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Preference orderings represented by coherent upper and lower conditional previsions

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Abstract
Preference orderings assigned by coherent lower and upper conditional previsions are defined and they are considered to define maximal random variables and Bayes random variables. Sufficient conditions are given such that a random variable is maximal if and only if it is a Bayes random variable. In a metric space preference orderings represented by coherent lower and upper conditional previsions defined by Hausdorff inner and outer measures are given.

Keywords Preference ordering · Coherent upper and lower conditional previsions · Choquet integral · Disintegration property · Hausdorff outer measures

1 Introduction

Complex decisions can be defined as decisions where a preference ordering $\succ$ between random variables cannot be represented by a linear functional, that is there exist no linear functional Γ such that

$$X \succ Y \Leftrightarrow \Gamma(X) > \Gamma(Y) \text{ and } X \approx Y \Leftrightarrow \Gamma(X) = \Gamma(Y).$$

Complex decisions arise also in decision making under ambiguity where aversion towards ambiguity can effect the preferences (Ellsberg 1961) and leads to a violation of Savage's sure think principle (Savage 1954) and cannot be described by subjective expected utility theory. The modeling of preferences and their representations have been investigated in Seidenfeld et al. (1995).

The Choquet expected utility theory has been introduced to modeling decision making under ambiguity using non-additive probabilities (Choquet 1953; Schmeidler 1989; Gilboa 1987; Mayag et al. 2011; Anscombe and Aumann 1963).

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Examples of non-additive measure are coherent upper and lower probabilities and non linear functional can be defined by coherent upper and lower prevision.

The vantages of using coherent upper and lower conditional probabilities instead of fuzzy measures, which are not required to be coherent, to define functional which represent a preference ordering are:

- coherent upper and lower conditional previsions define on a given class of random variables can be extended to the class of all random variables on Ω;
- coherent upper and lower conditional previsions are required to be fully conglomerable, i.e. to be conglomerable with respect to every partition even if they may fail the disintegration property.
- a new model of coherent upper conditional previsions based on Hausdorff outer measures has been introduced in Doria (2010, 2012) and it is proven to satisfy the disintegration property and to be fully conglomerable on every non-null partition (Doria 2017).

The problem to determine if a random variable is a maximal and a Bayes random variable in a given class is related to problem to verify if the coherent upper conditional prevision verifies the disintegration property.

Let Ω be a non-empty set and let $\mathbf{B}$ be a partition of Ω; disintegration property for linear prevision P requires that $P(X) = P(P(X|\mathbf{B}))$ and it has been studied in Seidenfeld et al. (1998); for coherent lower and upper previsions it has been investigated in Miranda et al. (2012) and Doria (2017).

For example in multi-criteria decision problem denoted by Ω the set of criteria, the elements of a partition $\mathbf{B}$ can represent clusters or macro-criteria—which are representative of the general objectives of the decision problem, as goals to pursue through the implementation of specific policies—and the elements in each B are the criteria.

To compare the random variables or the alternatives with respect to all criteria the disintegration property of the upper and lower conditional previsions can be applied to calculate $\overline{P}(X|\Omega)$ and $\underline{P}(X|\Omega)$ and to determine the maximal and/or the Bayes random variable with respect all criteria.

In Walley (1991) different preference orderings are defined with respect to lower and upper coherent previsions.

For each $B \in \mathbf{B}$ let $\underline{P}(\cdot|B)$ and $\overline{P}(\cdot|B)$ respectively a lower coherent conditional prevision and its conjugate upper coherent conditional prevision defined on the class $L(B)$ of all random variables on B and let K be a sub-class of $L(B)$.

For each $B \in \mathbf{B}$ a strict order $\succ_*$, (i.e. a complete antisymmetric and transitive binary relation) is defined with respect to a coherent lower conditional prevision and a weak order $\succ^*$, (i.e. a complete reflexive and transitive binary relation) is defined with respect to a coherent upper conditional prevision.

A random variable X_i is admissible in K under $\underline{P}(\cdot|B)$ if no random variable $X_j \in K$ with $i \neq j$ is preferable to X_i with respect to $\succ_*$.

An admissible random variable X_i in K is maximal under $\underline{P}(\cdot|B)$ if it is preferable to X_j according to $\succ^*$ for all $X_j \in K$, so also the coherent upper conditional prevision is involved to determine a maximal random variable in a class K.

A Bayes random variable under a coherent lower conditional prevision is a random variable which is maximal under a linear prevision on the class of all random variables defined on B.

Let μ be a submodular coherent upper conditional prevision on $\wp(B)$ and let $\overline{P}(\cdot|B)$ be the coherent upper conditional prevision represented as Choquet integral with respect to μ and let $\underline{P}(\cdot|B)$ its conjugate coherent lower prevision.

In this paper, sufficient conditions which assure that an admissible random variable is maximal if and only if it is a Bayes random variable are given. It is proven that:

- If K is a comonotonic class of random variables then X_i is maximal under $\overline{P}(\cdot|B)$ if and only if it is a Bayes.
- If K is a class of random variables such that the class $\mathbf{C} = \{X_i - X_j : X_j \in K\}$ is a comonotonic class, X_i is maximal under $\underline{P}(\cdot|B)$ if and only if it is a Bayes random variable.
- If K is a class containing only two random variables then X_i is maximal under $\underline{P}(\cdot|B)$ if and only if it is a Bayes random variable.
- Let μ be a submodular coherent upper conditional probability on $S \subset \wp(B)$. If K is a class containing μ-upper measurable random variables, then X_i is maximal under a $\underline{P}(\cdot|B)$ if and only if it is a Bayes random variable.

2 Orderings represented by coherent lower and upper conditional previsions

Let Ω be a non empty set and let $\mathbf{B}$ be an arbitrary partition of Ω. A bounded random variable is a function $X : \Omega \to \Re$ and $L(\Omega)$ is the class of all bounded random variables defined on Ω. A random variable is $\mathbf{B}$-measurable, or measurable with respect to a partition $\mathbf{B}$ if it is constant on the set $B \in \mathbf{B}$. For every $B \in \mathbf{B}$ denote by $X|B$ the restriction of X to B and by $\sup(X|B)$ the supremum value that X assumes on B. Let $L(B)$ be the class of all bounded random variables $X|B$. Denote by I_A the indicator function of any event $A \in \wp(B)$, i.e. $I_A(\omega) = 1$ if $\omega \in A$ and $I_A(\omega) = 0$ if $\omega \in A^c$.

A preference ordering is a binary comparison between random variables.

Definition 1 A preference ordering $\succ$ on the class $L(B)$ of random variables defined on B is represented by a functional Γ if and only if

$$X_i|B \succ X_j|B \Leftrightarrow \Gamma(X_i|B) > \Gamma(X_j|B)$$
and
$$X_i|B \approx X_j|B \Leftrightarrow \Gamma(X_i|B) = \Gamma(X_j|B)$$

Let $B = \{\omega_1, \ldots, \omega_n\}$ and let μ be a probability on $\wp(B)$; a classical linear functional on $L(B)$ to represent a preference ordering is the weighted sum $\Gamma(X) = \sum_{i=1}^{n} X(\omega_i)\mu(\omega_i)$.

