L) is the high (low) liquidity shock. The liquidity flow from a surplus bank would be $\delta = \gamma - \omega_L$, and the liquidity needed for a deficit bank would be $-\delta = -(\omega_H - \gamma)$. Notice that $\omega_H - \gamma = \gamma - \omega_L$. In our model, the fluctuations of customer deposits have an expected value equal to zero.

maximizes the payoffs of the final claimants of the banks, that is, shareholders and depositors. The objective of the planner is to avoid losses from the early liquidation of long-term assets. The planner's goal is then to achieve the complete coverage of liquidity risk by being able to reallocate liquidity from surplus to deficit banks.[6] The social planner acts as a network administrator that manages the liquidity flow within the interbank network. In order to obtain the largest liquidity flow (or carrying capacity) in a given network, the planner is assumed to coordinate the withdrawals among the banks. Herein, the planner decides which bank withdraws from which bank and the amount to be withdrawn. The coordination of withdrawals by the planner realizes the largest liquidity transfer from surplus to deficit banks.

We analyse and compare the efficiency of three classes of networks: *complete*, *incomplete regular* and *star-shaped* networks. To make a meaningful comparison of the performance of the three types of networks in providing the complete coverage of liquidity risk, we assume that each bank in the complete and incomplete networks have the same size of peripheral banks in the star-shaped network. In particular, these banks hold the same amount of customer deposits. This assumption guarantees that each of these banks is exposed to the same liquidity risk, that is, to the same shock, $(\delta, -\delta)$.

Given the size of a peripheral bank, we assume that the centre bank in the star-shaped network has the same balance sheet ratios of a peripheral bank but that it is $(n-1)$ times larger. That is, each balance sheet item of the centre bank is $(n-1)$ times larger than the same balance sheet item of a peripheral bank. This assumption is coherent with the observed core-periphery interbank networks. For example, Craig and von Peter (2014) estimate that "the median size of (German) banks in the core is 49 times that of banks in the periphery" (p. 337). It is important to highlight that the asymmetry in the size of the banks in the star-shaped network is not relevant in obtaining the results in Section 2.[7]

[6] Notice that this definition of efficiency is analogous to that given by Allen and Gale (2000). They characterized the first-best allocation by maximizing the utility of depositors (the only claimants of banks' liabilities) and avoiding the costly early liquidation of long-term assets. Our approach differs from that of Allen and Gale (2000) because we do not assume that neighbouring banks would be hit by alternate (opposite) shocks, instead considering a more general distribution of liquidity shocks.

[7] This assumption becomes instead relevant in Section 3, and we will discuss it again in this context (see footnote 22).

2.1 The Efficient Interbank Deposits

An interbank network provides complete coverage against liquidity risk if it guarantees that each deficit bank collects an amount of liquidity that is sufficient to meet its own shortage. This coverage has to be achieved for any realisation of liquidity shock. To evaluate and compare the efficiency of the three interbank networks, we characterise the minimum interbank deposit that ensures the feasibility of the complete coverage of liquidity risk. We begin by analysing the star-shaped network and then analyse the complete and the incomplete regular networks.

Star-Shaped Network. A star-shaped interbank network consists of a bank at the centre, ω_c, that places an interbank deposit in each of the $(n-1)$ peripheral banks, ω_p. The latter, in turn, place their interbank deposits in ω_c and exchange no deposits among themselves. Let $N^s = \{\Omega^s, \Lambda^s\}$ be a star-shaped interbank network with $\Lambda^s = \{c_s | \forall(\omega_p, \omega_c) \in N^s\}$. That is, the interbank deposit between each peripheral banks and the bank at the centre has a capacity equal to c_s. Therefore, the centre bank, ω_c, holds a total amount of interbank deposits equal to $(n-1)c_s$.

Proposition 1 *In a star-shaped interbank network, full coverage of liquidity risk is achieved with $c_s^* = \delta$ if interbank deposits withdrawals are coordinated.*

The proof is in the Appendix. The reasoning is as follows. Let x be the number of peripheral banks that experience a surplus. This implies that $(n-1-x)$ of peripheral banks face a liquidity deficit. By assumption, the liquidity shock leaves the total stock of customer deposits unchanged. Therefore, the change of the stock of customer deposits of the centre bank, Δh_c, is equal to the opposite of the change of the customer deposits held by all peripheral banks. That is, $\Delta h_c = (n-1-x)\delta - x\delta = (n-1-2x)\delta$. Thus, for $x > (n-1)/2$, the centre bank faces a liquidity shortage, while, for $x < (n-1)/2$, it experiences a liquidity surplus.

When a liquidity shock occurs, the $(n-1-x)$ peripheral deficit banks withdraw the interbank deposit, c_s, from the bank at the centre, while the latter withdraws c_s from each of the x peripheral surplus banks (withdrawals are coordinated by the social planner). Suppose that $c_s = \delta$. In this case, each peripheral bank in deficit collects an amount of liquidity equal to the needed liquidity shock, δ. If $x > (n-1)/2$, the centre bank faces a liquidity shortage equal to $(n-1-2x)\delta$ and collects a net amount of liquidity equal to

$xc_s - (n-1-x)c_s = (n-1-2x)\delta$. Similarly, if $x < (n-1)/2$, the liquidity surplus of the bank at the centre is enough to cover the liquidity shortages of the peripheral banks. The star-shaped network achieves the complete coverage of liquidity risk with coordinated withdrawals and $c_s^* = \delta$ (and, *a fortiori*, for any $c_s > c_s^*$).[8]

Complete Network. In a complete interbank network, each bank places a deposit in every other bank. Let $N^c = \{\Omega, \Lambda^c\}$ with $\Lambda^c = \{c_{ij} | i \neq j; i, j = 1, ..., n\}$ be a complete interbank network where all banks have the same size and the links in Λ^s have the same capacity, c_{ij}. Each bank holds, in this network, a total amount of interbank deposits equal to $c_c = (n-1)c_{ij}$.

Proposition 2 *In a complete interbank network, full coverage of liquidity risk is achieved with* $c_c^* = \frac{n-1}{n}2\delta$ *(or, equivalently, with* $c_{ij}^* = \frac{2\delta}{n}$*).*

The proof is in the Appendix. The reasoning is as follows. In a complete interbank network, there are $n/2$ deficit banks and the same number of surplus banks. Each deficit bank withdraws, from each of its $n/2$ neighbours in surplus, a deposit equal to c_{ij}, collecting a total amount of liquidity equal to $c_{ij}n/2$. The complete coverage of liquidity risk is achieved if $c_{ij}n/2$ is at least equal to the liquidity need, δ. The efficient interbank deposit is then $c_{ij}^* = 2\delta/n$ (and, *a fortiori*, any $c_{ij} > c_{ij}^*$), which implies $c_c^* = \frac{n-1}{n}2\delta$.[9] Note that, contrary to the star-shaped network, in the complete interbank network, the reallocation of liquidity is achieved with or without the coordination of deposit withdrawals. In the complete network, whether a deficit bank withdraws only from surplus banks or from all other banks, the efficient interbank deposit does not change. This is because each deficit bank is directly connected to all surplus banks, from which it collects the required liquidity, and mutual withdrawals among deficit banks offset one another.

Incomplete Regular Networks. Incomplete interbank networks are difficult to analyse, unless some restrictions are imposed on their shape. For the sake of tractability, we focus on

[8] If withdrawals were not coordinated, the size of the efficient interbank deposit, c_s^*, would be strictly larger than δ. For example, if the bank at the centre equally split withdrawals among all peripheral banks irrespective of their liquidity need (i.e., *pro rata*), the minimum efficient interbank deposits would be $c_s^* = 1.5\delta$.

[9] If $n = 4$, the interbank deposit c_c^* coincides with that obtained by Allen and Gale (2000). In their four-bank complete network, the efficient interbank deposit is equal to $3(\omega_H - \gamma)/2$ (where $\omega_H = \delta$ and $\gamma = 0$ in our set up).

regular incomplete networks. In such networks, all nodes have the same number of incoming and outgoing links. Let $N^r = \{\Omega, \Lambda^r\}$ with $\Lambda^r = \{c_{ij} | i \neq j;\ i = 1, ..., n;\ j = 1, ..., k \text{ with } k < n - 1\}$ be an incomplete regular interbank network where the links in Λ^r have the same capacity, c_{ij}. Let k be the number of incoming (or outgoing) links, then $c_r(k) = kc_{ij}$ is the total amount of interbank deposits held by each bank. To ensure that an incomplete network achieves the complete coverage of liquidity risk, it is necessary that the network is sufficiently connected. This minimum connectivity is guaranteed if each bank places interbank deposits in at least $n/2$ banks.[10]

Proposition 3 *In an incomplete regular interbank network with $k \geq n/2$, full coverage of liquidity risk is achieved with $c_r^*(k) = \frac{k}{k+1-n/2}\delta$ if interbank deposits withdrawals are coordinated.*

The proof is in the Appendix. The result shows how an incomplete network, even if sufficiently connected, needs a relatively high interbank deposit to achieve the full coverage of liquidity risk. Notice that the efficient interbank deposit, $c_r^*(k)$, decreases in k. The minimum amount of total interbank deposits held by each bank is then obtained when $k = n - 2$, which implies that $c_r^*(k = n - 2) = 2\delta$. Then, the efficient interbank deposit, $c_r^*(k)$, is larger than the efficient deposit $c_c^* = \frac{n-1}{n}2\delta$, for any $k \geq n/2$. The reasoning is as follows. In an incomplete network, a bank that experiences a liquidity deficit may not be connected to all the surplus banks (a feature that instead characterises the complete network). The number of interbank deposits that go from the surplus banks into the deficit banks may be relatively small. These deposits constitute the links over which the flow of surplus liquidity generated by the shock must pass to reach the deficit banks and achieve the complete reallocation of liquidity. To support this flow, the capacity of the interbank deposits that each bank has to hold in the incomplete network has to be larger than that determined in the complete network.[11]

[10] Allen and Gale (2000) analyzed an incomplete interbank network with $n = 4$ and $k = 1$, showing that it allows banks to cover their liquidity risk and implement the efficient allocation of deposits. The result hinges on the assumption that adjacent banks are hit by liquidity shocks of opposite sign (i.e., shocks are alternate). With a more general liquidity shock structure, incomplete interbank networks do not necessarily provide full coverage of liquidity risk when $k < n/2$.

[11] Notice that considering less connected regular networks (e.g., a circular one) would not change the result in Proposition 3. This result would actually hold *a fortiori*. Indeed, the sparser a regular interbank network is, the fewer the interbank deposits that channel liquidity from surplus to deficit banks are, and

Since the incomplete network is clearly dominated by the complete one, we do not take incomplete regular networks into account past this point. From this point on, we restrict our analysis to the comparison of the star-shaped and the complete networks.

Comparison. We notice that, with $n = 2$, the efficient bilateral interbank deposit held in the two networks is the same and equal to the liquidity shock, δ. The total amount of interbank deposits is also the same, equal to 2δ, in the two networks. This is intuitive since, in a two-bank network, the structure of the network does not play a role. When $n \geq 3$, the efficient interbank deposit may become different in the two networks.

Let us first consider the total amount of efficient interbank deposits, D^s and D^c, in the star-shaped and complete networks, respectively. In the star-shaped network, we have $(n-1)c_s^*$ deposits held by the $(n-1)$ peripheral banks and the $(n-1)c_s^*$ deposits held by the bank at the centre. In total, we have $D^s = 2(n-1)\delta$ deposits. In the complete network, we have a total of nc_c^* deposits, that is, $D^c = 2(n-1)\delta$. Therefore, the two networks have the same total amount of interbank deposits. However, the way the total amount of interbank deposits is split among each individual bank is very different in the two networks.

The total amount of interbank deposits of each bank (with the exception of the bank at the centre of the star-shaped network) is characterised as follows:

$$c_c^* = \frac{n-1}{n} 2\delta > c_s^* = \delta. \tag{1}$$

In the complete network, the total deposits, D^c, are split evenly among n banks. In the star-shaped network, the bank at the centre holds half of the total deposits, D^s, and the other half is equally split among the $(n-1)$ peripheral banks. If n is large enough, a peripheral bank in the star-shaped network holds a total amount of interbank deposits that is roughly half of the efficient amount that a bank holds in the complete network. This represents a sizable reduction.

The reason for this result lies in the different topologies of the two networks. The star-shaped network is characterised by a sparse and maximally centralized structure. This feature enables the bank at the centre to act as a liquidity hub through which it is possible to coordinate the interbank withdrawals in order to collect liquidity from surplus banks and

the larger the capacity of these deposits must be to allow the complete reallocation of liquidity within the network.

transfer it into deficit banks. Given the uneven allocation of the total amount of interbank deposits, the flow of liquidity that can cross the bank at the centre is $(n-1)$ times larger than the flow that can cross a peripheral bank, which is equal to δ. The carrying capacity of the bank at the centre, $(n-1)\delta$, can be used to reallocate liquidity among the peripheral banks. As a result, the efficient bilateral interbank deposit, c_s^*, is the smallest deposit that enables a deficit peripheral bank to collect the needed liquidity, δ.

Conversely, the complete network has a maximally connected and decentralised structure. Given the even allocation of the total amount of interbank deposits, the carrying capacity of each bank (i.e., the flow of liquidity that can cross a bank) is the same and equal to $c_c^* > \delta$. That is, a deficit bank has to deposit, in the other banks, more than what it needs in order to collect the needed liquidity, δ. This is because the deposits between banks that experience the same liquidity shock cannot be used in *ex-post* liquidity reallocation. Even if the withdrawals were coordinated, there is no way to use this interbank capacity. A planner, being constrained by the network structure, can use such interbank deposits to move liquidity among surplus banks or among deficit banks but not across both sets of banks (as would be needed). In a sense, part of the interbank deposits in a complete network is redundant with respect to covering liquidity risk.

The superior performance of the star-shaped interbank network in reallocating liquidity is also visible by comparing the carrying capacity of the two networks. The largest liquidity flow from surplus banks to deficit banks that the complete network can achieve is equal to $\delta n/2$. Conversely, the carrying capacity of the star-shaped network depends on the realisation of the shock. If the shock equally splits the peripheral banks into surplus and deficit banks (i.e., $x = (n-1)/2$), then we have a minimum carrying capacity equal to $\delta n/2$. If all the peripheral banks are either in surplus (i.e., $x = 0$) or in deficit (i.e., $x = n-1$), the star-shaped network reaches a maximum carrying capacity equal to $\delta(n-1)$, which coincides with the carrying capacity of the bank at the centre.

We conclude with an example to illustrate the different characteristics of the complete and star-shaped networks. Consider four banks, ω_1, ω_2, ω_3 and ω_4. Assume that they are linked in a complete network structure, and that each bank holds a total amount of interbank deposits, $c_c^* = \delta 3/2$, and a bilateral interbank deposit, $c_{ij}^* = \delta/2$. Upon the occurrence of the shock, two of these banks face a deficit, say ω_1 and ω_2, while the other two experience a surplus. Then, ω_1 and ω_2 withdraw their deposits from ω_3 and ω_4, each collecting $2c_{ij}^* = \delta$. The carrying capacity (or total flow of liquidity) of the complete network is then equal to 2δ. Note that both ω_1 and ω_2 collect δ while the total capacity of the incoming links of each of them is higher, equal to $3c^*_{ij} = 3\delta/2$. Similarly, the surplus banks, ω_3 and ω_4, have a bilateral interbank deposit with a capacity of c^*_{ij} that remains idle. Therefore, there is a spare capacity of two bilateral links (one link between the two deficit banks and one link between the two surplus banks) that is equal to $2c^*_{ij} = \delta$. Assume now that the four banks are disposed in a star-shaped network. Each peripheral bank holds an interbank deposit, $c^*_s = \delta$, with the bank at the centre. Suppose that all three peripheral banks face a deficit (or a surplus). This is the scenario characterised by the largest *ex-post* reallocation of liquidity. Then, each peripheral bank withdraws c^*_s from the bank at the centre (or the other way around, if all peripheral banks are in surplus). This means the carrying capacity of the star-shaped network is equal to 3δ, and it is fully used without spare capacity.

2.2 Systemic Risk

We assumed, so far, that reducing the magnitude of interbank deposits is valuable since it reduces the risk of contagion. In this section, we make this idea explicit by considering systemic risk, which is broadly defined as the risk that the network is affected by a domino effect that propagates the losses originating from the initial exogenous default of one or more banks, that is, a default cascade that involves otherwise solvent banks and possibly affects the entire network. We evaluate the resiliency of complete and star-shaped networks to systemic risk assuming that at least one of the banks in the network is bankrupt due to an exogenous solvency shock.