Nevertheless not all preference orderings can be represented by a linear functional.

Example 1 Let Ω be a non empty set, $\mathbf{B} = \{B_1, B_2\}$ and let μ be a probability measure defined on the field generated by $\mathbf{B}$. Let consider the class $K = \{X_1, X_2, X_3\}$ of bounded $\mathbf{B}$-measurable random variables defined on Ω by

Random variables	B_1	B_2
X_1	0.3	0.3
X_2	0.7	0.0
X_3	0.0	0.7

The preference ordering $X_1 \succ X_2$ and $X_2 \approx X_3$ cannot be represented by the linear functional $\Gamma = \int X d\mu$ since there exists no probability measure μ such that the following system has solution:

$$\begin{cases} X_1 \succ X_2 \\ X_2 \approx X_3 \end{cases} \Leftrightarrow \begin{cases} 0.3\mu(B_1) + 0.3\mu(B_2) > 0.7\mu(B_1) + 0.0\mu(B_2) \\ 0.7\mu(B_1) + 0.0\mu(B_2) = 0.0\mu(B_1) + 0.7\mu(B)_2. \end{cases}$$

In the next sections (see Examples 5 and 8) it is shown that the given ordering is represented by a coherent lower conditional prevision defined as the Choquet integral with respect to a coherent lower conditional probability μ such that $\mu(B_1) = \mu(B_2) = 0$.

Example 2 Let X_1 and X_2 be the random variables defined as in Example 1 than the preference ordering $X_1 \succ X_2$ can be represented by a linear functional, since

$$X_1 \succ X_2 \Leftrightarrow 0.3\mu(B_1) + 0.3\mu(B_2) > 0.7\mu(B_1) + 0.0\mu(B_2)$$

and the system has solution for all pair $(\mu(B_1), \mu(B_2))$ with $\mu(B_1) < \frac{3}{7}$ and $\mu(B_2) = 1 - \mu(B_1)$.

In Theorem 2 a sufficient condition is given such that a preference ordering represented by a coherent upper conditional prevision defined as the Choquet integral with respect to a submodular coherent upper probability can also be represented by a linear coherent prevision.

The previous examples put in evidence the necessity to introduce non-linear functionals to represent preference orderings, to investigate equivalent random variables and to manage null-events.

2.1 Coherent upper conditional previsions and their Choquet integral representation

For every $B \in \mathbf{B}$ coherent upper conditional previsions $\overline{P}(\cdot|B)$ are functionals defined on $L(B)$ (Walley 1991, 1981).

Definition 2 Coherent upper conditional previsions are functionals $\overline{P}(\cdot|B)$ defined on $L(B)$, such that the following conditions hold for every X and Y in $L(B)$ and every strictly positive constant λ:

1. $\overline{P}(X|B) \leq \sup(X|B)$;
2. $\overline{P}(\lambda X|B) = \lambda \overline{P}(X|B)$ (positive homogeneity);

3. $\overline{P}(X+Y)|B) \leq \overline{P}(X|B) + \overline{P}(Y|B)$ (subadditivity);
4. $\overline{P}(I_B|B) = 1$.

Definition 3 Given partition $\mathbf{B}$ and a random variable $X \in L(\Omega)$ a coherent upper conditional prevision $\overline{P}(X|\mathbf{B})$ is a random variable on Ω equal to $\overline{P}(X|B)$ if $\omega \in B$.

A coherent upper conditional prevision is continuous from below if for an increasing sequence X_n of random variables converging to X we have $lim_{n\to\infty}\overline{P}(X_n|B) = \overline{P}(X|B)$.

Suppose that $\overline{P}(X|B)$ is a coherent upper conditional prevision on $L(B)$ then its conjugate coherent lower conditional prevision is defined by $\underline{P}(X|B) = -\overline{P}(-X|B)$. Let K be a linear space contained in $L(B)$; if for every X belonging to K we have $P(X|B) = \underline{P}(X|B) = \overline{P}(X|B)$ then $P(X|B)$ is called a coherent linear conditional prevision (de Finetti 1972, 1974; Regazzini 1985, 1987) and it is a linear, positive and positively homogenous functional on $L(B)$.

The unconditional coherent upper prevision $\overline{P} = \overline{P}(\cdot|\Omega)$ is obtained as a particular case when the conditioning event is Ω. Coherent upper conditional probabilities are obtained when only 0-1 valued random variables are considered.

An upper prevision is a real-valued function defined on some class of bounded random variables K. A necessary and sufficient condition for an upper prevision $\overline{P}$ to be coherent is to be the upper envelope of linear previsions, i.e. there is a class M of linear previsions such that $\overline{P}$ =sup$\{P : P \in M\}$ (Walley 1991, 3.3.3). Let $\overline{P}$ be an upper prevision on an arbitrary domain K such that the class of all linear previsions dominated by $\overline{P}$, is non-empty. The maximal extension of $\overline{P}$ to $L(B)$, denoted by $\overline{E}$, is called (Walley 1991, 3.1.1) the natural extension of $\overline{P}$. Moreover $\overline{P}$ is coherent on K if and only if its natural extension $\overline{E}$ agrees with $\overline{P}$ on K.

A coherent upper conditional probability μ_B on $\wp(B)$ is

(a) *submodular* or *2-alternating* if $\mu(A \cup E) + \mu(A \cap E) \leq \mu(A) + \mu(E)$ for every $A, E \in \wp(B)$;
(b) *continuous from below* if $\lim_{i\to\infty} \mu(A_i) = \mu(\lim_{i\to\infty} A_i)$ for any increasing sequence of sets $\{A_i\}$, with $A_i \in \wp(B)$.

A coherent upper conditional prevision $\overline{P} : L(B) \to \Re$ can be represented as Choquet integral with respect to a coherent upper conditional probability μ on $\wp(B)$ if $\overline{P}(X) = \int X \mathrm{d}\mu \ \forall X \in L(B)$.

A necessary and sufficient condition (Doria 2014, Proposition 1) for the representation of a coherent upper conditional prevision on $L(B)$ as Choquet integral with respect to a coherent upper conditional probability μ is that μ is submodular. Then $\overline{P}(I_A) = \int I_A \mathrm{d}\mu = \mu(A)$. For every $x \in \Re$ let $\{X > x\} = \{\omega \in B : X(\omega) > x\}$.

The decreasing distribution function of X with respect to μ is the function

$$G_{\mu,X}(x) = \mu\{X > x\}.$$

It is unique except on a set with measure μ equal to zero. If μ is continuous from below then

$$G_{\mu,X}(x) = \mu\{X > x\} = \mu\{X \geq x\}.$$

Since X is a bounded random variable thus there exist a constant k such that $\widetilde{X} = X + k \geq 0$ and $G_{\mu,\widetilde{X}}(x) = G_{\mu,X}(x-k)$ for every real number x (Denneberg 1994, Proposition 4.1).

The Choquet integral (Denneberg 1994) of a bounded random variable X with respect to μ is defined by

$$\int X \mathrm{d}\mu = \int_0^{+\infty} G_{\mu,\widetilde{X}}(x)\mathrm{d}x = \int_0^{+\infty} \widehat{G}_{\mu,\widetilde{X}}(y)\mathrm{d}y$$

When a coherent upper probability is defined on a class S properly contained in $\wp(B)$ then a measurability condition for the random variable X is required to define the Choquet integral.

Definition 4 (Denneberg 1994 p. 49) Let μ_B be a coherent upper probability defined on a class S containing the empty set and properly contained in $\wp(B)$ and let $\mu_B^*(A)$ and $\underline{\mu}_B^*(A)$ be its outer and inner measures; a random variable X is upper μ-measurable if $\mu_B^*(\{X > x\}) = \underline{\mu}_B^*(\{X > x\})$ except on a countable set. A random variable X is lower μ-measurable if -X is upper μ-measurable. X is (upper, lower) S-measurable if it is (upper, lower) μ-measurable for any monotone set function on S. X is upper S-measurable if the class of upper level sets $\{\{X \geq x\} x \in \Re\}$ is contained in S. If S is a σ-field and the class of upper level sets of X and $-X$ belong to S then X is S-measurable, that is the inverse image $X^{-1}(A) = \{\omega \in B : X(\omega) \in A\}$ of every Borelian set A, belongs to S.

For a μ_B-upper measurable X the Choquet integral is defined by

$$\int_B^{\text{Cho}} X \mathrm{d}\mu_B = \int_B^{\text{Cho}} X \mathrm{d}\mu_B^* = \int_B^{\text{Cho}} X \mathrm{d}\underline{\mu}_B^*.$$

If Ω is finite and μ is defined on a field S, denote by $A_1, \ldots, A_n$ the atoms of S, which are the minimal elements of $S - \emptyset$. If the atoms A_i are enumerated so that $x_i = X(A_i)$ are in descending order, i.e. $x_1 \geq x_2 \geq \cdots \geq x_n$ and $x_{n+1} = 0$ the Choquet integral with respect to μ is given by

$$\int X \mathrm{d}\mu = \sum_{i=1}^{n} (x_i - x_{i+1})\mu(S_i)$$

where $S_i = A_1 \cup A_2 \cdots \cup A_i$.

3 Maximal random variables and Bayes random variables

In this section, sufficient conditions are given to assure that a random variable is maximal if and only if it is a Bayes random variable. A strict order $\succ_*$ with respect to a coherent lower conditional prevision and a weak order $\succ^*$ with respect to a coherent upper conditional prevision are considered to defined the maximal and the Bayes random variables.

3.1 Strict order with respect to a coherent lower prevision

Definition 5 X_i and X_j are equivalent given B with respect to $\underline{P}$, i.e. $X_i|B \approx_* X_j|B$ given B if and only if

$$\underline{P}(X_i|B) = \underline{P}(X_j|B).$$

A strict order on $L(B)$ is an antisymmetric and transitive binary relation on $L(B)$. Let X_i and X_j be two random variables belonging to $L(B)$.

A strict ordering, induced by a coherent lower conditional prevision $\underline{P}(\cdot|B)$ can be defined on the class of random variables belonging to $L(B)$ (Walley (Section 3.8.1)):

Definition 6 We say that the random variable $X_i|B$ is preferable to $X_j|B$ given B with respect to $\underline{P}(\cdot|B)$, i.e. $X_i \succ_* X_j$ given B if and only if

$$\underline{P}((X_i - X_j)|B) > 0 \text{ or } X \geq Y \text{ and } X \neq Y \tag{1}$$

Some information can be lost when a strict preference order is defined by $\underline{P}$ since $\underline{P}$ does not contain any information about which gambles, with $\underline{P}(X) = 0$ are really desirable.

3.2 Weak order with respect to a coherent upper conditional prevision

A weak order on $L(B)$ is a complete reflexive and transitive binary relation on $L(B)$.

Definition 7 X_i and X_j are equivalent given B with respect to $\overline{P}$, i.e. $X_i|B \approx^* X_j|B$ given B if and only if

$$\overline{P}(X_i|B) = \overline{P}(X_j|B).$$

Definition 8 We say that X_i is preferable to X_j given B , i.e. $X_i \succ^* X_j$ in B if and only if $\overline{P}((X_i - X_j)|B) > 0$ and X_i and X_j are indifferent given B, i.e. $X_i \approx X_j$ in B if and only if $\overline{P}((X_i - X_j)|B) = \overline{P}((X_j - X_i)|B) = 0$.

Definition 9 A random variable $X_i|B \in K$ is inadmissible in K given B if there is $X_j|B \in K$ such that $X_j|B \succ_* X_i|B$ with $j \neq i$. Otherwise $X_i|B$ is admissibile in K.

In Walley (section 3.9.4) the following definitions are given.

Definition 10 An admissible random variable $X_i|B \in K$ is maximal in K given B under the coherent lower prevision $\underline{P}(\cdot)|B$ if $\overline{P}(X_i - X_j)|B \geq 0\ \forall X_j|B \in K$

If $P(\cdot|B)$ is a linear prevision, the maximal random variables belonging to $L(B)$ under $P(\cdot|B)$ are the admissible random variables $X_i|B$ which satisfy $P(X_i|B) \geq P(X_j|B)$ for all $X_j|B \in K$. Any random variable which maximizes $P(X_j|B)$ over $X_j|B \in K$ is called a Bayes random variable under $P(\cdot|B)$.

A Bayes random variable under a coherent lower conditional prevision is a random variable which is maximal under a linear prevision on the class of all random variables defined on B.

Definition 11 An admissible random variable $X_i|B$ is defined to be a Bayes random variable under a coherent lower prevision $\underline{P}$ when, for each $X_j|B \in K$ there is $P \in M(\underline{P})$ such that $P(X_i|B) \geq P(X_j|B)\ \forall X_j|B \in K$.

If X_i is maximal under some $P \in M(\underline{P})$ then $P(X_i) \geq P(X_j)\ \forall\ X_j \in K$ so $\overline{P}(X_i - X_j) \geq P(X-Y) = P(X) - P(Y) \geq 0$ so X is maximal under $\underline{P}$.

Therefore, a Bayes random variable under a coherent lower prevision $\underline{P}$ is maximal under $\underline{P}$ but the converse is not true. The maximality theorem (Walley 3.9.5) claims that the converse holds if K is a convex subset of $L(\Omega)$.

In the next theorems, sufficient conditions are given such that a random variable is maximal in a class K if and only it is a Bayes random variable in K.

Definition 12 Two random variables X and $Y \in L(B)$ are comonotonic if,

$$(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0 \forall \omega_1, \omega_2 \in B.$$

A class **C** of random variables is comonotonic if and only if each pair of functions in **C** is comonotonic.