Definition 1 *Bank ω_i defaults on its creditors if $a_i + c_i < h_i + d_i$ (or, equivalently, if $e_i < 0$), that is, if a bank is not able to honour its debts.*

The initial exogenous solvency shock represents a loss of value of the long-term assets, a, that is able to cause the insolvency of one or more banks. Let Φ be the set of primary defaults, that is, the set of banks hit by the initial solvency shock. We assume that such exogenous shock does not affect the value of the long-term assets, a, of the banks that are not in set Φ. We also assume that all claimants of a defaulting bank have the same seniority, that is, the losses incurred by a defaulting bank above its equity are split equally (i.e., *pro rata*) among the neighbouring banks and the depositors. Default contagion occurs if the losses transmitted by the banks in Φ are large enough to cause secondary defaults, that is, the default of one or more banks in $\Omega \backslash \Phi$.

We characterise two thresholds of contagion as measures of the exposure of an interbank network to systemic risk. The first threshold of contagion τ_1 is the smallest shock that causes secondary defaults. The final threshold of contagion τ_2 is the smallest shock that induces the failure of all banks in the network. Therefore, *ceteris paribus*, the higher these two thresholds are and the more resilient the network is, the larger the size of the external shock needed to induce default contagion is. If the two thresholds coincide, the smallest shock that causes a secondary default is also sufficient to determine the collapse of the entire network. We characterise these thresholds for the complete and the star-shaped interbank networks.

In the complete network, the first and the final thresholds coincide.

Proposition 4 *In a complete network, $\tau_1^c = \tau_2^c$, and they are equal to*

$$\tau^c = ne_i + e_i \frac{h_i}{d_{ij}}, \tag{2}$$

where $d_{ij} = c_{ji}$ is the amount deposited by bank j in bank i.

The proof is in the Appendix. The reason why the first and the final thresholds of contagion are the same is as follows. In the complete network, each bank is exposed to every other bank with the same bilateral deposit, c_{ij}. Since the losses of a defaulting bank are borne *pro rata* by all its creditors, the losses caused by the primary defaults are evenly spread among all the other banks. As a consequence, the banks not hit by the original shock either all survive if their equity suffices to absorb the losses or they all fail if the original shock is larger than τ^c.

In a star-shaped network, the first and final thresholds of contagion may also coincide. This happens if the bank at the centre is in the set of primary defaults. Let us indicate, with e_c and e_p, the amount of equity, and, with h_c and h_p, the amount of customer deposits held by the centre bank and a peripheral bank, respectively.

Proposition 5 *If the bank at the centre, ω_c is in the set of primary defaults, Φ, then, in a star-shaped network, we have $\tau_1^s = \tau_2^s$, and their value is as follows:*

1. *For $\Phi = \omega_c$, we have*

$$\tau^s = (n-1)e_p + e_c + e_p \frac{h_c}{d_s}, \tag{3}$$

where $d_s = c_s$ is the amount deposited by a peripheral bank in the bank at the centre.

2. For $\Phi = \{\omega_c, \omega_p |$ for some $p \in \Omega \backslash \omega_c\}$ and $\sigma_c < (n-1)e_p + e_c + e_p \frac{h_c}{d_s} = \tau^s$, we have

$$\widetilde{\tau}^s = \left[(n-1)e_p + e_c + e_p \frac{h_c}{d_s}\right]\left(1 + \frac{h_p}{d_s}\right) - \sigma_c \frac{h_p}{d_s} = \tau^s + (\tau^s - \sigma_c)\frac{h_p}{d_s}, \qquad (4)$$

where σ_c is the loss of value of the external assets borne by the bank at the centre.

The proof is in the Appendix. This shows that the resiliency of the star-shaped network depends on whether the shock hits the centre bank only or some peripheral banks as well. The initial default of the bank at the centre alone (i.e., $\Phi = \omega_c$) causes contagion if the exogenous shock, σ_c, suffered by such a bank is larger than the threshold, τ^s. In this case, each peripheral bank receives a loss larger than its own equity, and they all default. Otherwise, if $\sigma_c < \tau^s$, contagion occurs only if bank ω_c receives additional (endogenous) flow of losses from peripheral banks, that is, only if one or more peripheral banks are in Φ, along with ω_c. In the latter case, the star-shaped network is more resilient.

When the solvency shock hits only the bank at the centre, the losses reach the peripheral banks in $\Omega \backslash \Phi$ directly. Debtholders (depositors) of the banks absorb losses only once. When peripheral banks are also included in Φ, the losses suffered by such banks are transferred *pro rata* to their depositors and to the bank at the centre. The latter, in turn, redirects these losses *pro rata* towards its own depositors and the peripheral banks in $\Omega \backslash \Phi$. Thus, there is an indirect passing of losses from the affected peripheral banks to the centre and from the centre to all peripheral banks, and, in such passages, depositors absorb part of the shock. This implies that, for a given exogenous shock, there is a larger flow of losses that reaches the peripheral banks in $\Omega \backslash \Phi$ when the shock is concentrated on the centre bank compared to the losses caused by the same shock if it is split between the bank at the centre and some peripheral banks. As a consequence, $\widetilde{\tau}^s > \tau^s$.

If, instead, the bank at the centre is not in the set of primary defaults, the first and final thresholds of contagion in the star-shaped network do not coincide.

Proposition 6 *If $\omega_c \notin \Phi$, then, in a star-shaped network, the first threshold of contagion is equal to*

$$\tau_1^s = m e_p + e_c \left(1 + \frac{h_p}{d_s}\right), \qquad (5)$$

where $m = e_c\left(\frac{1}{h_p} + \frac{1}{d_s}\right)$ is the minimum number of peripheral defaults, which is sufficient to induce the default of the bank at the centre. The final threshold of contagion is equal to

$$\tau_2^s = \left[(n-1)e_p + e_c + e_p \frac{h_c}{d_s}\right]\left(1 + \frac{h_p}{d_s}\right). \qquad (6)$$

The proof is in the Appendix. When the exogenous shock hits only peripheral banks, the failure of the entire network occurs if the shock is strictly larger than τ_2^s. By inspection, $\tau_2^s > \widetilde{\tau}^s$. Intuitively, when only peripheral banks are in the set of primary defaults, the star-shaped network is more resilient to external shocks. The results in Propositions 5 and 6 clearly highlight the sheltering role played by the bank at the centre.

The results on systemic risk show that both the complete and the star-shaped networks are robust-yet-fragile with respect to default contagion. Both structures are resilient to shocks smaller than their respective final thresholds of contagion, but, at the same time, they are exposed to the risk of a complete collapse if the initial shock is larger than such thresholds. However, the resiliency to external shocks and, therefore, the exposure to systemic risk of the two networks are different. Notice that the lowest final threshold of contagion in the star-shaped network is given by τ^s in Equation (3). In the complete network, the unique final threshold is given by τ^c in Equation (2). Assuming that the banks hold efficient interbank deposits in both networks, the inequality $\tau^s > \tau^c$ implies that

$$(n-1)e_p + e_c + e_p \frac{h_c}{c_s^*} > ne_i + e_i \frac{h_i}{c_{ij}^*}.$$

Recalling that $h_c = (n-1)h_i$, $e_p = e_i$, $c_s^* = \delta$ and $c_{ij}^* = \delta 2/n$, it follows that $e_p \frac{h_c}{c_s^*}$ is larger than $e_i \frac{h_i}{c_{ij}^*}$ for $n > 2$. Moreover, $(n-1)e_p + e_c > ne_i$; thus, $\tau^s > \tau^c$. Notice that the inequality, $\tau^s > \tau^c$, holds even assuming that the aggregate stock of equity is the same in both networks, that is, even if $(n-1)e_p + e_c = ne_i$. Summing up, $\tau_2^s > \widetilde{\tau}^s > \tau^s > \tau^c$, therefore, the star-shaped network is more resilient than the complete one.[12]

The key determinant of the higher resiliency of the star-shaped network is that it achieves the complete coverage of liquidity risk with the highest ratio between the external obligations, h (customer deposits), and intra-network obligations, c (interbank deposits), of each bank. This ratio dictates the *pro-rata* allocation of losses among the creditors of the defaulting banks. For any exogenous shock, the larger h/c is, the smaller the flow of losses that circulates within the interbank network is, and the larger the flow of losses that

[12] The ranking of contagion thresholds holds as well if we set, not only the total stocks of equity, but also the stocks of customer deposits and external assets to be the same in both networks. The proof is available upon request. We do not assume the same stock of customer deposits in the two networks because this would imply that a peripheral banks in the star-shaped network has roughly half the amount of deposits (hence, an exposure to liquidity risk) as that of a bank in the complete network. This would render trivial the results on the advantage of the star-shaped network in terms of the complete coverage of liquidity risk, obtained in Section 2.1.

exits the network to end up in the portfolios of depositors is. The ratio h/c in the star-shaped network is the same for a peripheral bank and the bank at the centre. Indeed, for a peripheral bank, $h_p/c_s^* = h_p/\delta$. For the bank at the centre, $h_c/(n-1)c_s^* = (n-1)h_p/(n-1)\delta = h_p/\delta$. In the complete network, this ratio is equal to $h_i/c_c^* = nh_i/2\delta(n-1)$. Given that $h_p = h_i$, the ratio h/c in the star-shaped network is roughly double that of the analogous ratio in the complete network.

We conclude with two observations on the containment of exogenously given systemic risk. First, the star-shaped network has the highest resiliency when the bank at the centre is not among the set of primary defaults. Any intervention that makes this systemic bank safer (like too-central-to-fail policies or stricter monitoring of its activities) increases the resiliency of the star-shaped network. Second, in the star-shaped network, it is possible to increase the first threshold of contagion τ_1^s in Equation (5) without affecting the second threshold τ_2^s in Equation (6). This can be obtained if every peripheral bank transfers x capital to the systemic bank. The equity of the bank at the centre, e_c, would increase by $(n-1)x$, while the equity of the defaulting peripheral banks, me_p, would decrease by mx. Since $m \leq (n-1)$, τ_1^s increases. In order to increase the resiliency of the network, the systemic bank should be relatively more capitalized than the peripheral banks. The higher capitalization of the systemic bank is not linked to its level of risk but to its position in the network. It intensifies the shelter role of the bank at the centre, improving the resiliency of the star-shaped network.

3. Decentralized Interbank Deposits

We analyse the bank's optimal (private) amount of interbank deposits. This choice depends on the type of network the bank belongs to. We study how each bank chooses the amount of interbank deposits taking the network structure as a datum. Thus, our objective is to compare how close the decentralised decisions in the complete and the star-shaped networks are to their respective efficient interbank deposits. On the *ex-post* withdrawal decision, we assume that deficit banks withdraw interbank deposits first from the neighbours that have a liquidity surplus.[13]

We assume that banks are profit maximizers and that each bank chooses the interbank

[13] This assumption is in agreement with the banking literature (e.g., Allen and Gale, 2000) where the bank's liquidity shock is observable (contrary to the depositor's shock, which is private information).

deposits by minimizing the expected loss induced by the random fluctuations of customer deposits. Recall that if a deficit bank cannot cover its liquidity need with interbank deposits, it has to liquidate its long-term illiquid asset a. We assume that banks face an expected net loss from the premature sale of illiquid assets a.[14] Avoiding such loss represents the benefit of holding interbank deposits.

We capture early liquidation loss in terms of the liquidity pricing of illiquid assets, that is, the premature liquidation of asset a occurs at the cash-in-the-market or fire-sale price, p. To determine the price, p, we follow Kyle (1985), assuming a linear relationship between p and the excess demand (or supply) of the asset:

$$p = p_o + \lambda(q + u),$$

where p_o is the price that reflects the fundamental value, v, of the asset, q is the quantity traded by informed traders, u is the quantity traded by liquidity traders (including the bank in case it faces a liquidity shortage) and λ captures the severity of fire-sale pricing. The larger λ is, the larger the deviation of the selling price p from p_0 is. Kyle's assumptions are that $E(u) = 0$, $q = f(v, u)$ and $E(v) = p_0$. It follows that $E(q) = 0$ as well.

Under these assumptions, we have $E(p) = p_0$ if the bank does not incur any asset sales. However, when the bank sells a_s of assets, the expected impact on the selling price is equal to $\Delta p = p - p_0 = -\lambda a_s$. Normalizing p_0 equal to 1, the bank expects to sell its assets at $p = 1 - \lambda a_s$. The expected revenue from the asset sale is therefore equal to $a_s(1 - \lambda a_s)$. This revenue is needed to cover the liquidity shortage, l_s, that the bank eventually faces, that is, $l_s = a_s - \lambda a_s^2$. The last function is a concave parabola with vertex $(1/2\lambda, 1/4\lambda)$, passing through the origin and is invertible in the relevant interval, $l_s \in [0, 1/4\lambda]$. Thus, we can determine the amount a_s to liquidate as follows:

$$a_s = \frac{1}{2\lambda}\left(1 - \sqrt{1 - 4\lambda l_s}\right).$$

Notice that $a_s \in [0, 1/2\lambda]$ since $\sqrt{1 - 4\lambda l_s} \in [0, 1]$. Moreover, $\partial a_s/\partial\lambda > 0$: the larger λ is, the larger the amount of assets to be sold to cover the liquidity shortage, l_s,is.

[14]Notice that if the expected liquidation cost faced by the deficit banks was equal to the expected gain obtained by the surplus banks, then risk-neutral banks would not necessarily hold interbank deposits since they would break even. However, there are reasons to believe that the expected liquidation costs of illiquid assets are larger than their expected gains. First, the sale of illiquid assets could trigger a chain of bank runs, including surplus banks. Second, some illiquid assets could be sold outside the banking system; in this case some, value would be lost for the surplus banks.

Let us consider a bank that holds no interbank deposits. Given a liquidity shortage of l_s, and given that the probability of facing a liquidity deficit is 1/2, the expected loss of premature asset liquidation, π, for such a bank is as follows:

$$\pi = \frac{1}{2}\lambda a_s = \frac{1}{4}\left(1 - \sqrt{1 - 4\lambda l_s}\right).$$

When a bank belongs to an interbank network, the expected loss of early liquidation is reduced by the amount of liquidity that the bank can fetch from its neighbouring banks. Let $l(\mathbf{c}_i)$ be the liquidity that bank ω_i obtains through its interbank deposits, where $\mathbf{c}_i = \{c_{ij} | j \in \mathcal{N}(i)\}$ is the vector composed of such interbank deposits and $\mathcal{N}(i) \subset \Omega$ is the set composed of the banks that have an interbank deposit with bank ω_i. The amount of liquidity that bank ω_i needs to raise through the sale of long-term assets is $l_s - l(\mathbf{c}_i)$, and the corresponding expected loss of asset liquidation is as follows:

$$\pi_i(l(\mathbf{c}_i)) = \begin{cases} \frac{1}{4}\left(1 - \sqrt{1 - 4\lambda(l_s - l(\mathbf{c}_i))}\right) & \text{for } l_s - l(\mathbf{c}_i) > 0 \\ 0 & \text{for } l_s - l(\mathbf{c}_i) \leq 0. \end{cases} \tag{7}$$

Notice that a liquidity surplus, that is, $l(\mathbf{c}_i) \geq l_s$, does not generate revenue for the bank. Moreover, since bank ω_i is indifferent among the vectors of interbank deposits, $\mathbf{c}_i$, such that $l(\mathbf{c}_i) \geq l_s$ (i.e., the interbank deposits that deliver the complete coverage from liquidity risk), we assume that a bank chooses the smallest of these deposits, that is, $\mathbf{c}_i$, such that $l(\mathbf{c}_i) = l_s$.[15]

Recall that interbank deposits have to be mutually agreed upon, that is, $c_{ij} = c_{ji}$, for all pairs of neighboring banks (ω_i, ω_j). Then, the amount of liquidity that bank ω_i collects from its neighbouring banks, and therefore the expected loss function $\pi_i(\mathbf{c}_i)$, depends also on the interbank deposit decision made by neighboring banks. We show that banks have the incentive to allocate interbank deposits evenly among their neighbours both in the complete and in the star-shaped networks (as this is done by the planner).

In the complete network, recalling that c_c is the total amount of interbank deposits that bank ω_i allocates among its neighbours, we have the following.

Lemma 1 *In the complete interbank network, N^c, we have $c_{ij} = \frac{c_c}{n-1}$ for all ω_i in Ω and for all ω_j in $\mathcal{N}(i)$.*

[15] Intuitively, $l(\mathbf{c}_i)$ grows monotonically in its argument. This is a feature that emerges from the characterization of bank's liquidity collection in both the interbank networks.