Theorem 1 *Let $K \subset L(B)$ be a class of comonotonic random variables and et μ be a submodular coherent upper conditional probability defined on $\wp(B)$ and let $\overline{P}(\cdot|B)$ a coherent upper conditional prevision defined as the Choquet integral with respect to μ. Then a random variable X is maximal in K with respect to the ordering $\succ$ in Definition 1 represented by $\overline{P}(\cdot|B)$ if and only it is a Bayes random variable in K.*

Proof Since K is a comonotonic class of random variables and μ is submodular then by Denneberg (1994, Proposition 10.1) there exists an additive measure α on $\wp(B)$ such that the Choquet integral with respect to μ is equal to the Choquet integral with respect to α and so X_i is a maximal random variable in K with respect to the ordering $\succ$ if and only if it is a Bayes random variable in K under a linear prevision.

$$\overline{P}(X_i) \geq \overline{P}(X_j) \Leftrightarrow \int X_i \mathrm{d}\mu \geq \int X_j \mathrm{d}\mu \forall X_j \in K \Leftrightarrow \int X_i \mathrm{d}\alpha \geq \int X_j \mathrm{d}\alpha \forall X_j \in K.$$

◇

Theorem 2 *Let $K \subset L(B)$ be a class of random variables and let $X_i \in K$, such that the class $\boldsymbol{C} = \left\{X_i - X_j : X_j \in K\right\}$ is a comonotonic class. Let μ be a submodular coherent upper conditional probability defined on $\wp(B)$ and let $\overline{P}(\cdot|B)$ a coherent upper conditional prevision defined as the Choquet integral with respect to μ. Then the random variable X_i is maximal in K under the conjugate lower conditional prevision $\underline{P}(\cdot|B)$ if and only it is a Bayes random variable in K.*

Proof Let X_i be a maximal random variable in K under $\underline{P}(\cdot)$. The coherent upper conditional probability μ defined on $\wp(B)$ is submodular and since the class **C** is comonotonic then by Denneberg (1994, Proposition 10.1) there exists an additive measure α on $\wp(B)$ such that

$$\int (X_i - X_j)\mathrm{d}\mu = \int (X_i - X_j)\mathrm{d}\alpha \forall X_j \in K.$$

Thus

$$X_i \text{ is maximal } \in K \text{ under } \underline{P}(\cdot|B) \Leftrightarrow \overline{P}((X_i - X_j)|B) \geq 0 \forall X_j \in K \Leftrightarrow$$
$$\Leftrightarrow \int_B (X_i - X_j)\mathrm{d}\mu \geq 0 \Leftrightarrow \int (X_i - X_j)\mathrm{d}\alpha \geq 0 \Leftrightarrow \int X_i \mathrm{d}\alpha \geq \int X_j \mathrm{d}\alpha.$$

Therefore, X_i is maximal in K under the linear prevision $P(X|B) = \int_B X\mathrm{d}\alpha$, that is X_i is a Bayes random variable in K. $\diamond$

Example 3 Let K be a class of random variables, $k_j \in \Re$ and let $X_i\ inK$ a maximal random variable such that $X_i - X_j = k_j$ for all $X_j \in K$. Thus the class $\mathbf{C} = \left\{X_i - X_j : X_j \in K\right\}$ is a comonotonic class since contains only constants. By Theorem 3 we have yhat X_i is a Bayes random variable in K.

Theorem 3 *Let $K = \{X_1, X_2\}$ be a class containing only two random variables of $L(B)$. Let μ be a submodular coherent upper conditional probability defined on $\wp(B)$ and let $\overline{P}(\cdot|B)$ a coherent upper conditional prevision defined as the Choquet integral with respect to μ. Then a random variable X_i is maximal in K under the conjugate lower conditional prevision $\underline{P}(\cdot|B)$ if and only it is a Bayes random variable in K.*

Proof Lets assume that X_1 is a maximal random variable in K under $\underline{P}(\cdot|B)$. The coherent upper conditional probability μ defined on $\wp(B)$ is submodular and since any constant c and any random variable are comonotonic thus the class $\mathbf{C} = \{X_1 - X_2; c\}$ is comonotonic; then by Denneberg (1994, Proposition 10.1) there exists an additive measure α on $\wp(B)$ such that

$$\int ((X_1 - X_2)|B)\mathrm{d}\mu = \int ((X_1 - X_2)|B)\mathrm{d}\alpha.$$

Thus

$$X_i \text{ is maximal in } K \text{ under } \underline{P}(\cdot|B) \Leftrightarrow \overline{P}((X_1 - X_2)|B) \geq 0 \Leftrightarrow$$
$$\Leftrightarrow \int_B (X_1 - X_2)\mathrm{d}\mu \geq 0 \Leftrightarrow \int (X_1 - X_2)\mathrm{d}\alpha \geq 0 \Leftrightarrow \int X_1 \mathrm{d}\alpha \geq \int X_2 \mathrm{d}\alpha.$$

Therefore, X_1 is maximal in K under the linear prevision $P(X|B) = \int_B X\mathrm{d}\alpha$, that is X_1 is a Bayes random variable in K. $\diamond$

Theorem 4 *Let μ be a submodular coherent upper probability defined on a class S containing the empty set and properly contained in $\wp(B)$ and let $\mu^*(A)$ and $\underline{\mu}^*(A)$ be its outer and inner measures; Let $K \subset L(B)$ be a class of μ-upper measurable random variables. Then a random variable X_i is maximal in K under the conjugate lower conditional prevision $\underline{P}(\cdot|B)$ if and only it is a Bayes random variable in K.*

Proof By submodularity and coherence of μ_B there exists a linear prevision P such that

$$\int (X_i - X_j)d\underline{\mu}^* = \underline{P}(X_i - X_j) \leq P(X_i - X_j) \leq \overline{P}(X_i - X_j) = \int (X_i - X_j)\mathrm{d}\mu^*.$$

Let $X_i \in K$ be an admissibile μ-upper random variable; thus by Definition 4

$$X_i \text{ is maximal in } K \text{ under } \underline{P}(\cdot|B) \Leftrightarrow \overline{P}((X_i - X_j)|B) \geq 0 \forall X_j \in K \Leftrightarrow$$
$$\Leftrightarrow \int_B (X_i - X_j)\mathrm{d}\mu = \int (X_i - X_j)\mathrm{d}\mu^* = \int (X_i - X_j)\mathrm{d}\underline{\mu}^* \geq 0 \forall X_j \in K \Leftrightarrow .$$

So there exists a linear prevision P such that
X_i is maximal in K under $\underline{P}(\cdot|B) \Leftrightarrow$
$\Leftrightarrow \underline{P}(X_i - X_j) = P(X_i - X_j) = \overline{P}(X_i - X_j) \geq 0 \Leftrightarrow$
$\Leftrightarrow X_i$ is a Bayes random variable in K $\diamond$

Example 4 Let $\mathbf{B} = \{B_1, B_2, B_3\}$ and let $\{P_1, P_2, P_3\}$ be a class of additive probabilities defined on $\mathbf{B}$ by $P_1(B_1) = 0.5$, $P_1(B_2) = 0.4$, $P_1(B_3) = 0.1$
$P_2(B_1) = 0.3$, $P_2(B_2) = 0.55$, $P_1(B_3) = 0.15$
$P_3(B_1) = 0.3$, $P_3(B_2) = .4$, $P_3(B_3) = 0.3$

Let $\overline{\mu}_B$ be the coherent upper conditional probability defined by $\overline{\mu}_B(A) = max\{P_1(A); P_2(A); P_3(A)\}\ \forall A \in \wp(B)$ and let $\underline{\mu}_B$ be the coherent upper conditional probability defined by $\underline{\mu}_B(A) = min\{P_1(A); P_2(A), P_3(A)\}\ \forall A \in \wp(B)$.

Let $K = \{X_1, X_2, X_3\}$ be the class of $\mathbf{B}$-measurable random variables defined by:

Random variables	B_1	B_2	B_3
X_1	8	5	7
X_2	9	8	9
X_3	5	3	7

If the lower conditional prevision is defined as Choquet integral with respect to $\underline{\mu}_B$ and the upper conditional prevision as the Choquet integral with respect to $\overline{\mu}_B$ we have

$$\underline{P}(x_2 - x_1) = (3-2)(0,3) + (2-1)(0,7) + (1-0)(1) = 2,1,$$
$$\underline{P}(x_3 - x_1) = (0+2)(0,1) + (-2+3)(0,5) + (-3-0)(1) = -2,3$$
$$\underline{P}(X_1 - X_2) = (-1+2)(0,3) + (-2+3)(0,45) + (-3-0)(1) = -2,25$$
$$\underline{P}(X_3 - X_2) = (-2+4)(0,1) + (-4+5)(0,45) + (-5-0)(1) = -4,35$$
$$\underline{P}(X_1 - X_3) = (3-2)(0,3) + (2-0)(0,7) + (0-0)(1) = 1,7$$
$$\underline{P}(X_2 - X_3) = (5-4)(0,4) + (4-2)(0,7) + (2-0)(1) = 3,8$$

then the random variable X_1 and X_3 are not admissible in K since

$$\underline{P}((X_2 - X_1)) > 0 \text{ and } \underline{P}(X_1 - X_3) > 0\underline{P}(X_2 - X_3) > 0.$$

The only admissible random variable in K is X_2.

4 Conglomerability and disintegration property of coherent lower and upper conditional previsions

Let B and B' two sets of a partition $\mathbf{B}$; if a random variable is maximal in the class K under $\underline{P}(\cdot|B)$ it could not to be maximal under $\underline{P}(\cdot|B')$ so to determine if a random variable is maximal in K under $\underline{P}(\cdot|\Omega)$ the disintegration property could be used to aggregate preferences on different conditioning events B.

In Walley (1991, 6.8) full conglomerability is required as a rationality axiom for a coherent lower prevision since it assures that it can be coherently extended to coherent conditional previsions for any partition $\mathbf{B}$ of Ω. If the partition $\mathbf{B}$ represents an experiment that could be performed it is necessary to update the unconditional upper prevision after observing a set B of $\mathbf{B}$. Conglomerability is based on the following conglomerability principle: if a random variable X is B-desirable, i.e. we have a disposition to accept X for every set B in the partition $\mathbf{B}$, then X is desirable. If there is no coherent way of updating the initial prevision after learning the outcome of the experiment the lower prevision, which represents our knowledge, is unreasonable. If a coherent lower probability $\underline{P}$ is such that all the sets in the partition have zero probability, then its minimal coherent conditional prevision extension is the lower vacuous coherent conditional prevision. So to verify the existence of lower coherent conditional previsions coherent with $\underline{P}$ is equivalent to verify the conglomerability for all countable partition (Walley 1991, 6.8.2). A consequence is that a fully conglomerable coherent conditional previsions in the sense of Walley may fail the disintegration property on a null partition.

In Walley an unconditional prevision is defined to be coherent with the coherent conditional prevision if and only if the Conglomerative axiom and the Generalized Bayes Rule are satisfied.

Linear conditional and unconditional previsions defined on the class of all bounded random variables are coherent if and only if (Walley 1991, 6.5.7) they satisfy the disintegration property introduced by Dubins (1975) which is a generalization to the class of all bounded random variables of the conglomerative principle, introduced by de Finetti (1974, p.99), de Finetti (1974) for probabilities.

Walley (1991, Section 6.3) discusses some consequences of the coherence of the lower unconditional prevision $\underline{P}$ with the lower conditional prevision $\underline{P}(\cdot|\mathbf{B})$.

Proposition 1 *If $\underline{P}$ and $\underline{P}(\cdot|\boldsymbol{B})$ defined on $\mathbf{L}(\Omega)$ are coherent then the following conditions hold for every X in $\mathbf{L}(\Omega)$ and $B \in \boldsymbol{B}$:*

(i) $\underline{P}(X) \leq \underline{P}(\underline{P}(X|\boldsymbol{B})$; $\underline{P}(X) \leq \sup \overline{P}(X|\boldsymbol{B})$
and
(ii) $\overline{P}(X) \geq \overline{P}(\overline{P}(X|\boldsymbol{B})$; $\overline{P}(X) \geq \inf \underline{P}(X|\boldsymbol{B})$

If the random variable X is $\mathbf{B}$-measurable, i.e. it is constant on the atoms of the partition $\mathbf{B}$, then $\underline{P}(X|\mathbf{B}) = \overline{P}(X|\mathbf{B}) = X$ so that conditions i) and ii) are always satisfied.

In some special cases coherence of $\underline{P}$ and $\underline{P}(\cdot|\mathbf{B})$ can be characterized by simpler conditions. In particular in Walley (1991, section 6.5.3 and section 6.5.7) it has been proven that if P and $P(\cdot|\mathbf{B})$ are respectively linear unconditional and conditional previsions on the class of all bounded random variables and $P(\cdot|\mathbf{B})$ are separately coherent, then P and $P(\cdot|\mathbf{B})$ are coherent if and only if the following conglomerative property is satisfied $P(X) = P(P(X|\mathbf{B}))$.

The notions of disintegrability and conglomerability given by Dubins (1975) can be extended to coherent upper conditional previsions.

Definition 13 A coherent upper conditional prevision $\overline{P}(X|\mathbf{B})$ is disintegrable with respect to a partition $\mathbf{B}$ if the following equality is satisfied for every bounded variable $X \in L(\Omega)$

$$\overline{P}(X) = \overline{P}(\overline{P}(X|\mathbf{B})).$$

Definition 14 A coherent upper conditional prevision $\overline{P}(X|\mathbf{B})$ is defined to be conglomerative with respect to a partition $\mathbf{B}$ of Ω if the following condition is satisfied: for every bounded variable $X \in L(\Omega)$

$$\overline{P}(X|\mathbf{B}) \geq 0 \text{ implies } \overline{P}(X) \geq 0.$$

Remark 1 If a coherent upper conditional prevision is conglomerative with respect to a partition then the compatibility of conditioned maximality and unconditioned maximality is assured even if the disintegration property fails to be satisfied.

The following example shows that a coherent lower conditional prevision which is coherent with an unconditional lower prevision may fail the disintegration property; it is shown that a preference ordering can be represented by a coherent lower prevision but it may be not represented by the conjugate coherent upper prevision.