The proof is in the Appendix. The reason is as follows. If bank ω_i deposits the same amount, c_{ij}, in each of its neighbours, then the liquidity, $l(\mathbf{c}_i)$, that it can collect is equal to $\frac{n}{2}c_{ij}$ independently from the realisation of the random liquidity shock. Conversely, if the interbank deposits of bank ω_i are not evenly split, then liquidity collection, $l(\mathbf{c}_i)$, becomes a random variable that depends on the realisation of the liquidity shock. This random variable has a mean equal to $\frac{n}{2}c_{ij}$ with positive variance. Given that the loss function $\pi_i(l(\mathbf{c}_i))$ in Equation (7) is strictly convex, Jensen's inequality implies that the expected loss of asset liquidation is minimal when bank ω_i distributes its interbank deposits evenly among its neighbours, that is, when the allocations of interbank deposits between banks is perfectly symmetric. We have $c_{ij} = c_{ik}$ for all ω_i in Ω and for all ω_j and ω_k in $\mathcal{N}(i)$. This implies that the bilateral interbank deposits are all equal to c_{ij} and that the vector of interbank deposits is the same in each bank, that is, $\mathbf{c}_i = \mathbf{c}_j$, for all ω_i and ω_j in Ω.

In the star-shaped network, the decision on the bilateral interbank deposit depends on whether the bank is a peripheral bank, ω_p, or the bank at the centre, ω_c. In principle, the centre bank could have different bilateral deposits in each peripheral bank. However, it is optimal for the centre bank to allocate its deposits evenly among the peripheral banks. Let us indicate, with c_{st}, the total amount of interbank deposits that the centre bank allocates among the $(n-1)$ peripheral banks.

Lemma 2 *In the star-shaped interbank network, N^s, we have $c_s = \frac{c_{st}}{n-1}$ for all ω_p in $\Omega \backslash \omega_c$.*

The proof is in the Appendix, and the reasoning is the same as in the complete network. If the centre bank, ω_c, deposits the same amount in each of the peripheral banks, then the liquidity that it can collect is certain independently from the realisation of random liquidity shock. Conversely, if the centre bank does not allocate its interbank deposits evenly, the liquidity collected becomes a random variable that depends on the realisation of liquidity shock. The strict convexity of the expected loss function implies that the expected loss of asset liquidation is the lowest if bank ω_c distributes its interbank deposits evenly among the peripheral banks.[16] As long as the peripheral banks agree on c_s, in this network, the allocations of interbank deposits between banks is perfectly symmetric. Bilateral interbank deposits are all equal to c_s.

Finally, in order to analyse banks' individual incentives to hold interbank deposits, we need to define an equilibrium. Since establishing an interbank deposit requires the consent

[16] The exact expression of the loss function for the bank at the centre is established in Lemma 4; however, it has the same property of strict convexity of the loss function in (7).

of two banks (while the breaking up of a bilateral deposit can be unilaterally decided by one bank), we adopt the equilibrium notion of pairwise stability by Jackson and Wolinsky (1996).[17] Given bank ω_i's expected loss function, $\pi_i(\mathbf{c}_i)$, with $\mathbf{c}_i = \{c_{ij}|j \in \mathcal{N}(i)\} = (c_{ij}, c_{-ij})$ the vector of bilateral interbank deposits, we obtain the following definition.

Definition 2 *A vector of interbank deposits, $\mathbf{c}_i^d = \left\{c_{ij}^d|j \in \mathcal{N}(i)\right\}$, forms a pairwise-stable equilibrium interbank network, N, if:*

1. *For all ω_i and $\omega_j \in \Omega$, with $c_{ij}^d > 0$, we have*

$$\pi_i(c_{ij}^d, c_{-ij}^d) < \pi_i(0, c_{-ij}^d) \text{ and } \pi_j(c_{ij}^d, c_{-ij}^d) < \pi_j(0, c_{-ij}^d).$$

2. *For all ω_i and $\omega_j \in \Omega$ and for all $\widetilde{c}_{ij} \neq c_{ij}^d$ we have:*

$$\text{if } \pi_i(c_{ij}^d, c_{-ij}^d) > \pi_i(\widetilde{c}_{ij}, c_{-ij}^d) \text{ then } \pi_j(c_{ij}^d, c_{-ij}^d) < \pi_j(\widetilde{c}_{ij}, c_{-ij}^d).$$

Condition (1) states that, in equilibrium, it is not possible for any bank to profit from severing the bilateral interbank deposits unilaterally with a neighbouring bank. Condition (2), in addition, requires that there are no bilateral interbank deposits that lead to a Pareto-improvement. That is, if there are alternative interbank deposits, $\widetilde{c}_{ij}$, that make bank ω_i strictly better off, they must make the other bank, ω_j, strictly worse off.

With the symmetric bilateral interbank deposits established in Lemmae 1 and 2, it is easy to check whether a vector of interbank deposits forms an equilibrium.

Lemma 3 *A vector of symmetric interbank deposits forms an equilibrium if and only if, for all ω_i and $\omega_j \in \Omega$ and for all $\widetilde{c}_{ij} \neq c_{ij}^d$, $\pi_i(c_{ij}^d, c_{-ij}^d) < \pi_i(\widetilde{c}_{ij}, c_{-ij}^d)$.*

To prove this, assume that banks ω_i and ω_j agree to deviate and hold a bilateral deposit, $\widetilde{c}_{ij} \neq c_{ij}^d$. The rest of the bilateral deposits with the other banks still have the equilibrium value of c_{-ij}^d. Thanks to the symmetric distribution of interbank deposits, the values of c_{-ij}^d are the same for banks ω_i and ω_j. Therefore, $\pi_i(\widetilde{c}_{ij}, c_{-ij}^d) = \pi_j(\widetilde{c}_{ij}, c_{-ij}^d)$. This implies that if two banks deviate from a symmetric distribution of the interbank

[17] Pairwise stability is a relatively weak equilibrium concept. It is independent from any particular procedure through which interbank deposits are formed (and that we do not model). Modeling the formation process explicitly might lead to a more restrictive definition of the equilibrium (see Section 5 in Jackson and Wolinsky, 1996).

deposits, their expected loss is the same. Note that, in Definition 2, Condition (1) is equivalent to $\pi_i(c_{ij}^d, c_{-ij}^d) < \pi_i(0, c_{-ij}^d)$, and, likewise, Condition (2) is equivalent to $\pi_i(c_{ij}^d, c_{-ij}^d) < \pi_i(\widetilde{c}_{ij}, c_{-ij}^d)$. With symmetric interbank deposits, when two banks have to choose their bilateral interbank deposits, their expected payoff coincides. Thus, to identify an equilibrium interbank deposit, it suffices to verify that for one of the two banks there is no other interbank deposit that makes the bank better off. We characterise the decentralised interbank deposits for the complete interbank network and then do the same for the star-shaped network.

Complete Interbank Network. In the complete interbank network, upon the occurrence of liquidity shock, a deficit bank, ω_i, withdraws $\frac{n}{2}$ deposits c_{ij}, from banks in liquidity surplus, collecting liquidity, $l(\mathbf{c}_i) = \frac{n}{2}c_{ij}$. Thus, the shortage of liquidity faced by a deficit bank is $\left(\delta - \frac{n}{2}c_{ij}\right)$. The expected loss for early asset liquidation that each bank, ω_i, faces is equal to

$$\pi_i(\mathbf{c}_i) = \frac{1}{4}\left(1 - \sqrt{1 - 4\lambda\left(\delta - \frac{n}{2}c_{ij}\right)}\right). \tag{8}$$

Proposition 7 *The vector of interbank deposits,* $\mathbf{c}_i^d = \left\{c_{ij}^d = \delta\frac{2}{n} | \forall j \in \mathcal{N}(i)\right\}$, *is the unique pairwise-stable equilibrium of the complete interbank network.*

The proof is in the Appendix. In the complete network, banks avoid liquidation loss by holding the efficient amount of bilateral interbank deposits. Indeed, $c_{ij}^d = c_{ij}^* = 2\delta/n$, which implies that $c_c^d = c_c^* = 2(n-1)\delta/n$. The reason is that the holding of interbank deposits brings only benefits to banks and there is no reason for banks to hold an amount of deposits smaller than the efficient one.

Star-Shaped Interbank Network. We first characterise the expected loss function of the early asset liquidation of a generic peripheral bank, ω_p, and then we characterise the same function of the bank at the centre, ω_c. A peripheral bank that suffers a negative liquidity shock retrieves an amount of liquidity equal to c_s from the bank at the centre. The liquidity shortage faced by a peripheral deficit bank is $(\delta - c_s)$, and its expected loss function is as follows:

$$\pi_p(c_s) = \frac{1}{4}\left[1 - \sqrt{1 - 4\lambda\left(\delta - c_s\right)}\right].$$

For the bank at the centre, recall that the amount of customer deposits held by ω_c is $h_c = (n-1)h_p$ and the liquidity shock leaves the total stock of customer deposits unchanged. The reduction of customer deposits, $-\Delta h_c$, that ω_c can face is a random variable that takes on a value $y\delta$, where $y = 2x - (n-1)$ and x is the number of peripheral surplus banks. As the shock occurs, the $(n-1-x)$ peripheral deficit banks withdraw their deposits, c_s, from the bank at the centre, while the latter withdraws its deposits, c_s, from x peripheral surplus banks. Thus, if $y > 0$, the amount of liquidity that ω_c collects is equal to $xc_s - (n-1-x)c_s = yc_s$. For a given realisation of x, the liquidity shortage that bank ω_c needs to cover through early asset liquidation is equal to $y(\delta - c_s)$.[18]

Lemma 4 *In the star-shaped network, the expected loss of the bank at the centre, ω_c, due to early asset liquidation is as follows:*

$$\pi_c(c_s) = \sum_{x=\left[\frac{n-1}{2}\right]+1}^{n-1} \binom{n-1}{x} 0.5^{n-1} \left\{ \frac{1}{2} \left[1 - \sqrt{1 - 4\lambda(\delta - c_s)\, y} \right] \right\},$$

where $x = |(\Omega \backslash \omega_c)^+|$ and $y = 2x - (n-1)$.

The proof is in the Appendix. With the expected pay-off functions, $\pi_p(c_s)$ and $\pi_c(c_s)$, we obtain the following.

Proposition 8 *The vector of interbank deposits, $\mathbf{c}^d = [c_s^d = \delta | \forall \omega_p, \omega_c \in \Omega]$, is the unique pairwise-stable equilibrium of the star-shaped interbank network.*

The proof is in the Appendix. As in the complete network, banks choose efficient interbank deposits, that is, $c_s^d = c_s^* = \delta$. The reason is that the holding of interbank deposits only benefits banks. Overall, banks make the efficient choice of interbank deposits when they face an expected loss due to the premature sale of illiquid assets.[19] To compare the two networks in terms of efficiency we consider the explicit costs of holding interbank deposits.

3.1 Costly Interbank Deposits

We assumed, so far, that liquidity can be reshuffled inside the network at no cost using interbank deposits. That is, liquidity insurance among banks occurs in the form of pure

[18] For the sake of simplicity, in what follows, we restrict the number of banks, n, to be an even integer.

[19] This result resembles that obtained by Allen and Gale (2000), where the interbank market is able to decentralise the efficient allocation. Without unexpected liquidity shocks that induce costly early asset liquidations, banks internalise efficient interbank deposit decisions since holding deposits is beneficial in preventing the expected cost of early asset liquidation.

transfers, where banks in need of liquidity withdraw from banks in excess without any repayment. In the context of imperfect capital markets, however, the exchange of interbank deposits entails repayments. Castiglionesi and Wagner (2013) showed that the existence of frictions, such as moral hazard (i.e., the magnitude of the shock δ depends on the effort made by banks) or liquidity shocks that are not verifiable/contractible (i.e., surplus banks have to be induced to not withdraw when in surplus), make the presence of minimum repayments necessary to establish interbank relationships.[20]

The existence of repayments may represent a cost for the banks in presence of a spread between the interest rate at which banks can borrow and the interest rate at which banks can lend, with the former being higher than the latter.[21] Because of this spread, banks face a positive expected cost when they insure the liquidity risk through interbank deposits. Let us normalise the net lending rate to zero, and let us indicate, with $\gamma > 0$, the net borrowing rate (which also coincides with the spread). Then, $C_i(\gamma, \mathbf{c}_i)$ is the expected cost faced by bank ω_i that depends on γ and the vector of interbank deposits, which is determined by the position of the bank in the network.

The spread between borrowing and lending rates represents an intermediation cost for banks. The assumption is that banks determine the amount of their interbank exposition bilaterally, but the exchange of liquidity is settled through a clearing house, which charges different rates to borrowing and lending banks. The spread is an intermediation revenue for the clearing house, and this revenue is used to cover the operating costs of the intermediation activity (e.g., of electronic trading platforms).

We take the reason for the existence of these frictions as given, and we use them to compare how distant the two networks are from the efficient full coverage of liquidity risk. Given the benefit (captured by the expected liquidation loss, π_i) and the cost (represented by the expected cost of intermediation, C_i) of holding interbank deposits, every bank ω_i minimises $\psi_i(\mathbf{c}_i) = C_i + \pi_i$. Let us analyse again the complete and the star-shaped networks

[20] Castiglionesi and Wagner (2013) show in a three-bank model, that banks insure each other less than the efficient amount when interbank insurance is provided with repayments. They do not compare how such friction affects the efficiency of different network structures.

[21] In the London wholesale money market, the LIBOR and the LIBID are the rates at which banks exchange liquidity. The LIBOR is an ask rate at which banks are willing to lend (i.e., the interest at which banks borrow), and the LIBID is a bid rate at which banks are willing to borrow (i.e., the rate at which banks lend). There is a spread between these two rates with the LIBOR being higher than the LIBID. While the LIBID is not officially announced, the LIBOR is published daily by the British Bankers' Association.

in turn.

Complete Interbank Network. Recall that each bank, ω_i, is equally likely to be a surplus or deficit bank. Moreover, every deficit bank has $n/2$ bilateral deposits with surplus neighbours each of them equal to c_{ij}. Similarly, every surplus bank has $c_{ij}n/2$ bilateral deposits with deficit neighbours. Therefore, the expected intermediation cost is as follows:

$$C_i = \frac{1}{2}c_{ij}\frac{n}{2}\gamma$$

Given the expected loss function, π_i, in (8), a bank, ω_i, minimises the following objective function:

$$\psi_i(\mathbf{c}_i) = C_i + \pi_i = c_{ij}\frac{n}{4}\gamma + \frac{1}{4}\left[1 - \sqrt{1 - 4\lambda\left(\delta - \frac{n}{2}c_{ij}\right)}\right]. \tag{9}$$

Proposition 9 *The vector of interbank deposits,*

$$\mathbf{c}_i^d = \left\{c_{ij}^d = \frac{2}{n}\delta - \frac{1}{2\lambda n}\left[1 - \left(\frac{\lambda}{\gamma}\right)^2\right] | \forall j \in \mathcal{N}(i)\right\},$$

of each bank, ω_i, is the unique pairwise stable equilibrium of the complete interbank network.

The proof is in the Appendix. The result shows that if the cost of holding interbank deposits is sufficiently small, namely, if $\lambda \geq \gamma > 0$, the banks hold the efficient amount of interbank deposits, $c_{ij}^d = c_{ij}^* = \frac{2}{n}\delta$. This occurs when the marginal cost of holding one unit of deposits is not higher than the marginal benefit. Conversely, for

$$\gamma \in \left(\lambda, \frac{\lambda}{\sqrt{1 - 4\lambda\delta}}\right),$$

banks hold a positive but inefficient amount of interbank deposits, $0 < c_{ij}^d < c_{ij}^*$. Finally, if $\gamma \geq \lambda/\sqrt{1 - 4\lambda\delta}$, then $c_{ij}^d \leq 0$, that is, banks do not hold interbank deposits since it is too costly. The equilibrium decentralised interbank deposit, c_{ij}^d, depends on the severity of the losses caused by early asset liquidation, λ, and the cost of borrowing in the interbank market, γ. As intuition suggests, c_{ij}^d is increasing in λ and decreasing in γ. Accordingly, the difference, $c_{ij}^* - c_{ij}^d$, is decreasing in λ and increasing in γ.