Example 5 Let $\mathbf{B}$ and $K = \{X_1, X_2, X_3\}$ as in Example 1. The preference ordering $X_1 \succ X_2$ and $X_2 \approx X_3$ can be represented by the lower vacuous conditional prevision defined by $\underline{P}(X|\Omega) = \inf\{X(\omega) : \omega \in \Omega\}$ since

$$\underline{P}(X_1|\Omega) = 0.3 \text{ and } \underline{P}(X_2|\Omega) = \underline{P}(X_3|\Omega) = 0$$

but is not represented by the upper vacuous conditional prevision $\overline{P}(X|\Omega) = \sup\{X(\omega) : \omega \in \Omega\}$ because

$$\overline{P}(X_1|\Omega) = 0.3 \text{ and } \overline{P}(X_2|\Omega) = \overline{P}(X_3|\Omega) = 0.7.$$

The vacuous lower conditional prevision does not satisfy the disintegration property on the class K since

$$\underline{P}(\underline{P}(X_2|\mathbf{B})) = \underline{P}(B_1)\underline{P}(X_2|B_1) + \underline{P}(B_2)\underline{P}(X_2|B_2) = \underline{P}(B_1)0.7 + \underline{P}(B_2)0$$
$$\underline{P}(\underline{P}(X_3|\mathbf{B})) = \underline{P}(B_1)\underline{P}(X_3|B_1) + \underline{P}(B_2)\underline{P}(X_3|B_2) = \underline{P}(B_1)0 + \underline{P}(B_2)0.7$$

so that

$$\underline{P}(\underline{P}(X_2|\mathbf{B})) = \underline{P}(X_2) \Leftrightarrow \underline{P}(B_1) = 0 \text{ and } \underline{P}(\underline{P}(X_3|\mathbf{B})) = \underline{P}(X_3) \Leftrightarrow \underline{P}(B_2) = 0$$

But if we assume $\underline{P}(B_1) = \underline{P}(B_2) = 0$ we have that

$$\underline{P}(X_1) = 0.3 \neq 0 = \underline{P}(B_1)\underline{P}(X_1|B_1) + \underline{P}(B_2)\underline{P}(X_1|B_2).$$

5 Coherent upper conditional previsions defined as Choquet integral with respect to Hausdorff outer measures

Examples of coherent upper and lower conditional previsions which satisfy Theorems 2, 3, 4 of Sect. 3 are coherent lower and upper conditional previsions defined as Choquet integral with respect to Hausdorff inner and outer measures.

A new model of coherent upper conditional previsions defined as Choquet integral with respect to Hausdorff outer measures has been introduced in Doria (2007, 2008, 2011, 2012) because the Radon–Nikodym derivative, the mathematical tool to define conditional expectation in the axiomatic approach may be not used to define a coherent conditional prevision. Coherent upper conditional previsions defined with respect to Hausdorff outer measures are proven satisfy the disintegration property on every non-null partition Doria (2017) and they are proven to be symmetric in the sense that they are invariant with respect to equimeasurable random variables Doria (2014).

The applications of the model have been investigated in Doria (2015) and in Di Cencio and Doria (2017).

5.1 Coherent conditional prevision and the Radon–Nikodym derivative

In the axiomatic approach (Billingsley 1986 Section 34) conditional expectation is defined with respect to a σ-field of conditioning event by the Radon–Nikodym derivative.

Let $\mathbf{F}$ and $\mathbf{G}$ be two σ-field of subsets of Ω with $\mathbf{G}$ contained in $\mathbf{F}$ and let X be an integral random variable. Let P be a probability measure on $\mathbf{F}$; defined a measure ν on $\mathbf{G}$ by $\nu(g) = \int_G X \mathrm{d}P$. This measure is finite and absolutely continuous with respect to P. Thus there exists a function, the Radon–Nikodym derivative denoted by $E[X|G]_\omega$, defined on Ω, $\mathbf{G}$-measurable, integrable and satisfying the functional equation:

$$\int_G E[X|G]_\omega \mathrm{d}P = \int_G X \mathrm{d}P \text{ with } G \in \mathbf{G}$$

This function is unique up to a set of P-measure zero and it is a version of the conditional expected value.

The next theorem shows that every time the σ-field $\mathbf{G}$ is properly contained in $\mathbf{F}$ and it contains all singletons of $[0, 1]^n$ then the conditional prevision defined by the Radon–Nikodym derivative is not coherent. It occurs because one of the defining property of the Radon–Nikodym derivative, that is to be measurable with respect to the σ-field of the conditioning events contradicts the following necessary condition for the coherence of a linear conditional prevision.

If for every B belongs to $\mathbf{B}$ $P(X|B)$ are coherent linear previsions (Walley 1991, p. 292) and X is $\mathbf{B}$-measurable then $P(X|B) = X$. This necessary condition is not satisfied if $P(X|\mathbf{B})$ is defined by the Radon–Nikodym derivative.

Theorem 5 *Let $\Omega = [0, 1]^n$ and let $\boldsymbol{F}$ and $\boldsymbol{G}$ be two σ-field of subsets of Ω such that $\boldsymbol{G}$ is properly contained in $\boldsymbol{F}$ and it contains all singletons of Ω. Let $\boldsymbol{B}$ be the partition of singletons and let X be the indicator function of an event A belonging to $\boldsymbol{F} - \boldsymbol{G}$. If we define the conditional prevision $P(X|\boldsymbol{B})$ equal to the Radon–Nikodym derivative with probability 1, that is*

$$P(X|\boldsymbol{B}) = E[X|\boldsymbol{G}]_\omega$$

except on a subset N of $[0, 1]^n$ of P-measure zero, then the conditional prevision $P(X|\boldsymbol{B})$ is not coherent.

Proof If the equality $P(X|\mathbf{B}) = E[X|\mathbf{G}]_\omega$ holds with probability 1, then the linear conditional prevision $P(X|\mathbf{B})$ is different from X, the indicator function of A; in fact having fixed $A \in \mathbf{F} - \mathbf{G}$, the indicator function X is not $\mathbf{G}$-measurable so it does not satisfies a property of the Radon–Nikodym derivative and it cannot be assumed as conditional expectation according to the axiomatic definition. Therefore the linear prevision $P(X|\mathbf{B})$ does not satisfy the necessary condition for being coherent, $P(X|\mathbf{B}) = P(X|\{\omega\}) = X$ for every singleton $\{\omega\}$ of $\mathbf{G}$. $\diamond$

Example 6 Let $\Omega = [0, 1]^n$ and let $\mathbf{F}$ and $\mathbf{G}$ be respectively the Lebesgue σ-field and the Borel σ-field. Let $\mathbf{B}$ be the partition of singletons and let X be the indicator function of an event belonging to $\mathbf{F} - \mathbf{G}$; if the linear conditional prevision is defined by the Radon–Nikodym derivative, by Theorem 10 we have

$$P(X|\mathbf{B}) = E[X|\mathbf{G}]_\omega \neq X$$

and so it is not coherent.