Star-Shaped Interbank Network. Because of the asymmetric structure of the star-shaped network, the expected cost, $\pi_c(c_s)$, faced by the bank at the centre is different from that faced by the banks at the periphery, $\pi_p(c_s)$. However, this difference does not affect the decentralised equilibrium interbank deposit when interbank deposits are exchanged without intermediation costs. Both the bank at the centre and the ones at the periphery hold the efficient amount of interbank deposits, δ. However, when the exchange of interbank deposits is costly, the difference between functions π_p and π_c becomes relevant for the determination of the equilibrium interbank deposit.[22]

Let us compute the expected intermediation cost, C_{sp}, for a peripheral bank, ω_p. For this bank it is equally likely to be in surplus or in deficit, and, in both cases, it holds an amount of interbank deposits equal to c_{sp}. The expected cost is then

$$C_{sp} = \frac{1}{2} c_{sp} \gamma,$$

and bank ω_p's objective function is

$$\psi_p(\mathbf{c}_{sp}) = C_{sp} + \pi_p(c_{sp}) = \frac{1}{2} c_{sp} \gamma + \frac{1}{4} \left[1 - \sqrt{1 - 4\lambda \left(\delta - c_{sp} \right)} \right]. \tag{10}$$

Let us consider now the bank at the centre, ω_c. Its expected cost is not only determined by its liquidity shock, but also by its central position in the network. Indeed, the bank at the centre has to withdraw its deposit, c_{sc}, from the peripheral surplus banks not only when it experiences a deficit (to serve its own liquidity needs) but also when it experiences a surplus. In the latter case, the centre bank has to withdraw from surplus peripheral banks since the liquidity surplus of the centre bank is not enough to serve the peripheral deficit banks. The centre bank then has to pay the intermediation cost on these withdrawals. The expected cost, C_{sc}, depends on the number of peripheral banks in surplus (i.e., on the realisation of random variable x).

$$C_{sc} = \sum_{x=0}^{n-1} \binom{n-1}{x} 0.5^{n-1} \gamma c_{sc} x = \gamma c_{sc} \sum_{x=0}^{n-1} \binom{n-1}{x} 0.5^{n-1} x = \gamma c_{sc} \frac{n-1}{2},$$

[22] The presence of the intermediation cost makes the assumption regarding the asymmetric composition of the star-shaped network crucial. The centre bank bears an intermediation cost that is $n-1$ times larger than that borne by a peripheral bank. In a symmetric star-shaped network, the centre bank is endowed with the same stock of customer deposits of a peripheral bank. Both types of banks face the same liquidity risk, and no bank would find it convenient to be at the center by bearing the much higher intermediation cost. A symmetric star-shaped network would likely be sustainable if the centre bank is compensated by the peripheral banks to cover the higher costs (a result that is obtained in the context of OTC markets by Babus and Hu, 2017). In what follows, we show that an asymmetric star-shaped network can be stable even without such compensation.

and bank ω_c minimises the objective function

$$\psi_c(\mathbf{c}_{sc}) = C_{sc}+\pi_c(c_{sc}) = \gamma c_{sc}\frac{n-1}{2}+\sum_{x=\frac{n-1}{2}+1}^{n-1}\binom{n-1}{x}0.5^{n-1}\left\{\frac{1}{2}\left[1-\sqrt{1-4\lambda\left(\delta-c_{sc}\right)y}\right]\right\}. \tag{11}$$

The presence of the intermediation cost, γ, makes the interbank deposit, c^d_{sp}, that minimises Equation (10) different, in general, from the interbank deposit, c^d_{sc}, that minimises Equation (11).

Lemma 5 *The vector of optimal interbank deposits for a peripheral bank is*

$$\mathbf{c}^d_{sp} = \left\{c^d_{sp} = \delta - \frac{1}{4\lambda}\left[1-\left(\frac{\lambda}{\gamma}\right)^2\right] | \forall\omega_p \in N^s\right\},$$

and, for the bank at the centre, is as follows:

$$\mathbf{c}^d_{sc} = \left\{c^d_{sc} > \widetilde{c}_{sc} = \delta - \frac{1}{4\lambda\widetilde{y}}\left[1-\left(\frac{\lambda\widetilde{y}}{\gamma(n-1)}\right)^2\right] | \omega_c \in N^s\right\},$$

where $\widetilde{y}$ is the expected value of y conditional on $y > 0$. Moreover, assume $\gamma \geq \eta\lambda$ with $\eta > 1$, then, $c^d_{sp} \leq \widetilde{c}_{sc} < c^d_{sc}$.

The proof is in the Appendix. The optimal decision of the peripheral banks turns out to be identical to a generic bank, ω_i, in the complete network. That is, if $\lambda \geq \gamma > 0$, the peripheral banks would hold the efficient amount, $c^d_{sp} = c^*_s = \delta$. Conversely, for

$$\gamma \in \left(\lambda, \frac{\lambda}{\sqrt{1-4\lambda\delta}}\right),$$

peripheral banks hold a positive but inefficient amount, $0 < c^d_{sp} < c^*_s$. If $\gamma \geq \lambda/\sqrt{1-4\lambda\delta}$, then $c^d_{sp} \leq 0$. Concerning the bank at the centre, the optimal interbank deposit cannot be explicitly characterised. For this bank, the benefit of holding interbank deposits is a random variable that, in turn, is a non-linear function of binomial random variable x. To overcome this issue, we use Jensen's inequality to obtain $\widetilde{c}_{sc}$, which is a lower bound of the actual optimal interbank deposit, c^d_{sc}. Notice that, as in the complete network, the optimal deposit, c^d_{sp}, and the lower bound, $\widetilde{c}_{sc}$, are increasing in λ and decreasing in γ.

Finally, Lemma 5 establishes a sufficient condition that allows c^d_{sp} and c^d_{sc} to be ordered. In particular, the condition $\gamma \geq \eta\lambda$ with $\eta > 1$ guarantees that $c^d_{sp} \leq \widetilde{c}_{sc}$, which implies that $c^d_{sp} < c^d_{sc}$ since $\widetilde{c}_{sc} < c^d_{sc}$. The parameter η gets close to 1 as the number of banks,

n, increases.[23] Why does a sufficiently high γ guarantee that the optimal deposit for the peripheral banks is smaller than the corresponding deposit of the bank at the centre? First, notice that, for $c_s < \delta$, the expected intermediation cost is linear in c_s (and is $n-1$ times larger for the centre than for the peripheral banks). However, the expected benefit (i.e., the avoided losses of early asset liquidation) is increasing and strictly convex in c_s. Second, recall that the bank at the centre, conditional on being in deficit, has a liquidity shortage equal to $\widetilde{y}(\delta - c_s)$, which is larger than that faced by a deficit peripheral bank, $\delta - c_s$. With everything else equal, a higher intermediation cost, γ, induces to hold a smaller deposit, c_s, and to face a larger liquidity shortage, $(\delta - c_s)$. Then, the higher γ is, the higher the marginal benefit of holding the deposits of both a peripheral and the centre bank is. However, the marginal benefit of the centre bank grows faster in γ than that of a peripheral bank. This implies that, for sufficiently high γ, the bank at the centre is willing to hold a larger amount of interbank deposits to avoid the higher expected losses of early asset liquidation.

Notice that $\widetilde{y}$ is increasing in n. With everything else equal, a larger n increases the liquidity shortage that the bank at the centre faces in case of liquidity deficit, and it makes the holding of interbank deposits more valuable to such a bank. Therefore, for higher n, a lower γ is needed to induce the bank at the centre to hold a larger amount of interbank deposits than the peripheral banks (whose liquidity shortage does not depend on n).

The following proposition characterises the equilibrium interbank deposits in the star-shaped network.

Proposition 10 *Assume* $\gamma \in \left[\eta\lambda, \frac{\lambda}{\sqrt{1-4\lambda\delta}}\right)$*; then, the vectors of interbank deposits,* $\mathbf{c}_s^d \in [\mathbf{c}_{sp}^d, \mathbf{c}_{sc}^d]$*, are the pairwise-stable equilibria of the star-shaped network.*

The proof is in the Appendix. The reason that the pairwise equilibrium is not unique is because the optimal interbank deposits of the peripheral banks and the bank at the centre do not coincide anymore. The condition $\gamma \geq \lambda\eta$ guarantees that $c_{sp}^d < c_{sc}^d$, and any deposits in the interval $[c_{sp}^d, c_{sc}^d]$ is pairwise-stable. Indeed, any bilateral deviation between a peripheral bank and the bank at the centre is not Pareto-improving; while the peripheral bank would like to reduce the deposit, the bank at the centre would like to increase it. Moreover, both banks do not have the incentive to sever the deposit unilaterally as long

[23] For $n = 10$, $\eta = 1.201$, for $n = 25$, $\eta = 1.125$, for $n = 30$, $\eta = 1.096$, for $n = 50$, $\eta = 1.091$ and for $n = 100$, $\eta = 1.062$.

as $\gamma < \lambda/\sqrt{1-4\lambda\delta}$. This is the condition that guarantees the optimal c^d_{sp} to be strictly positive, and, therefore, it also guarantees that $c^d_{sc} > 0$. Finally, notice that $c_s \notin [c^d_{sp}, c^d_{sc}]$ does not represent a pairwise-stable equilibrium since there would be Pareto-improving bilateral deviations.

The interbank deposit that actually emerges as the equilibrium in the star-shaped network depends on historical conditions and on the bargaining power of the two types of banks. However, in order to compare the relative efficiency of the complete versus the star-shaped networks, we do not need to precisely determine which will emerge as the equilibrium deposit. It is enough to consider the smallest deposit, c^d_{sp}, among the possible equilibria and, therefore, the largest possible inefficiency of the star-shaped network.

Comparison. We compare how close the decentralised deposits held by the banks in the two networks are to their respective efficient values. Recall that the total amount of efficient interbank deposits in the star-shaped network, D^s, and in the complete network, D^c, is the same and equal to $2(n-1)\delta$. Let $D^d_s = 2(n-1)c^d_s$ and $D^d_c = n(n-1)c^d_{ij}$ be the total amount of decentralised interbank deposits in the star-shaped and complete networks, respectively. To compare the relative efficiency of the two networks, we look at the differences between the inefficient decentralised deposits and the efficient ones, that is, we compare $D^s - D^d_s$ with $D^c - D^d_c$.

Assume that $\lambda/\sqrt{1-4\lambda\delta} > \gamma \geq \lambda\eta$. In the star-shaped network, we consider the smallest among the equilibrium deposits, c^d_{sp}.

$$D^c - D^d_c = 2(n-1)\delta - n(n-1)c^d_{ij} = \frac{n-1}{2\lambda}\left[1 - \frac{1}{\gamma^2}\right]$$

and

$$D^s - D^d_s = 2(n-1)\delta - 2(n-1)c^d_{sp} = \frac{n-1}{2\lambda}\left[1 - \frac{1}{\gamma^2}\right].$$

Therefore the complete and star-shaped networks show the same inefficiency when, in the latter, the smallest equilibrium deposit is considered. This implies that, whenever $c^d_s \in (c^d_{sp}, c^d_{sc}]$, the star-shaped network is characterised by a level of interbank deposits that is closer to the efficient amount compared to the complete network.

Proposition 11 *Assume $\gamma \in \left[\eta\lambda, \frac{\lambda}{\sqrt{1-4\lambda\delta}}\right)$; then, the decentralised star-shaped network is closer to the efficient coverage of liquidity risk than the decentralised complete network.*

If $\gamma < \lambda\eta$, we cannot establish whether $c^d_{sp} < c^d_{sc}$ or $c^d_{sp} > c^d_{sc}$; thus, we cannot fully characterise the equilibrium deposit, c^d_s, and comparing the two networks becomes more difficult. Clearly, if the inequality, $c^d_{sp} < c^d_{sc}$, still holds, then the result in Proposition 11 would carry over under this condition as well. However, we cannot exclude the possibility that $c^d_{sp} > c^d_{sc}$. In such a case, similarly to Proposition 10, any vector $\mathbf{c}^d_s \in [\mathbf{c}^d_{sc}, \mathbf{c}^d_{sp}]$ represents a pairwise-stable interbank deposit equilibrium, and the star-shaped network is farther from efficiency than the complete network. Indeed, consider $\lambda < \gamma < \lambda\eta$; then, the deposits c^d_{sp} and c^d_{ij} induce the same inefficiency in the two networks as before. However, in the range $[c^d_{sc}, c^d_{sp})$, the star-shaped network is more inefficient than the complete network.

The problem is represented by the bank at the centre that could require a relatively small interbank deposit when the intermediation cost is not sufficiently high. Recall that $\gamma \geq \eta\lambda$ guarantees that the bank at the centre is willing to hold an amount of interbank deposits that is larger than the peripheral banks in order to avoid the higher expected losses of early asset liquidation. When $\gamma < \lambda\eta$, the interbank deposit, c_s, gets closer to the efficient amount δ; thus, the liquidity shortage, $(\delta - c_s)$, becomes smaller, and the liquidation losses faced by the bank at the centre may not compensate its higher intermediation costs. In this case, the bank at the centre may not be willing to hold an amount of deposits larger than the peripheral banks.

One feasible solution is to exploit the position of the bank at the centre of the star-shaped network. Notice that the centre bank is the only bank that can intermediate all bilateral interbank transactions. This is clearly not feasible for a peripheral bank in N^s and is equally unfeasible for a bank in the complete network.[24] We assume that the centre bank operates as the clearing house as well. The spread represents a source of revenue for the bank at the centre rather than an intermediation cost that compensates the operating expenses of the clearing house activity.[25]

By acting as the clearing house, the bank at the centre repays the amount borrowed from

[24] Consider the complete network with $c^d_{ij} < \delta 2/n = c^*_{ij}$. A deficit bank can collect, from $n/2$ surplus banks, an amount strictly smaller that its own liquidity shortage, δ. Hence, its links with other deficit banks cannot be used to transfer liquidity to them. Similarly, a surplus bank can transfer, to deficit banks, an amount strictly smaller than its own surplus, δ. Then, its links with other surplus banks cannot be exploited to intermediate liquidity.

[25] This assumption is in agreement with empirical evidence based on loan-level data from the Euro area interbank market provided by Gabrieli and Georg (2016). They showed that banks with a more central position in the interbank network charge larger intermediation spreads because these banks have access to cheap borrowing.

the surplus peripheral banks at the lending interest rate (which is zero), and it charges the borrowing rate, $\gamma > 0$, to each peripheral deficit bank. The bank at the centre collects an expected amount of intermediation revenue, R_{sc}, that depends on the number of peripheral banks in deficit (i.e., on the realisation of random variable $(n-1-x)$).

$$R_{sc} = \sum_{x=0}^{n-1} \binom{n-1}{x} 0.5^{n-1} \gamma c_{sc}(n-1-x) = \gamma c_{sc} \sum_{x=0}^{n-1} \binom{n-1}{x} 0.5^{n-1}(n-1-x) = \gamma c_{sc} \frac{n-1}{2}.$$

For simplicity, let us assume that the expected revenue, R_{sc}, cancels out the operating costs of acting as clearing house. That is, the expected losses (or profits) are $C_{sc} = 0$. The objective function of the bank at the centre then becomes $\psi_c(\mathbf{c}_s) = C_{sc} + \pi_c(c_{sc}) = \pi_c(c_{sc})$ and reaches a minimum when $c_{sc}^d = c_s^* = \delta$ (with no intermediation cost, the efficient deposit is the one that minimises $\pi_c(c_{sc})$).[26] It follows that $c_{sp}^d \leq c_{sc}^d = \delta$. Similar to Proposition 10, if $\lambda < \gamma < \lambda/\sqrt{1-4\lambda\delta}$, any vector of interbank deposit $\mathbf{c}_s^d \in [\mathbf{c}_{sp}^d, \mathbf{c}_{sc}^d = \boldsymbol{\delta}]$ is a pairwise-stable equilibrium in the star-shaped network. Since the star-shaped and the complete networks have the same inefficiency if the equilibrium deposit is c_{sp}^d, it follows that the decentralised star-shaped network is closer to the efficient coverage of liquidity risk than the decentralised complete network. For $\gamma \leq \lambda$, both networks induce banks to hold the efficient amount of interbank deposits.

Proposition 12 *Assume that the bank at the centre of the star-shaped network acts as a clearing house. Then, i) if $\lambda < \gamma < \lambda/\sqrt{1-4\lambda\delta}$, the decentralised star-shaped network is closer to the efficient coverage of liquidity risk than the decentralised complete network, and ii) if $0 < \gamma \leq \lambda$, both networks provide the efficient coverage of liquidity risk.*

Efficiency versus Systemic Risk. Finally, let us analyse how the (inefficient) decentralised interbank networks perform in terms of systemic risk. Recall that, in Section 2.2, we obtain the ranking of the contagion thresholds in the two networks by considering the efficient interbank deposits. We seek to determine if that ranking still holds if the inefficient decentralised deposits characterised in this Section are taken into account.