5.2 Coherent upper conditional previsions defined by Hausdorff outer measures and their integral representations

As example of coherent upper and lower conditional previsions which satisfy the results proven in the previous section we consider a new model of coherent upper conditional prevision. In a metric space, coherent upper conditional previsions based on Hausdorff outer measures are introduced and their properties have been studied. For the definition of Hausdorff outer measure and its basic properties see Rogers (1970) and Falconer (1986).

Let (Ω, d) be a metric space and let $\mathbf{B}$ be partition of Ω.

Let $\delta > 0$ and let s be a non-negative number. The diameter of a non empty set U of Ω is defined as $|U| = \sup\{d(x, y) : x, y \in U\}$ and if a subset A of Ω is such that $A \subseteq \bigcup_i U_i$ and $0 < |U_i| \leq \delta$ for each i, the class $\{U_i\}$ is called a δ-cover of A.

The Hausdorff s-dimensional outer measure of A, denoted by $h^s(A)$, is defined on $\wp(\Omega)$, the class of all subsets of Ω, as

$$h^s(A) = \lim_{\delta\to 0} \inf \sum_{i=1}^{+\infty} |U_i|^s .$$

where the infimum is over all δ-covers $\{U_i\}$.

A subset A of Ω is called measurable with respect to the outer measure h^s if it decomposes every subset of Ω additively, that is if $h^s(E) = h^s(A \cap E) + h^s(E - A)$ for all sets $E \subseteq \Omega$.

Hausdorff s-dimensional outer measures are submodular, continuous from below and their restriction on the Borel σ-field is countably additive.

The Hausdorff dimension of a set A, $dim_H(A)$, is defined as the unique value, such that

$$h^s(A) = +\infty \text{ if } 0 \le s < \dim_H(A),$$
$$h^s(A) = 0 \text{ if } \dim_H(A) < s < +\infty.$$

For every $B \in \mathbf{B}$ denote by s the Hausdorff dimension of B and let h^s be the Hausdorff s-dimensional Hausdorff outer measure associated to the coherent upper conditional prevision. For every bounded random variable X a coherent upper conditional prevision $\overline{P}(X|B)$ is defined by the Choquet integral with respect to its associated Hausdorff outer measure if the conditioning event has positive and finite Hausdorff outer measure in its Hausdorff dimension. Otherwise if the conditioning event has Hausdorff outer measure in its Hausdorff dimension equal to zero or infinity it is defined by a 0–1 valued finitely, but not countably, additive probability.

Theorem 6 (Doria 2012, Theorem 3) *Let (Ω, d) be a metric space and let $\boldsymbol{B}$ be a partition of Ω. For every $B \in \boldsymbol{B}$ denote by s the Hausdorff dimension of the conditioning event B and by h^s the Hausdorff s-dimensional outer measure. Let m be a 0–1 valued finitely additive, but not countably additive, probability on $\wp(B)$. Thus, for each $B \in \boldsymbol{B}$, the function defined on $\wp(B)$ by*

$$\overline{P}(A|B) = \frac{h^s(A \cap B)}{h^s(B)} \; \textit{if}\; 0 < h^s(B) < +\infty$$

and by

$$\overline{P}(A|B) = m(A \cap B) \; \textit{if}\; h^s(B) \in \{0, +\infty\}$$

is a coherent upper conditional probability.

If $B \in \mathbf{B}$ is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension s the fuzzy measure μ_B^* defined for every $A \in \wp(B)$ by $\mu_B^*(A) = \frac{h^s(AB)}{h^s(B)}$ is a coherent upper conditional probability, which is submodular, continuous from below and such that its restriction to the σ-field of all μ_B^* measurable sets is a Borel regular countably additive probability.

The coherent upper unconditional probability $\overline{P} = \mu^*_\Omega$ defined on $\wp(\Omega)$ is obtained for B equal to Ω.

In Doria (2012, Theorem 2) a new model of coherent upper conditional prevision is given.

Theorem 7 *Let (Ω, d) be a metric space and let $\mathbf{B}$ be a partition of Ω. For every $B \in \mathbf{B}$ denote by s the Hausdorff dimension of the conditioning event B and by h^s the Hausdorff s-dimensional outer measure. Let m be a 0–1 valued finitely additive, but not countably additive, probability on $\wp(B)$. Then for each $B \in \mathbf{B}$ the functional $\overline{P}(X|B)$ defined on $L(B)$ by*

$$\overline{P}(X|B) = \frac{1}{h^s(B)} \int_B X dh^s \text{ if } 0 < h^s(B) < +\infty$$

and by

$$\overline{P}(X|B) = m_B \text{ if } h^s(B) \in \{0, +\infty\}$$

is a coherent upper conditional prevision.

When the conditioning event B has Hausdorff outer measure in its Hausdorff dimension equal to zero or infinity, an additive conditional probability is coherent if and only if it takes only 0–1 values. Because linear previsions on $L(B)$ are uniquely determined by their restrictions to events, the class of linear previsions on $L(B)$ whose restrictions to events take only the values 0 and 1 can be identified with the class of 0–1 valued additive probability defined on all subsets of B (Walley 1991). In Theorem 6 and Theorem 7 a different m is chosen for each B.

If the conditioning event B has positive and finite Hausdorff outer measure in its Hausdorff dimension in Doria (2012) the functional $\overline{P}(X|B)$ is proven to be monotone, comonotonically additive, submodular and continuous from below.

The models of coherent upper conditional previsions and probabilities proposed in Theorem 6 and in Theorem 7 satisfy the theorems given in the Section 3 when Ω is a set with finite and positive Hausdorff measure in its Hausdorff dimension since Hausdorff outer measures are submodular.

Example 7 Let (Ω, d) be a metric space where Ω is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension and let K be a class of random variable such that it satisfies the condition given in Theorem 2 (or in Theorem 3 or in Theorem 4). Then a random variable X_i is maximal in K under the conjugate of the upper conditional prevision defined in Theorem 8 if and only if it is a Bayes random variable

In the next theorem, it is proven that if a random variable is maximal under the coherent lower conditional prevision which is the conjugate of the coherent upper conditional prevision defined in Theorem 7 when the conditioning event has positive and finite Hausdorff outer measure in its Hausdorff dimension then it is maximal with respect to a functional representable as Choquet integral by any submodular monotone set function which is continuous from below.