[26] If the expected revenue is higher than the operating expenses, then the expected profit from intermediation is positive, or, equivalently, the expected cost, C_{sc}, is negative. In such a case, the bank at the centre would like to increase the interbank deposits. However, the maximum feasible amount a deficit peripheral bank would withdraw is δ (for peripheral banks intermediation is still costly). As long as $C_{sc} \leq 0$, the centre bank acting as clearing house would choose the corner solution given by the efficient deposit, δ.

Let us consider the lowest final threshold of contagion in the star-shaped network, τ^s, and the unique final threshold of contagion in the complete network, τ^c. When banks hold inefficient interbank deposits in both networks, that is when $\lambda < \gamma < \lambda/\sqrt{1-4\lambda\delta}$, the two thresholds are as follows:[27]

$$\begin{aligned}\tau^s &= (n-1)e_p + e_c + e_p\frac{h_c}{c_s^d} \\ \tau^c &= ne_i + e_i\frac{h_i}{c_{ij}^d}.\end{aligned}$$

Recall that the comparison between the two thresholds did not depend on the total amount of equity that characterises the networks; therefore, we can assume that $(n-1)e_p+e_c = ne_i$. Moreover, $e_p = e_i$ and $h_c = (n-1)h_i$. Under the assumed parametrisation of γ, the equilibrium interbank deposits are $c_s^d \in [c_{sp}^d, c_{sc}^d = \delta]$, with c_{sp}^d given in Lemma 5 and c_{ij}^d given in Proposition 9. To compare the two thresholds, assume that the equilibrium interbank deposit in the star-shaped network is $c_s^d = c_{sp}^d + \varepsilon < \delta$.

We have that $\tau^c > \tau^s$, that is, the complete network is more resilient than the star-shaped network, if and only if

$$\delta - \frac{1}{4\lambda}\left[1-\left(\frac{\lambda}{\gamma}\right)^2\right] + \varepsilon > (n-1)\left\{\frac{2}{n}\delta - \frac{1}{2\lambda n}\left[1-\left(\frac{\lambda}{\gamma}\right)^2\right]\right\}.$$

After rearranging, we get

$$\gamma > \frac{\lambda}{\sqrt{1-4\lambda\delta+\frac{n}{n-1}4\lambda\varepsilon}}.$$

Notice that, if $\varepsilon = 0$ (i.e., $c_s^d = c_{sp}^d$), the previous condition does not hold and the star-shaped network is still more resilient than the complete network (as with the efficient interbank deposits). However, for $\varepsilon > 0$ and for γ comprised in the range

$$\frac{\lambda}{\sqrt{1-4\lambda\delta}} > \gamma > \frac{\lambda}{\sqrt{1-4\lambda\delta+\frac{n}{n-1}4\lambda\varepsilon}},$$

we have that $\tau^c > \tau^s$, that is, the complete network is more resilient than the star-shaped network.

The comparison between τ^c and the other two thresholds, τ_2^s and $\widetilde{\tau}^s$, is more difficult to establish, but the result that the star-shaped network is more resilient than the complete

[27] Notice that we consider the case in which the centre bank acts as a clearing house as well. A similar analysis and conclusions apply if $\eta\lambda < \gamma < \lambda/\sqrt{1-4\lambda\delta}$ (i.e., if the bank at the centre faces intermediation costs).

one does not hold unequivocally in the decentralised analysis. On one hand, decentralised interbank deposits in the star-shaped network are closer to efficient ones than the analogous deposits in the complete network; therefore, banks in the latter network incur higher costs in terms of premature asset liquidation. On the other hand, though, the decentralised interbank deposits in the complete network could be sufficiently small to render this network less exposed to the risk of financial contagion. This result displays the existence of a trade-off between the coverage of liquidity risk and the exposure to systemic risk. We did not fully exploit this trade-off in this paper since we took the network structure as given both in the social planner solution and in the decentralised one. A challenging avenue of research is to fully endogenise such a trade-off.

4. Conclusions

In this paper, we compare the performance of three classes of networks: star-shaped, complete and incomplete regular networks. We define efficiency as the complete transfer of liquidity from banks in surplus to banks in deficit in order to prevent the costly early liquidation of long-term assets. We show that the complete network achieves the full coverage of liquidity risk if each bank holds an amount of deposits that is roughly twice the amount held by the peripheral banks in the star-shaped network. Incomplete regular networks provide complete insurance against liquidity risk only with interbank deposits that are larger than those required in the complete and star-shaped networks. The benefits of holding a smaller amount of interbank deposits lies in the containment of systemic risk. We show that the star-shaped network is less exposed to systemic risk than the complete network, as it holds the smallest interbank deposits and because of the shelter role of the bank at the centre.

We then study the decentralised interbank decision while taking the network structure as given. When banks do not bear an intermediation cost of holding interbank deposits, they make the efficient decision in both the complete and the star-shaped networks. When banks face such a cost, in both types of networks, banks hold an amount of interbank deposits that is smaller than their respective efficient amount. However, the star-shaped network induces to hold an amount of interbank deposits that is closer to its efficient amount. This implies that the decentralised star-shaped network induces smaller losses due to the premature liquidation of long-term assets. However, this feature could expose the system to higher systemic risk than the decentralised complete network because of the larger interbank deposits. A complete study of the trade-off between the coverage of liquidity risk and the risk of contagion is a challenging area for future research.

Appendix

To demonstrate Propositions 1, 2 and 3, we turn an interbank network $N = \{\Omega, \Lambda\}$, as defined above, into a flow network. We then apply some results of flow network theory. To transform N into a flow network, it is sufficient to add to it the liquidity shock. Formally, this means to add a set of source nodes (i.e., nodes with no incoming links) and a set of sink nodes (i.e., nodes with no outgoing links). To each surplus bank, ω_i in $\Omega^+ \subseteq \Omega$, we attach a *source node,* s_i and a link, l_{si}, that connects the surplus bank to the source node. Correspondingly, to each deficit bank, ω_i in $\Omega^- \subseteq \Omega$, we attach a *sink node,* t_i, and a link, l_{it}, that connects the deficit bank to the sink node. A liquidity shock that hits an interbank network is defined as a four-tuple $\Delta = \{S, T, \Lambda^+, \Lambda^-\}$ where $S = \{s_i | \forall i \in \Omega^+\}$ is the set of source nodes, $T = \{t_i | \forall i \in \Omega^- \}$ is the set of sink nodes, $\Lambda^+ = \{l_{si}\}$ and $\Lambda^- = \{l_{it}\}$ are the sets of links that connect sources and sinks to the surplus and deficit banks, respectively.

Adding the liquidity shock Δ to an interbank liquidity network, N, we obtain an *interbank liquidity flow* network, L, which is an n-tuple: $L = \{N, \Delta\} = \{\Omega, S, T, \Lambda, \Lambda^+, \Lambda^-\}$. An interbank liquidity flow, L, is a value assignment to the links in Λ, Λ^+ and Λ^- such that: i) no link carries a flow larger than its own capacity (*capacity constraint*); ii) the divergence of a node, i.e., the difference between its inflow and its outflow, is null for all nodes in Ω (*flow conservation property*). A flow that complies with these two requirements is *feasible.* In other words, a flow is feasible if it comes out of the sources, crosses the network and ends entirely in the sinks, without exceeding the capacity of the links that carry the flow. The value of the largest feasible flow that can cross a flow network is called the *carrying capacity* of the network.

Finding the carrying capacity of a network is a fundamental problem in the theory of flow networks – known as the *maximum flow problem.* A solution to this problem is given by the *minimum cut-maximum flow* theorem provided by Ford and Fulkerson (1956).

Theorem 1 (Ford and Fulkerson 1956) *In every flow network, the maximum value of a flow equals the capacity of a cut of minimum capacity.*

The theorem states that the carrying capacity of a network is equal to the capacity of the *cut* which has the smallest *capacity* among all possible cuts of the network. A *cut* is a partition, $\{U, \overline{U}\}$, of the set of nodes, $\{\Omega, S, T\}$, of a flow network such that $S \subseteq U$ and $T \subseteq \overline{U}$, i.e. all source nodes are in U and all sink nodes are in $\overline{U}$. The *capacity* of a cut is the sum of the capacities of its *forward links*, which are the links going from U into $\overline{U}$. In other words, the cut of smallest capacity is the bottleneck of a network and sets the upper bound to the magnitude of the flows that such a network can transfer from sources to sinks. The maximum feasible flow of a network is achievable by social planner with a proper value assignment to the flows carried by each link in Λ.[28]

The three following propositions use the Ford-Fulkerson theorem to characterise the minimum interbank deposit that enables the social planner to obtain a complete *ex-post* reallocation of liquidity in the star-shaped, the complete and incomplete networks.

Proof of Proposition 1. Let $L^s = \{\Omega, S, T, \Lambda^s, \Lambda^+, \Lambda^-\}$ be a star-shaped interbank liquidity flow network. For the sake of notational simplicity, we assume that $n = |\Omega|$ is an odd number. Let $(U, \overline{U})$ be a cut of L^s, i.e. $S \subseteq U$ and $T \subseteq \overline{U}$. Let $Z = U \backslash S$ be the set of banks in U and, correspondingly, let $Y = \overline{U} \backslash T$ be the set of banks in $\overline{U}$. Let z^- be the number of deficit peripheral banks in Z, and let y^+ be the number of surplus peripheral banks in Y. Recall that the variation of the customer deposits of the centre bank ω_c is equal to $\Delta h_c = (n - 1 - 2x)\delta$, where x is the number of deficit peripheral banks in L^s. Correspondingly, we attach to the bank at the centre a source node if $\Delta h_c > 0$ and a sink node if $\Delta h_c < 0$. In the former case, the capacity is equal to $(n - 1 - 2x)\delta$, while in the latter case the capacity is equal to $-(n - 1 - 2x)\delta = (2x - n + 1)\delta$. Thus the structure of L^s, and the capacities of its possible cuts, also depend on the realisation of the shock.

We now seek to characterise the cut of L^s with the minimum capacity. We have three possible scenarios:

1. $\Delta h_c > 0$, i.e. $x < (n-1)/2$ and $\omega_c \in \Omega^+$. The formula that characterises the capacity of a cut of L^s depends on whether the centre bank is in Z or Y. In this case, we have

$$\Gamma(U, \overline{U}) = |Y|\, c_s + \left(z^- + y^+\right) \delta \text{ if } \omega_c \in Z, \text{ and} \tag{12}$$

$$\Gamma(U, \overline{U}) = |Z|\, c_s + \left(z^- + y^+\right) \delta + (n - 1 - 2x)\delta \text{ if } \omega_c \in Y. \tag{13}$$

The first addenda of these equations are the sums of the capacities of the links that go

[28] To obtain the maximum flow, the planner must ensure that all forward links that cross the minimum cut are filled to capacity, while all backward links that cross such a cut must carry no flow.

from Z into Y, $\Gamma(Z,Y)$, where each link has capacity c_s. The second addenda are the sums of the capacities of the links starting from deficit peripheral banks in Z and ending in the sink nodes in T, plus the sums of the capacities of the links starting from the source nodes in S and ending in surplus peripheral banks in Y, each with capacity δ. Finally, the third addendum of (13) is the capacity of the link that goes from the source node attached to the centre bank to the latter (that, in this case, lies in the set Y).

Let us analyse equation (12). Note that $(z^- + y^+)$ in (12) is minimal for sets Y such that $Y \subseteq \Omega^-$, for $|Y| \leq (n-1)/2$, and $Y \supset \Omega^-$, for $|Y| > (n-1)/2$. This is so because $(z^- + y^+)$ diminishes of one unit as we move one deficit peripheral bank from Z into Y, while $(z^- + y^+)$ increases of one unit if we move one surplus peripheral bank from Z into Y. Since we seek to minimize (12), we restrict the attention to sets Y such that $Y \subseteq \Omega^-$ or $Y \supset \Omega^-$. With this facilitating restriction, it can be checked by inspection that:

a) if $c_s > \delta$, then with $Y = \varnothing$ (12) is minimized and equal to $(n-1-x)\delta$;

b) if $c_s < \delta$, then with $Y = \Omega^-$ (12) is minimized and equal to $(n-1-x)c_s < (n-1-x)\delta$;

c) if $c_s = \delta$, then with $Y = \Omega^-$ (12) is minimized and equal to $(n-1-x)c_s = (n-1-x)\delta$.

Consider now equation (13). If $\omega_c \in Y$ we have that $(z^- + y^+)$ is minimal for sets Z such that $Z \subseteq \Omega^+$, for $|Z| \leq n/2$, and $Z \supset \Omega^+$, for $|Z| > n/2$, for the same reason expounded above. Again, since we seek to minimize (13), we restrict the attention to the sets Z such that $Z \subseteq \Omega^+$ and $Z \supset \Omega^+$. Then it can be checked by inspection that:

a) if $c_s > \delta$, then (13) is minimized for $Z = \varnothing$ and is equal to $(n-1-x)\delta$;

b) if $c_s < \delta$, then (13) is minimized for $Z = \Omega^+$ and is equal to $xc_s + (n-1-2x)\delta < (n-1-x)\delta$;

c) if $c_s = \delta$, then (13) is minimized for $Z = \Omega^+$ and is equal to $xc_s + (n-1-2x)\delta = (n-1-x)\delta$.

Therefore, the planner achieves the complete reallocation of liquidity in L^s if $c_s \geq \delta$, filling the deficit δ of all the $n-1-x$ deficit banks with a flow equal to the flow out of the source nodes (i.e. the sum of the surpluses), $xc_s + (n-1-2x)\delta$.

2. $\Delta h_c < 0$, i.e. $x > (n-1)/2$ and $\omega_c \in \Omega^-$. Again, the formula that characterises the capacity of a cut of L^s depends on whether the centre bank is in Z or Y. We have

$$\Gamma(U,\overline{U}) = |Y|\, c_s + \left(z^- + y^+\right)\delta + (2x-n+1)\delta \text{ if } \omega_c \in Z, \text{ and} \tag{14}$$

$$\Gamma(U,\overline{U}) = |Z|\, c_s + \left(z^- + y^+\right)\delta \text{ if } \omega_c \in Y. \tag{15}$$

The third addendum of (14) is the capacity of the link that goes from the centre bank to the sink node attached to it (since, in this case, ω_c lies in the set Z). Like above, and for the same reasons, we restrict the attention to sets Z and Y such that $Z \subseteq \Omega^+$ or $Z \supset \Omega^+$, and $Y \subseteq \Omega^-$ or $Y \supset \Omega^-$. Then it can be checked by inspection that:

a) if $c_s > \delta$, then (14) is minimized for $Y = \varnothing$ and is equal to $(n-1-x)\,\delta + (2x - n + 1)\delta = x\delta$;

b) if $c_s < \delta$, then (14) is minimized for $Y = \Omega^-$ and is equal to $(n-1-x)c_s + (2x - n + 1)\delta < x\delta$;

c) if $c_s = \delta$, then (14) is minimized for $Y = \Omega^-$ and is equal to $(n-1-x)c_s + (2x - n + 1)\delta = x\delta$.

Consider now equation (15). We have that:

a) if $c_s > \delta$, then (15) is minimized for $Z = \varnothing$ and is equal to $(n-1-x)\,\delta + (2x - n + 1)\delta = x\delta$;

b) if $c_s < \delta$, then (15) is minimized for $Z = \Omega^+$ and is equal to $(n-1-x)c_s + (2x - n + 1)\delta < x\delta$;

c) if $c_s = \delta$, then (15) is minimized for $Z = \Omega^+$ and is equal to $(n-1-x)c_s + (2x - n + 1)\delta = x\delta$.

Therefore, the planner achieves the complete reallocation of liquidity in L^s if $c_s \geq \delta$, filling the deficit of all deficit banks with a flow equal to the flow coming out of the source nodes: $(n-1-x)\delta + (2x-n+1)\delta = x\delta$.

3. $\Delta h_c = 0$, i.e. $x = (n-1)/2$, which means that there is an equal number of surplus peripheral banks and of deficit peripheral banks. Since the centre bank is neither in surplus nor in deficit, we attach no source or sink node to it. We have

$$\Gamma(U, \overline{U}) = |Y|\, c_s + \left(z^- + y^+\right) \delta \text{ if } \omega_c \in Z, \text{ and} \tag{16}$$

$$\Gamma(U, \overline{U}) = |Z|\, c_s + \left(z^- + y^+\right) \delta \text{ if } \omega_c \in Y. \tag{17}$$

Like above, and for the same reasons, we restrict the attention to sets Z and Y such that $Z \subseteq \Omega^+$ or $Z \supset \Omega^+$, and $Y \subseteq \Omega^-$ or $Y \supset \Omega^-$.

Note that, for $|Y| \leq (n-1)/2$,we have $y^+ = 0$ and $z^- = ((n-1)/2 - |Y|)$, while for $|Y| > n/2$, $y^+ = (|Y| - (n-1)/2)$ and $z^- = 0$. Likewise, for $|Z| \leq (n-1)/2$, we have $y^+ = ((n-1)/2 - |Z|)$ and $z^- = 0$, while for $|Z| > (n-1)/2$, $y^+ = 0$ and $z^- = (|Z| - (n-1)/2)$. Thus we rewrite the above equations respectively as:

$$\Gamma(U, \overline{U}) = |Y|\, c_s + |(n-1)/2 - |Y||\, \delta \text{ if } \omega_c \in Z, \text{ and} \tag{18}$$

$$\Gamma(U, \overline{U}) = |Z|\, c_s + |n/2 - 1 - |Z||\, \delta \text{ if } \omega_c \in Y. \tag{19}$$

Therefore, if $\omega_c \in Z$:

i) if $c_s > \delta$, then with $Y = \varnothing$ (18) is minimized and equal to $\delta(n-1)/2$;

ii) if $c_s < \delta$, then with $Y = \Omega^-$ (18) is minimized and equal $c_s(n-1)/2 < \delta(n-1)/2$;

iii) if $c_s = \delta$, then with $Y = \Omega^-$ (18) is minimized and equal $c_s(n-1)/2 = \delta(n-1)/2$.

Likewise, if $\omega_c \in Y$:

i) if $c_s = \delta$, then with $Z = \varnothing$ (19) is minimized and equal to $\delta(n-1)/2$;

ii) if $c_s < \delta$, then with $Z = \Omega^+$ (19) is minimized and equal $c_s(n-1)/2 < \delta(n-1)/2$;

iii) if $c_s = \delta$, then with $Z = \Omega^+$ (19) is minimized and equal $c_s(n-1)/2 = \delta(n-1)/2$.

Therefore, the planner achieves the complete reallocation of liquidity in L^s if $c_s \geq \delta$, filling the deficit of all the $(n-1)/2$ deficit banks with a flow equal to the flow coming out of the source nodes: $\delta(n-1)/2$. The planner achieves the complete coverage of liquidity risk in L^s by coordinating interbank deposits withdrawals and with the smallest interbank deposit $c_s = \delta$. ∎

Proof of Proposition 2. Let $L^c = \{\Omega, S, T, \Lambda^c, \Lambda^+, \Lambda^-\}$ be a complete interbank liquidity flow network. Let $(U, \overline{U})$ be a cut of L^c and let Z, Y, z^- and y^+ be defined as in the proof of Proposition 1. Then the capacity $\Gamma(U, \overline{U})$ of a cut in L^c is

$$\begin{aligned}\Gamma(U,\overline{U}) &= |Z|\,|Y|\,c_{ij} + \left(z^- + y^+\right)\delta \\ &= |Z|\,(n - |Z|)c_{ij} + \left(z^- + y^+\right)\delta,\end{aligned}$$

where $|Z|\,|Y|\,c_{ij}$ is the sum of the capacities of the links starting from banks in Z and ending in banks in Y. Like in the proof of Proposition 1, and for the same reasons, we restrict the attention to sets Z and Y such that $Z \subseteq \Omega^+$ or $Z \supset \Omega^+$, and $Y \subseteq \Omega^-$ or $Y \supset \Omega^-$. Under this restriction, we have $y^+ = (n/2 - |Z|)$ and $z^- = 0$ for $|Z| \leq n/2$,, while for $|Z| > n/2$, we have $y^+ = 0$ and $z^- = (|Z| - n/2)$. Hence we rewrite the capacity $\Gamma(U, \overline{U})$ as:

$$\Gamma(U,\overline{U}) = |Z|\,(n - |Z|)c_{ij} + |n/2 - |Z||\,\delta. \tag{20}$$

The first addendum of equation (20) is a concave parabola with two minima at the extremes of the range of $|Z|$, i.e. it is minimal and equal to zero for $|Z| = 0$ and $|Z| = n$. The second addendum is a piecewise linear and convex function with minimum equal to zero for $|Z| = n/2$. It can be checked by inspection that, for all $c_{ij} < \delta$, equation (20) is m-shaped, with three local minima corresponding to $|Z| = 0$, $|Z| = n/2$, and $|Z| = n$. More precisely:

1. for $c_{ij} = \frac{2}{n}\delta$, i.e. for $c_c = \frac{n-1}{n}2\delta$, (20) has three global minima, for $|Z| = 0, |Z| = n/2$, and $|Z| = n$, all equal to $\frac{n}{2}\delta$;

2. for $c_{ij} > \frac{2}{n}\delta$, i.e. for $c_c > \frac{n-1}{n}2\delta$, (20) has two global minima, for $|Z| = 0$ and $|Z| = n$, both equal to $\frac{n}{2}\delta$;

3. for $c_{ij} < \frac{2}{n}\delta$, i.e. for $c_c < \frac{n-1}{n}2\delta$, (20) has a minimum for $|Z| = n/2$ and equal to $\left(\frac{n}{2}\right)^2 c_{ij} = \frac{n^2}{4(n-1)}c_c < \frac{n}{2}\delta$.

Thus, a complete interbank network N^c with an interbank deposit $c_c \geq \frac{n-1}{n}2\delta$ has a carrying capacity sufficient to provide full coverage of liquidity risk $\frac{n}{2}\delta$. Note that this capacity is achievable even without the planner's withdrawal coordination. The planner achieves the complete coverage of liquidity risk in L^c with the smallest interbank deposit $c_c = \frac{n-1}{n}2\delta$ (or $c_{ij} = \frac{2}{n}\delta$). Notice that the coordination of withdrawals by the planner is sufficient but not necessary to obtain the full coverage of liquidity risk. Given the symmetric structure of the network, the same result would obtain if all deficit banks withdraw from all their neighbouring banks (not only the surplus banks). ■

Proof of Proposition 3. Let $L^r = \{\Omega, S, T, \Lambda^r, \Lambda^+, \Lambda^-\}$ be an *incomplete regular* interbank liquidity flow network with degree $k \geq n/2$. Let $(U, \overline{U})$ be a cut of L^c and let Z, Y, z^- and y^+ be defined as in the proof of Proposition 1.

The capacity $\Gamma(U, \overline{U})$ of a cut in L^r is $\Gamma(U, \overline{U}) = \Gamma(X, Y) + (z^- + y^+)\,\delta$. Like in the proofs of propositions 1 and 2, we restrict the attention to sets Z and Y that minimize $(z^- + y^+)$, i.e. such that $Z \subseteq \Omega^+$ or $Z \supset \Omega^+$, and $Y \subseteq \Omega^-$ or $Y \supset \Omega^-$. Note that:

1. By the assumption of bilateral obligations among the banks in Ω, we have that $\Gamma(X, Y) = \Gamma(Y, X)$. For convenience, below we write eq. (22) exploiting the fact that, for $|X| = n/2, ..., n$, $\Gamma(X, Y)$ is equal to $\Gamma(Y, X)$ for $|Y| = n - |X|$.

2. For $|X| = 0, ..., n/2$, $\Gamma(X, Y)$ is minimal for sets X which are maximally connected, i.e. sets X such that each node in X is connected to all other nodes in X (recall that $k \geq n/2$). The same applies to $\Gamma(Y, X)$ for $|Y| = 0, ...n/2$: it is minimal for sets Y in which each node is connected to all other nodes in Y.

Hence, since we are seeking the cut that minimises $\Gamma(U, \overline{U})$, we further restrict our attention to partitions (X, Y) of Ω where: i) for $|X| \leq n/2$, X is maximally connected and $X \subseteq \Omega^+$; ii) for $|Y| \leq n/2$, Y is maximally connected and $Y \subseteq \Omega^-$. With this restriction, we minimize the number of links that go from the set of surplus banks into the set of deficit banks (i.e. the *bridge* that connects the two sets). Under this restriction we have that i) for $|X| \leq n/2$, $y^+ = (n/2 - |X|)$, $x^- = 0$ and each bank in X has $k - |X| - 1$ links with banks in Y, while ii) for $|X| \geq n/2$, $x^- = (|X| - n/2) = (n/2 - |Y|)$, $y^+ = 0$ and each bank in Y has $k - |Y| - 1$ links with banks in X. Hence we write the capacity $\Gamma(U, \overline{U})$ of a cut in L^r as

$$\Gamma(U, \overline{U}) = |X| \left[(k+1) - |X|\right] c_{ij} + \left(\frac{n}{2} - |X|\right) \delta \text{ for } |X| \leq n/2, \tag{21}$$

$$\Gamma(U, \overline{U}) = |Y| \left[(k+1) - |Y|\right] c_{ij} + \left(\frac{n}{2} - |Y|\right) \delta \text{ for } |Y| \leq n/2. \tag{22}$$

It can be checked by inspection that, for $c_{ij} < \frac{1}{k}\delta$, both equations (21) and (22) are strictly concave, with local minima at the extremes of the ranges of their respective arguments. More specifically, we have that:

1. for $c_{ij}(k) = \frac{1}{k+1-n/2}\delta$, i.e. for $c_r(k) = \frac{k}{k+1-n/2}\delta$, we have i) equation (21) has two global minima, for $|X| = 0$, and $|X| = n/2$; and ii) equation (22) has two global minima, for $|Y| = 0$, and $|Y| = n/2$. Each of such minima is equal to $\frac{n}{2}\delta$. Thus, with these interbank deposits, the carrying capacity of N^r is equal to $\frac{n}{2}\delta$.

2. for $c_{ij}(k) > \frac{1}{k+1-n/2}\delta$, i.e. for $c_r(k) > \frac{k}{k+1-n/2}\delta$, both equations (21) and (22) are minimal and equal to $\frac{n}{2}\delta$ for, respectively, $|X| = 0$ and for $|Y| = 0$. With such deposits the upper bound to the carrying capacity is set by the value of the liquidity shock. This means that the interbank deposits are larger than what is necessary for the coverage of liquidity risk, if interbank deposits are coordinated.

3. for $c_{ij} < \frac{1}{k+1-n/2}\delta$, i.e. for $c_r(k) < \frac{k}{k+1-n/2}\delta$, both equations (21) and (22) are minimal and equal to $\frac{n}{2}(k + 1 - \frac{n}{2})c_{ij} < \frac{n}{2}\delta$ for, respectively, $|X| = n/2$ and for $|Y| = n/2$. In this case the interbank deposits are not sufficiently large to support a complete post-shock reallocation of liquidity from surplus banks to deficit banks.

To sum up, an incomplete interbank network L^r with an interbank deposit $c_r(k) \geq \frac{k}{k+1-n/2}\delta$ has a carrying capacity sufficient to provide full coverage of liquidity risk $(n/2)\delta$. The planner achieves the complete coverage of liquidity risk in L^r by coordinating interbank deposits withdrawals and with the smallest interbank deposit $c_r(k) = \frac{k}{k+1-n/2}\delta$. ■

To compare the exposure to systemic risk of different network structures, we represent them as *financial flow networks*. We turn an interbank network, $N = (\Omega, \Lambda)$, into a financial flow network, F, by adding to it: i) a set $A = \{a\}$ of *source nodes*, that represent the external assets a held by the banks in Ω; ii) two sets, Q and T, of *sink nodes*, where Q represents the equity holders of the banks in Ω and T represents the households who hold debt claims (customer deposits) against the banks in Ω; iii) a set of links, $\Lambda^a = \{l_i^a\}$, that connect the external assets a to the banks in Ω that own them; iv) a set of links,

$\Lambda^q = \{l^i_Q\}$, that connect the banks in Ω to their shareholders in Q; and v) a set of links, $\Lambda^h = \{l^i_T\}$, that connect the banks in Ω to their bondholders and depositors in T. Let $F^c = \{\Omega, A, Q, T, \Lambda^c, \Lambda^a, \Lambda^q, \Lambda^h\}$ and $F^s = \{\Omega, A, Q, T, \Lambda^s, \Lambda^a, \Lambda^q, \Lambda^h\}$ be the financial flow networks corresponding to the complete and star-shaped interbank networks, respectively. The flows that cross the two financial flow networks represents a flow of value going from the external assets (the source nodes A) into the portfolios of the final claimants: shareholders (sink Q) and depositors (sink T).

Propositions 4, 5 and 6, make use of the following property of network flows: the value of the net forward flow that crosses a cut is the same for all the cuts of the network.[29] Applying this property to a financial flow network, F, we have that the value of the net forward flow that crosses all cuts of F equals the value of the exogenous shock, i.e. the flow that crosses the cut $\{A, (\Omega, T, Q)\}$. It follows that the value of the exogenous shock is equal to the forward flow that goes from the set of defaulting nodes into the rest of the network, i.e. the flow that crosses the cut $\{(A, \Phi), (\Omega\backslash\Phi, Q, T)\}$.

Proof of Propositon 4. Let m be the number of primary defaults caused by a shock $m = |\Phi|$ and let $b_i \in [0, 1]$ be a parameter that measures the percentage *loss-given-default* of a node, i.e. it measures the share of the value of the liabilities issued by the i-th bank which is lost upon its default. Each node $\omega_i \in \Phi$ sends a flow equal to its own equity e_i to the sink Q, a flow equal to $b_i h_i$ to the sink T and a flow equal to $b_i d_{ij}$ to each of its $(n-m)$ creditors in $\Omega\backslash\Phi$. Thus the forward flow that crosses the cut $\{(A, \Phi), (\Omega\backslash\Phi, Q, T)\}$, which is equal to the shock that comes out of the source nodes, is

$$me_i + mb_i h_i + mb_i d_{ij}(n - m), \tag{23}$$

where the term me_i is the value of the flow of losses going from Φ to Q, the term $mb_i h_i$ is the flow of losses that goes from Φ to T, and the sum $mb_i d_{ij}(n - m)$ is the flow of losses going from Φ to $\Omega\backslash\Phi$. In a complete network N^c, each node in $\Omega\backslash\Phi$ receives a flow of losses equal to $mb_i d_{ij}$ from its defaulting debtors. For default contagion to occur, this flow of losses must be larger than or equal to the absorbing capacity of a node that is given by its capital: $mb_i d_{ij} \geq e_j$. The value of an exogenous shock that is exactly large enough to cause such a condition to be fulfilled, i.e. such that $mb_i d_{ij} = e_j$, constitutes both the first and the final threshold of contagion of a network N^c. All nodes in $\Omega\backslash\Phi$ default together if such a threshold is reached. This condition requires that $mb_i = e_j/d_{ij}$. Substituting this

[29] See Ahuja, Magnanti and Orlin (1993) page 179.

value in (23), and recalling that e_i is the same for all banks in Ω, we obtain the first and final contagion threshold of a complete network:

$$\tau^c = me_i + \frac{e_i}{d_{ij}}h_i + \frac{e_i}{d_{ij}}d_{ij}(n-m) = ne_i + e_i\frac{h_i}{d_{ij}}.$$

■

Proof of Proposition 5. We use again the result that the shock out of the source nodes is equal to the forward flow that crosses the cut $\{(A, \Phi), (\Omega\backslash\Phi, Q, T)\}$. Considering the two cases in the proposition, we have

1) if $\Phi = \omega_c$, the flow that crosses the cut $\{(A, \omega_c), (\Omega\backslash\omega_c, T, Q)\}$ is equal to

$$e_c + b_c h_c + b_c d_s(n-1), \tag{24}$$

where b_c is the *loss-given-default* parameter of ω_c. Contagion occurs for any shock such that $b_c d_s(n-1) \geq e_p(n-1)$. The smallest of such shocks is the one that causes $b_c d_s(n-1) = e_p(n-1)$, hence $b_c = e_p/d_s$. This condition characterises both the first and the final threshold of contagion. Substituting $b_c = e_p/d_s$ into equation (24), we obtain

$$\tau^s = e_c + \frac{e_p}{d_s}h_c + \frac{e_p}{d_s}d_s(n-1) = e_c + (n-1)e_p + e_p\frac{h_c}{d_s}.$$