Theorem 8 *Let (Ω, d) be a metric space where Ω is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension, let $\mathbf{B}$ be a partition of Ω and let K be a class of random variables on Ω. Let $B \in \mathbf{B}$ be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension and let $\overline{P}(\cdot|B)$ be the coherent upper conditional prevision defined in Theorem 7 and let $\underline{P}(\cdot|B)$ its conjugate lower conditional prevision. Then for every submodular monotone set function ν defined on $\wp(B)$ which is continuous from below the following condition holds:*

$$\text{an admissible random variable } X_i \in K \text{ is maximal in } K \text{ under } \underline{P}(\cdot|B)$$
$$\Leftrightarrow \int (X_i - X_j)d\nu \geq 0 \forall X_j \in K$$

Proof Since B is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension the coherent upper conditional prevision defined as in Theorem 7 is a functional which is monotone, submodular, continuous from below and representable by μ_B^*. By Proposition 13.5 of Denneberg (1994) all submodular monotone set functions ν on $\wp(B)$ which are continuous from below agree on the set system of weak upper level sets with μ_B^* so that the Choquet integral with respect to μ_B^* is equal to the Choquet integral with respect to ν. So

an admissible random variable $X_i \in K$ is maximal in K under $\underline{P}(\cdot|B) \Leftrightarrow \int(X_i - X_j)\mathrm{d}\mu_B^* \geq 0\ \forall X_j \in K \Leftrightarrow \int(X_i - X_j)\mathrm{d}\mu_B^* = \int(X_i - X_j)\mathrm{d}\nu \geq 0\ \forall X_j \in K$ ⋄

The given preference ordering in Example 1 can be represented by the lower coherent prevision defined as Choquet integral with respect to the lower conditional probability obtained as the minimum of the class of 0–1 valued probability defined on $\mathbf{B}$. By Theorem 6 these probabilities can be used to asses probability when the the set Ω has Hausdorff outer measure in its Hausdorff dimension equal to zero or infinity.

Example 8 Let (Ω, d) be a metric space with $\Omega = N$ so that $dim_H(\Omega) = 0$ and $h^0 = (\Omega) = +\infty$. Let $\mathbf{B} = \{B_1, B_2\}$ be the partition of Ω where $B_1 = \{p \in N : p = 2n; n \in N\}$ and $B_2 = \{d \in N : d = 2n - 1; n \in N\}$ so that $dim_H(B_1) = dim_H(B_2) = 0$ and $h^0(B_1) = h^0(B_2) = +\infty$.

By Theorem 6 a class $\{P_1, P_2\}$ of additive probabilities can be defined on $\mathbf{B}$ by

$$P_1(B_1) = 0,\ P_1(B_2) = 1 \text{ and } P_2(B_1) = 1,\ P_2(B_2) = 0.$$

Let $\overline{\mu}_B$ be the coherent upper conditional probability defined by

$\overline{\mu}_B(A) = max\{P_1(A); P_2(A)\}\ \forall A \in \wp(B)$ and let $\underline{\mu}_B$ be the coherent upper conditional probability defined by

$\underline{\mu}_B(A) = min\{P_1(A); P_2(A)\}\ \forall A \in \wp(B)$.

Let $K = \{X_1, X_2, X_3\}$ be the class of $\mathbf{B}$-measurable random variables as in Example 1:

Random variables	B_1	B_2
X_1	0.3	0.3
X_2	0.7	0.0
X_3	0.0	0.7

if the lower conditional prevision is defined as Choquet integral with respect to $\underline{\mu}_B$ and the upper conditional prevision as the Choquet integral with respect to $\overline{\mu}_B$ then each random variable X_i is admissible in K since $\underline{P}((X_i - X_j)) < 0$ for every $i, j \in \{1, 2, 3\}$ with $i \neq j$ and all the random variables X_i for $i = 1, 2, 3$ are maximal in K with respect to $\underline{P}$ because

$$\overline{P}((X_1 - X_3)) = \frac{3}{10} \text{ and } \overline{P}((X_1 - X_2)) = \frac{3}{10},$$
$$\overline{P}((X_2 - X_3)) = \frac{7}{10} \text{ and } \overline{P}((X_2 - X_1)) = \frac{4}{10},$$
$$\overline{P}((X_3 - X_1)) = \frac{4}{10} \text{ and } \overline{P}((X_3 - X_2)) = \frac{7}{10}.$$

X_2 and X_3 are not indifferent with respect $\underline{P}$ and with respect to $\overline{P}$ but they are equivalent with respect to $\underline{P}$ and with respect to $\overline{P}$ since $\underline{P}(X_2) = \underline{P}(X_3) = 0$ and $\overline{P}(X_2) = \overline{P}(X_3) = \frac{7}{10}$.

Moreover, the ordering $X_1 \succ X_2$ and $X_2 \approx X_3$ considered in Example 1 can be represented by $\underline{P}$ since $\underline{P}(X_1) = 0.3 > 0 = \underline{P}(X_2) = \underline{P}(X_3)$.

But X_1 not $\succ_* X_2$ with respect to $\underline{P}$ since $\underline{P}(X_1 - X_2) < 0$; nevertheless $X_1 \succ^* X_2$ with respect to $\overline{P}$ since $\overline{P}(X_1 - X_2) > 0$.

No random variable X_i for $i = 1, 2, 3$ is a Bayes random variable under $\underline{P}$.

In Theorem 14 of Doria (2008) it has been proved that if the conditioning event has positive and finite Hausdorff measure in its Hausdorff dimension then indifferent random variable are equivalent with respect to the coherent upper prevision defined in Theorem 7. Example 8 shows that if the conditioning event has Hausdorff outer measure in its Hausdorff dimension equal to infinity the implication does not hold.

In Doria (2017) it has been proven that coherent upper conditional prevision defined by Hausdorff outer measures as in Theorem 7 satisfy the disintegration property on every non-null partition.

Definition 15 Denoted by t the Hausdorff dimension of Ω, a partition $\mathbf{B}$ is non-null if the complement of the union of sets $B \in \mathbf{B}$ with positive h^t-measure sure, has zero h^t-measure.

Therefore, if in Example 8 conditional previsions are defined by Theorem 7 we obtain that the disintegration property holds since $\mathbf{B}$ is a non-null partition.

Example 9 Let (Ω, d), $\mathbf{B}$ and K be as in Example 8. By Theorem 7 we can obtain

$$P(X_1|B_2) = 1,\ P(X_1|B_1) = 0,\ P(B_1) = 0$$
and
$$P(X_1|B_3) = 1,\ P(X_3|B_2) = 0,\ P(B_2) = 1$$

so that the disintegration property is satisfied and the ordering $X_1 \succ X_1$ and $X_2 \approx X_2$ can be represented by the given coherent conditional prevision

$$P(X_1|\Omega) = P(B_1)P(X_1|B_1) + P(B_2)P(X_1|B_2) = 0 \cdot 1 + 1 \cdot 1 = 1$$
$$P(X_2|\Omega) = P(B_1)P(X_2|B_1) + P(B_2)P(X_2|B_2) = 0 \cdot 1 + 1 \cdot 0 = 0$$
$$P(X_3|\Omega) = P(B_1)P(X_3|B_1) + P(B_2)P(X_3|B_2) = 0 \cdot 1 + 1 \cdot 0 = 0.$$

6 Conclusions

Preference orderings represented by coherent lower previsions which asses null previsions to the atoms of the partition are analyzed. In these cases coherent lower previsions are fully conglomerable but may fail the disintegration property. It implies that a maximal random variable on an atom of a null partition is not a maximal random variable on the set Ω. Examples of classes of random variables are given such that there are not maximal random variables and Bayes random variables with respect to preferences orderings defined by coherent lower and upper conditional previsions.

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