2) if $\Phi = \{\omega_c, \omega_p |$ for some $p \in (1, ..., n-1)\}$, the flow that crosses the cut $\{(A, \Phi), (\Omega\backslash\Phi, T, Q)\}$ is equal to

$$(m-1)e_p + e_c + (m-1)b_p h_p + b_c h_c + b_c d_s(n-m),$$

where $m = |\Phi|$, the sum $(m-1)e_p + e_c$ is the flow of losses that goes from the set of primary defaults into the sink Q, the sum $(m-1)b_p h_p + b_c h_c$ is the flow of losses that goes from Φ into the sink T, and the term $b_c d_s(n-m)$ is the flow of losses that goes from the central node to its creditors in $\Omega\backslash\Phi$. Both first and complete contagion occur for any shock such that $b_c d_s(n-m) \geq e_p(n-m)$, hence $b_c \geq e_p/d_s$. Taking the smallest of such shocks – i.e., $b_c = e_p/d_s$ – we obtain the first and final threshold

$$\widetilde{\tau}^s = e_c + (n-1)e_p + e_p\frac{h_c}{d_s} + (m-1)b_p h_p. \tag{25}$$

To determine $(n-1)b_p$ we apply the conservation property to the bank at the centre, and given σ_c is the loss of value of the external assets borne by ω_c, we have

$$\begin{aligned} \sigma_c + (m-1)b_p d_s &= e_c + (n-1)b_c d_s + b_c h_c \\ &= e_c + (n-1)e_p + e_p\frac{h_c}{d_s} \end{aligned}$$

thus

$$(m-1)b_p = \left[e_c + (n-1)e_p + e_p\frac{h_c}{d_s}\right]\frac{1}{d_s} - \frac{\sigma_c}{d_s}.$$

Since $(m-1)b_p > 0$ it has to be $\sigma_c < \tau^s$. Plugging $(m-1)b_p$ in equation (25), we have the result

$$\widetilde{\tau}^s = \left[e_c + (n-1)e_p + e_p\frac{h_c}{d_s}\right]\left(1 + \frac{h_p}{d_s}\right) - \sigma_c\frac{h_p}{d_s}.$$

■

Proof of Proposition 6. If $\Phi = \{\omega_p | \text{for some } p \in (1, ..., n-1)\}$ and $\omega_c \notin \Phi$, the flow that crosses the cut $\{(A, \Phi), (\Omega \backslash \Phi, T, Q)\}$ is equal to

$$me_p + mb_ph_p + mb_pd_s, \tag{26}$$

where me_p and mb_ph_p are the flows that Φ sends into Q and T, respectively, and mb_pd_p is the flow that the central node ω_c receives from the defaulting nodes in Φ. The smallest shock that reaches the first threshold of contagion is such that $mb_pd_s = e_c$, hence $mb_p = e_c/d_s$. Substituting this value into equation (26), we obtain $\tau_1^s = me_p + e_c(1 + h_p/d_s)$.

The number m of peripheral banks in the set of primary defaults (which is necessary and sufficient to induce the default of the bank at the centre) is minimal when the shock borne by each of these defaulting banks is maximal, i.e. when the shock is equal the external assets a_p held by a peripheral bank. Thus, the loss-given-default of a peripheral bank is $b_p = \frac{a_p - e_p}{h_p + d_s}$. By the balance sheet identities we have that $a_p = h_p + e_p$, hence $b_p = \frac{h_p}{h_p + d_s}$. Plugging b_p in $mb_pd_s = e_c$ we obtain $m = e_c\left(\frac{1}{h_p} + \frac{1}{d_s}\right)$.

The final threshold of contagion is set by the flow that crosses the cut $\{(A, \Phi, \omega_c), (\Omega \backslash (\Phi, \omega_c), T, Q)\}$ which is equal to

$$e_c + me_p + mb_ph_p + b_ch_c + b_cd_s(n-m-1), \tag{27}$$

where $me_p + e_c$ is the flow of losses going from the primary defaults into the sink Q; $mb_ph_p + b_ch_c$ is the flow of losses going from the primary defaults into the sink T; and $b_cd_s(n-m-1)$ is the flow of losses that goes from the bank at the centre into the peripheral banks not in the set of primary defaults $\Omega \backslash (\Phi, \omega_c)$. All banks in $\Omega \backslash (\Phi, \omega_c)$ default if the bank at the centre sends to each of them a flow larger than or equal to e_p. The final threshold of contagion is equal to the smallest of such shocks, i.e. $b_c = e_p/d_s$. Plugging b_c into equation (27) we get

$$\tau_2^s = e_c + (n-1)e_p + e_p\frac{h_c}{d_s} + mb_ph_p. \tag{28}$$

To obtain mb_p, we apply again the flow conservation property to the bank at the centre (i.e., the flow that enters the central node has to be equal to the flow that exits from it). we have

$$\begin{aligned} mb_p d_s &= e_c + (n-1)b_c d_s + b_c h_c \\ &= e_c + (n-1)e_p + e_p \frac{h_c}{d_s}. \end{aligned}$$

Thus

$$mb_p = \left[e_c + (n-1)e_p + e_p \frac{h_c}{d_s} \right] \frac{1}{d_s}.$$

Substituting mb_p in equation (28), we obtain the final threshold τ_2^s

$$\tau_2^s = \left[e_c + (n-1)e_p + e_p \frac{h_c}{d_s} \right] \left(1 + \frac{h_p}{d_s} \right).$$

■

Proof of Lemma 1. Suppose that a bank ω_i in N^c deposits the same amount $\overline{c}_{ij}$ in each of its $n-1$ neighbours. Then $c_c = (n-1)\overline{c}_{ij}$. Assume that bank ω_i faces a liquidity deficit. In this case, bank ω_i withdraws $\overline{c}_{ij}$ from each of its $n/2$ neighbours that have a liquidity surplus. The liquidity $l(\overline{\mathbf{c}}_i)$ collected through the complete interbank network is equal to

$$l(\overline{\mathbf{c}}_i) = \frac{n}{2}\overline{c}_{ij} = \frac{n}{2(n-1)} c_c$$

with certainty. Hence, the expected cost of asset liquidation faced by bank ω_i is

$$\pi_i \left(l(\overline{\mathbf{c}}_i) \right) = \pi_i \left(\frac{n}{2(n-1)} c_c \right).$$

Suppose now that bank ω_i allocates its interbank deposits among its neighbours in an heterogeneous fashion. That is, assume that the vector of interbank deposits $\mathbf{c}_i$ is such that: i) $\sum_{j \in \mathcal{N}(i)} c_{ij} = c_c$; and ii) $c_{ij} \neq c_{ik}$ for at least one pair (ω_k, ω_j) in $\mathcal{N}(i)$. In this case, the amount of liquidity $l(\mathbf{c}_i)$ collected by bank ω_i is equal to $\sum_{j \in \Omega^+} c_{ij}$ and it is not certain any longer. It depends on the realisation of the liquidity shock and on which banks happen to belong to the set of surplus banks Ω^+. That is, the amount $l(\mathbf{c}_i)$ is now a random variable, and it is the sum of $n/2$ interbank deposits (not equal to one another) withdrawn from a set of $n/2$ surplus banks $(\omega_j | j \in \Omega^+)$. The set $(c_{ij} | j \in \Omega^+)$ of these deposits is a random sample out of the set composed of all elements of $\mathbf{c}_i$, i.e. out of the set $(c_{ij} | j \in \mathcal{N}(i))$. Thus the expected value of the mean of the sample $(c_{ij} | j \in \Omega^+)$ is equal to the mean of the elements of $\mathbf{c}_i$, i.e.

$$E\left(\frac{\sum_{j \in \Omega^+} c_{ij}}{n/2} \right) = \frac{\sum_{j \in \mathcal{N}(i)} c_{ij}}{n-1} = \frac{c_c}{n-1} = \overline{c}_{ij}.$$

Therefore, the liquidity $l(\mathbf{c}_i)$ the bank ω_i expect to collect is a random variable with expected value equal to

$$E(l(\mathbf{c}_i)) = \frac{n}{2}E\left(\frac{\sum_{j\in\Omega^+} c_{ij}}{n/2}\right) = \frac{n}{2(n-1)}c_c = \frac{n}{2}\bar{c}_{ij} = l(\bar{\mathbf{c}}_i),$$

and with strictly positive variance.

Recall that the expected cost of premature asset liquidation $\pi_i(l(\mathbf{c}_i))$ in (7) is a strictly convex function in $l(\mathbf{c}_i)$. By applying the Jensen inequality we have, for any amount of total interbank deposit c_c, the following

$$E(\pi_i(l(\mathbf{c}_i))) > \pi_i(E(l(\mathbf{c}_i))) = \pi_i\left(l(\bar{\mathbf{c}}_i)\right) = \pi_i\left(\frac{n}{2(n-1)}c_c\right)$$

The expected cost of asset liquidation faced by bank ω_i that allocates its interbank deposits evenly among its neighbours is strictly smaller than the expected cost faced by a bank that does not do so. ■

Proof of Lemma 2. Let $(\Omega\backslash\omega_c)^+$ be the set of peripheral banks that experience a surplus as the liquidity shock occurs and let $x = |(\Omega\backslash\omega_c)^+|$. The natural number x is a binomial random variable that takes on values $0, 1, 2, ..., n-1$ with probability

$$\binom{n-1}{x}0.5^x 0.5^{n-1-x} = \binom{n-1}{x}0.5^{n-1}.$$

Given the definition of the liquidity shock, if x peripheral banks experience a liquidity surplus then $n-1-x$ of them face a liquidity deficit. As the liquidity shock occurs, the centre bank withdraws its deposits from the peripheral banks in surplus.

Suppose that the centre bank ω_c splits its total interbank deposits c_{st} equally among the $n-1$ peripheral banks. The centre deposits $\widetilde{c}_s$ in each of the peripheral banks. Then, for any given realisation of x, the net amount of liquidity that ω_c collects through the network is $l(\widetilde{\mathbf{c}}_s) = x\widetilde{c}_s$ with certainty. The expected cost of asset liquidation that ω_c faces is then $\pi_c(x\widetilde{c}_s) = \pi_c(x\frac{c_{st}}{n-1})$.

Assume now that ω_c allocates its interbank deposits in the peripheral banks in a heterogeneous fashion. That is, $c_{st} = \sum_{p\in\Omega\backslash\omega_c} c_{sp}$, where c_{sp} is the amount that ω_c deposits in the peripheral bank ω_p, and $c_{sj} \neq c_{sk}$ for at least one pair $(\omega_k, \omega_j) \in \Omega\backslash\omega_c$. In this case, the amount of liquidity $l(\mathbf{c}_s)$ collected by the bank at the centre is equal to $\sum_{p\in(\Omega\backslash\omega_c)^+} c_{sp}$. The value $l(\mathbf{c}_s)$ depends both on the number x of peripheral banks in surplus and on which ones of the $n-1$ peripheral banks happen to be in surplus. The $l(\mathbf{c}_s)$ depends on both the cardinality and on the composition of the set $(\Omega\backslash\omega_c)^+$ which, in turn, depends on the realisation of the liquidity shock. The set of deposits that the bank at the centre has in the peripheral surplus banks $[c_{sp}|p \in (\Omega\backslash\omega_c)^+]$ is, therefore, a random sample of x elements out of the set of the $n-1$ deposits of ω_c in the peripheral banks $[c_{sp}|p \in (\Omega\backslash\omega_c)]$. The expected mean of such a sample of deposits, $E\left(\frac{1}{x}\sum_{p\in(\Omega\backslash\omega_c)^+} c_{sp}\right)$ is equal to the mean of the population from which it is drawn, i.e. the mean of all deposits of ω_c in the peripheral banks. It follows that the expected value of the mean of the sample $[c_{sp}|p \in (\Omega\backslash\omega_c)^+]$ is equal to the mean of the elements of $\mathbf{c}_s$, i.e.

$$E\left(\frac{1}{x}\sum_{p\in(\Omega\backslash\omega_c)^+} c_{sp}\right) = \frac{1}{n-1}\sum_{p\in\Omega\backslash\omega_c} c_{sp} = \frac{c_{st}}{n-1}.$$

In the case of uneven allocation of interbank deposits, the liquidity collection of ω_c, for any given value of x, is a random variable with mean

$$E(l(\mathbf{c}_s)) = xE\left(\frac{1}{x}\sum_{p\in(\Omega\backslash\omega_c)^+} c_{sp}\right) = x\frac{c_{st}}{n-1} = l(\widetilde{\mathbf{c}}_s)$$

and with strictly positive variance.

Given that the objective function of the centre bank $\pi_c(l(\mathbf{c}_s))$ is strictly convex in its argument, Jensen's inequality implies that for any amount c_s and any given realisation of x, we have

$$E(\pi_c(l(\mathbf{c}_s))) > \pi_c(E(l(\mathbf{c}_s))) = \pi_c\left(l(\widetilde{\mathbf{c}}_s)\right) = \pi_c\left(x\frac{c_{st}}{n-1}\right)$$

The expected loss of asset liquidation faced by ω_c if it allocates evenly its interbank deposits is strictly smaller than the expected cost if it does not do so. ■

Proof of Proposition 7. First notice that $\pi_i(\mathbf{c}_i) = 0$ for $\left(\delta - \frac{n}{2}c_{ij}\right) \leq 0$ and, given that a bank is assumed to choose the smallest deposit that guarantees the complete coverage from liquidity risk, the vector of bilateral interbank deposits $\mathbf{c}_i^d = \left\{c_{ij}^d = \delta\frac{2}{n}|\forall j \in \mathcal{N}(i)\right\}$ is chosen to obtain a zero expected liquidation loss. The vector $\mathbf{c}_i^d$ corresponds to a pairwise-stable equilibrium because no pair of banks benefits by choosing a bilateral interbank deposit $\widetilde{c}_{ij} < c_{ij}^d = \delta\frac{2}{n}$. Given Lemma 3, it suffices to consider the deviation of one bank. Assume bank ω_i deviates by lowering the amount of the bilateral interbank deposit with bank ω_j. That is, $\widetilde{\mathbf{c}}_i = (\widetilde{c}_{ij}, c_{-ij}^d)$. If bank ω_i experiences the liquidity deficit δ, it expects to collect from its surplus neighbouring banks an amount strictly less than $\frac{n}{2}c_{ij}^d = \delta$ since there is now the possibility that bank ω_j will be in surplus. In that case, bank ω_i collects an amount of liquidity equal to $(\frac{n}{2}-1)c^d_{ij}+\widetilde{c}_{ij}=\delta-(c^d_{ij}-\widetilde{c}_{ij})<\delta$. Therefore bank ω_i faces a strictly positive expected liquidation loss $\pi_i(\widetilde{c}_{ij},c^d_{-ij})>0=\pi_i(c^d_{ij},c^d_{-ij})$ and it will not deviate from $\mathbf{c}^d_i$. The vector $\mathbf{c}^d_i=\left\{c^d_{ij}=\delta\frac{2}{n}|\forall j\in\mathcal{N}(i)\right\}$ is also the unique equilibrium since any vector different from $\mathbf{c}^d_i$ would induce a deviation. Suppose banks hold a vector of interbank deposits equal to $\widetilde{\mathbf{c}}_i=\left\{0\leq\widetilde{c}_{ij}<c^d_{ij}|\forall j\in\mathcal{N}(i)\right\}$. When each bilateral interbank deposit is strictly less than $\delta\frac{2}{n}$ all banks expect to collect an amount of liquidity $l(\widetilde{\mathbf{c}}_i)=\frac{n}{2}\widetilde{c}_{ij}<\delta$. Banks, therefore, face a strictly positive expected loss of liquidation: $\pi(\frac{n}{2}\widetilde{c}_{ij})>0$. Suppose that bank ω_i deviates by slightly increasing the bilateral interbank deposit with its neighbouring bank ω_j. That is, $\widetilde{\mathbf{c}}^\varepsilon_i=(\widetilde{c}_{ij}+\varepsilon,\widetilde{c}_{-ij})$. In this way, bank ω_i increases the amount of liquidity that expects to collect from bank ω_j when it experiences a surplus. This reduces the expected loss of asset liquidation, that is $\pi_i\left(\widetilde{\mathbf{c}}^\varepsilon_i\right)<\pi_i\left(\widetilde{\mathbf{c}}_i\right)$. It follows, from Lemma 3, that any bank would deviate from the vector of interbank deposits $\widetilde{\mathbf{c}}_i$ by slightly increasing the bilateral exposure. By continuity, banks do not have incentives to deviate anymore when the vector of interbank deposits reaches $\mathbf{c}^d_i$. ■

Proof of Lemma 4. Let $x=|(\Omega\backslash\omega_c)^+|$ be the number of peripheral banks that experience a liquidity surplus δ. The number x is a binomial random variable that takes on values $0,1,2,...,n-1$ with probability $\binom{n-1}{x}0.5^x0.5^{n-1-x}=\binom{n-1}{x}0.5^{n-1}$. By the above definition of liquidity shock, if x peripheral banks experience a liquidity surplus then $n-1-x$ peripheral banks face a liquidity deficit. Thus the variation of the customer deposits of the peripheral banks as a whole is equal to $x\delta-(n-1-x)\delta$. Consequently, the variation of the stock of customer deposits of the bank at the centre is $\Delta h_c=(n-1-x)\delta-x\delta=(n-1-2x)\delta$.

Let y be the difference between the number of peripheral banks in surplus and the number of peripheral banks in deficit: $y\equiv|(\Omega\backslash\omega_c)^+|-|(\Omega\backslash\omega_c)^-|=x-(n-1-x)=2x-(n-1)$. For $1\leq y\leq(n-1)$ the bank at the centre faces a decrease of its customer deposits equal to $y\delta$, while it experiences a liquidity surplus if $y<0$. As the shock occurs, all the $(n-1-x)$ peripheral deficit banks withdraw c_s from the bank at the centre, while the latter withdraws its deposit c_s from each of the x peripheral banks in surplus. Thus, the amount of liquidity that ω_c collects through the network, if $y>0$, is equal to $xc_s-(n-1-x)c_s=yc_s$. If $y>0$ then the overall shortage of liquidity of the bank at the centre is equal to $y(\delta-c_s)$. Note that, for all $y\in(1,(n-1))$, the expected loss due to early asset liquidation is $\frac{1}{2}\left[1-\sqrt{1-4\lambda\left(\delta-c_s\right)y}\right]$, while it is equal to zero for all $y\in(-(n-1),0)$, i.e. for all $x\leq\left[\frac{n-1}{2}\right]$. We can express the expected losses due to asset liquidation of the bank at the centre as:

$$\begin{aligned}\pi_c(c_s) &= \sum_{x=0}^{\left[\frac{n-1}{2}\right]} \binom{n-1}{x} 0.5^{n-1} \left\{ \frac{1}{2} \left[1 - \sqrt{1 - 4\lambda\left(\delta - c_s\right)\left[2x - (n-1)\right]} \right] \right\} \\ &\quad + \sum_{x=\left[\frac{n-1}{2}\right]+1}^{n-1} \binom{n-1}{x} 0.5^{n-1} \left\{ \frac{1}{2} \left[1 - \sqrt{1 - 4\lambda\left(\delta - c_s\right)\left[2x - (n-1)\right]} \right] \right\} \\ &= \sum_{x=\left[\frac{n-1}{2}\right]+1}^{n-1} \binom{n-1}{x} 0.5^{n-1} \left\{ \frac{1}{2} \left[1 - \sqrt{1 - 4\lambda\left(\delta - c_s\right) y} \right] \right\}.\end{aligned}$$

■

Proof of Proposition 8. The proof follows the same logic of Proposition 7. Notice that for any $c_s \geq \delta$, we have $\pi_p(c_s) = \pi_c(c_s) = 0$. Under the assumption that banks choose the smallest bilateral interbank deposit that minimises their objective functions, we have $c_s^d = \delta$ for both the centre and the peripheral banks. The corresponding vector of interbank deposits $\mathbf{c}^d = [c_s^d = \delta | \forall \omega_p, \omega_c \in \Omega]$ forms a pairwise stable equilibrium since no pair of banks, composed by the centre bank and by a peripheral bank, benefits by choosing an amount of interbank deposit $\widetilde{c}_s < c_s^d = \delta$. Since deposits are symmetric, according to Lemma 3 it suffices to consider the deviation of one bank. Assume the peripheral bank ω_p' deviates by lowering the amount of the bilateral interbank deposit with bank ω_c. That is, $\widetilde{\mathbf{c}}_s' = (\widetilde{c}_s, c_{-s}^d)$. If bank ω_p' experiences the liquidity deficit δ, it expects to collect from the centre bank the amount $\widetilde{c}_s$ which is strictly less than δ. Therefore bank ω_p' faces a strictly positive expected liquidation loss $\pi_p'(\widetilde{c}_s, c_{-s}^d) > \pi_p'(c_s^d) = 0$. Then bank ω_p' will not deviate from $\mathbf{c}^d$. (a similar reasoning applies if we consider the centre bank as the deviating bank). The vector of interbank deposits $\mathbf{c}^d$ is also the unique pairwise stable equilibrium since any vector different from $\mathbf{c}^d$ would induce a deviation. Suppose banks hold a vector of interbank deposits equal to $\widetilde{\mathbf{c}} = \left\{0 \leq \widetilde{c}_s < c_s^d | \forall j \in \mathcal{N}(i)\right\}$. When each bilateral interbank deposit is strictly less than δ all banks expect to collect an amount of liquidity less than δ. Banks, therefore, face a strictly positive expected loss from the asset liquidation: $\pi(\widetilde{c}_s) > 0$. Suppose now that the peripheral bank ω_p' deviates by slightly increasing the bilateral interbank deposit with the centre bank ω_c. That is, $\widetilde{\mathbf{c}}_s' = (\widetilde{c}_s + \varepsilon, \widetilde{c}_{-s})$. In this way, bank ω_p' increases the amount of liquidity that expects to collect from the centre bank. This reduces the expected loss of asset liquidation, that is $\pi_p'(\widetilde{\mathbf{c}}_s') < \pi_p'(\widetilde{\mathbf{c}})$. (a similar reasoning applies if we consider the centre bank as the deviating bank). It follows, from Lemma 3, that any bank would deviate from the vector of interbank deposits $\widetilde{\mathbf{c}}$ by slightly increasing the bilateral exposure. By continuity, banks do not have incentives to deviate anymore when the vector of interbank deposits reaches $\mathbf{c}^d$. ■

Proof of Proposition 9. The first order condition for the minimization of the function $\psi_i(\mathbf{c}_i)$ in (9) is

$$\frac{n\gamma}{4} = \frac{n\lambda}{4\sqrt{1 - 4\lambda\left(\delta - \frac{n}{2}c_{ij}\right)}},$$

and, after rearranging, we get the optimal solution

$$c_{ij}^d = \frac{2}{n}\delta - \frac{1}{2\lambda n}\left[1 - \left(\frac{\lambda}{\gamma}\right)^2\right].$$

Note that the strict convexity of the function $\psi_i(\mathbf{c}_i)$ ensures that the solution admits at most one minimum, which has to be the one characterised by the f.o.c. For this reason, the vector of interbank deposits $\mathbf{c}_i^d = \left\{c_{ij}^d | \forall j \in \mathcal{N}(i)\right\}$ represents a pairwise stable equilibrium. That is, no pair of banks benefit by choosing a bilateral interbank deposit $\widetilde{c}_{ij} \neq c_{ij}^d$. Given Lemma 3, it suffices to consider the deviation of one bank. Assume bank ω_i deviates by choosing a bilateral interbank deposit $\widetilde{c}_{ij} \neq c_{ij}^d$ with bank ω_j. That is, $\widetilde{\mathbf{c}}_i = (\widetilde{c}_{ij}, c_{-ij}^d)$. The strict convexity of $\psi_i(\mathbf{c}_i)$ ensures that $\psi_i(\widetilde{\mathbf{c}}_i) > \psi_i(\mathbf{c}_i^d)$ and bank ω_i does not deviate from $\mathbf{c}_i^d$. The vector $\mathbf{c}_i^d = \left\{c_{ij}^d | \forall j \in \mathcal{N}(i)\right\}$ is also the unique equilibrium since any vector different from $\mathbf{c}_i^d$ would induce a Pareto-improving deviation. Suppose banks hold a vector of interbank deposits equal to $\widetilde{\mathbf{c}}_i = \left\{\widetilde{c}_{ij} \neq c_{ij}^d | \forall j \in \mathcal{N}(i)\right\}$. Banks therefore face an expected loss equal to $\psi_i(\widetilde{\mathbf{c}}_i)$ that, again, it is strictly larger than $\psi_i(\mathbf{c}_i^d)$. If $\widetilde{c}_{ij} < c_{ij}^d$ bank ω_i has a profitable deviation by increasing the bilateral interbank deposit with its neighboring bank ω_j. This reduces the expected loss. It follows that any bank would deviate from the vector of interbank deposits $\widetilde{\mathbf{c}}_i$ by increasing the bilateral exposure. A similar argument applies if $\widetilde{c}_{ij} < c_{ij}^d$. Banks do not have incentives to deviate anymore when the vector of interbank deposits reaches $\mathbf{c}_i^d$. ■

Proof of Lemma 5. The f.o.c. with respect c_{sp} of the function ψ_p in (10) is

$$\frac{\gamma}{2} = \frac{1}{2}\frac{\lambda}{\sqrt{1 - 4\lambda\left(\delta - c_{sp}\right)}},$$

and, after rearranging, we get the optimal deposit

$$c_{sp}^d = \delta - \frac{1}{4\lambda}\left[1 - \left(\frac{\lambda}{\gamma}\right)^2\right].$$

The f.o.c. with respect c_{sc} of the function ψ_c in (11) is

$$\gamma\frac{(n-1)}{2}=\sum_{x=\frac{n-1}{2}+1}^{n-1}\binom{n-1}{x}0.5^{n-1}\frac{y\lambda}{\sqrt{1-4\lambda\left(\delta-c_{sc}\right)y}}. \tag{29}$$

Let c_{sc}^{d} be the deposit that satisfies (29). Note that the r.h.s. of (29) - that is, the expected marginal benefit of holding interbank deposits - is a non-linear function of the random variable $y=2x-(n-1)$. Therefore it is impossible to simplify (29) and to get an explicit characterization of c_{sc}^{d}. To overcome this issue, we use Jensen's inequality to obtain a lower bound of the r.h.s. of (29).

Let

$$\widetilde{y}=\frac{\sum_{x=\frac{n-1}{2}+1}^{n-1}\binom{n-1}{x}0.5^{n-1}\left[2x-(n-1)\right]}{\sum_{x=\frac{n-1}{2}+1}^{n-1}\binom{n-1}{x}0.5^{n-1}}$$

be the expected value of y conditional on $y>0$ (i.e., on $x\geq\frac{n-1}{2}+1$). Let us then compute the expected value of the r.h.s. of (29) conditional on $y>0$. We have

$$\frac{\sum_{x=\frac{n-1}{2}+1}^{n-1}\binom{n-1}{x}0.5^{n-1}\frac{y\lambda}{\sqrt{1-4\lambda(\delta-c_{sc})y}}}{\sum_{x=\frac{n-1}{2}+1}^{n-1}\binom{n-1}{x}0.5^{n-1}}=2\sum_{x=\frac{n-1}{2}+1}^{n-1}\binom{n-1}{x}0.5^{n-1}\frac{y\lambda}{\sqrt{1-4\lambda\left(\delta-c_{sc}\right)y}}$$

since $\sum_{x=\frac{n-1}{2}+1}^{n-1}\binom{n-1}{x}0.5^{n-1}=0.5$ when n is even. Then, by applying the generalized Jensen inequality, we have

$$2\sum_{x=\frac{n-1}{2}+1}^{n-1}\binom{n-1}{x}0.5^{n-1}\frac{y\lambda}{\sqrt{1-4\lambda\left(\delta-c_{sc}\right)y}}>\frac{\widetilde{y}\lambda}{\sqrt{1-4\lambda\left(\delta-c_{sc}\right)\widetilde{y}}}$$

because the marginal benefit of holding interbank deposits is increasing and convex in x. The previous inequality implies

$$\sum_{x=\frac{n-1}{2}+1}^{n-1}\binom{n-1}{x}0.5^{n-1}\frac{y\lambda}{\sqrt{1-4\lambda\left(\delta-c_{sc}\right)y}}>\frac{\widetilde{y}\lambda}{2\sqrt{1-4\lambda\left(\delta-c_{sc}\right)\widetilde{y}}}.$$

Let us substitute $\widetilde{y}\lambda/2\sqrt{1-4\lambda\left(\delta-c_{sc}\right)\widetilde{y}}$ in the r.h.s. of the f.o.c. in (29) to obtain the following

$$\gamma\frac{(n-1)}{2}=\frac{\widetilde{y}\lambda}{2\sqrt{1-4\lambda\left(\delta-c_{sc}\right)\widetilde{y}}}. \tag{30}$$

Solving for c_{sc} equation (30), we obtain a deposit $\widetilde{c}_{sc}$ strictly smaller than the optimal deposit c_{sc}^{d} that would solve the f.o.c. in (29). This is because the r.h.s. of equation (30) is strictly smaller than the r.h.s. of the f.o.c. in (29), and both of them are monotonically decreasing in c_{sc}. We have

$$\widetilde{c}_{sc} = \delta - \frac{1}{4\lambda\widetilde{y}}\left[1 - \left(\frac{\widetilde{y}\lambda}{\gamma(n-1)}\right)^2\right] < c^d_{sc}.$$

To establish an order between c^d_{sp} and c^d_{sc}, and given that c^d_{sc} cannot be characterised, we compare c^d_{sp} with $\widetilde{c}_{sc}$. We have that

$$c^d_{sp} = \delta - \frac{1}{4\lambda}\left[1 - \left(\frac{\lambda}{\gamma}\right)^2\right] \leq \delta - \frac{1}{4\lambda\widetilde{y}}\left[1 - \left(\frac{\lambda\widetilde{y}}{\gamma(n-1)}\right)^2\right] = \widetilde{c}_{sc}$$

if

$$1 - \left(\frac{\lambda}{\gamma}\right)^2 \geq \frac{1}{\widetilde{y}}\left[1 - \left(\frac{\lambda\widetilde{y}}{\gamma(n-1)}\right)^2\right],$$

which, after rearranging, implies

$$\frac{\gamma}{\lambda} \geq \sqrt{\frac{\widetilde{y}}{\widetilde{y}-1} - \frac{\widetilde{y}^2}{(\widetilde{y}-1)(n-1)^2}} \equiv \eta > 1$$

■

Proof of Proposition 10. If $\gamma \geq \lambda\eta$ then $c^d_{sp} \leq \widetilde{c}_{sc} < c^d_{sc}$ and any interbank deposit $c_s \in [c^d_{sp}, c^d_{sc}]$ is a pairwise-stable equilibrium. Notice that now there are no symmetric interbank deposits (peripheral banks and the centre bank have different optimal values) so Lemma 3 does not apply. We have to look for bilateral deviations and unilateral severance from the equilibrium deposit. Let us assume banks exchange a deposit $c_s \in (c^d_{sp}, c^d_{sc})$. In this case, the peripheral bank would like to deviate and reduce the interbank deposit and its loss function ψ_p. However, the centre bank does not find profitable such deviation since it would increase its loss function ψ_c. The bank a the centre would actually increase the deposit c_s (so to reduce ψ_c). No Pareto-superior bilateral deviation is profitable. Consider now the deposit $c_s = c^d_{sp}$. The centre bank would like to increase the deposit but the peripheral banks are not willing to do so. Again, no bilateral deviation is profitable for both banks. A similar argument applies for $c_s = c^d_{sc}$. We have to check now that both types of bank do not unilaterally sever the interbank deposit. This is guaranteed by the condition

$$\gamma < \frac{\lambda}{\sqrt{1-4\lambda\delta}}.$$

The optimal deposit of the peripheral bank c^d_{sp} is strictly positive when $\gamma < \lambda/\sqrt{1-4\lambda\delta}$. Therefore, under such condition there is no incentive for the peripheral banks to unilaterally severe the deposit. For the bank at the centre, we need that the optimal c_{sc}^{d} is also strictly positive. Recall that since $\gamma \geq \lambda\eta$, we have $c_{sp}^{d} < c_{sc}^{d}$. Therefore the condition that guarantees $c_{sp}^{d} > 0$ it is also sufficient to guarantee that $c_{sc}^{d} > 0$, and there is no incentive for the bank at the centre to unilaterally severe the deposit.

To complete the proof, we need to show that any deposit $c_s \notin [c_{sp}^{d}, c_{sc}^{d}]$ does not represent an equilibrium. Take a deposit $c_s < c_{sp}^{d}$. Both the peripheral bank and the bank at the centre find profitable to increase the interbank deposit because they would reduce their loss functions ψ_p and ψ_c. The bilateral deviation is for both banks profitable until $c_s = c_{sp}^{d}$. A similar argument applies if we assume a deposit $c_s > c_{sc}^{d}$. The bilateral deviation is profitable until $c_s = c_{sc}^{d}$. ■

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