Bollettino
Unione Matematica Italiana

Rinaldo M. Colombo, Massimiliano D. Rosini

# Well Posedness of Balance Laws with Non-Characteristic Boundary

*Bollettino dell'Unione Matematica Italiana, Serie 8*, Vol. **10-B** (2007), n.3, p. 875–894.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2007_8_10B_3_875_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2007.

Bollettino U. M. I.
(8) **10**-B (2007), 875-894

# Well Posedness of Balance Laws with Non–Characteristic Boundary

RINALDO M. COLOMBO - MASSIMILIANO D. ROSINI

**Sunto.** – *Questa nota presenta un risultato di buona positura per un problema ai valori iniziali ed al contorno per un sistema non lineare di leggi di bilancio, nel caso non caratteristico.*

**Summary.** – *This note presents a well posedness result for the initial–boundary value problem consisting of a nonlinear system of hyperbolic balance laws with boundary, in the non–characteristic case.*

## 1. – Introduction.

In this paper we study the well posedness of the following initial–boundary value problem for a nonlinear system of balance laws

$$\left\{\begin{array}{rcll} \partial_t u + \partial_x f(u) & = & g(t,x,u) & (t,x) \in \Omega \\ u(t_o,x) & = & \bar{u}(x) & x \leq \Psi(t_o) \\ b(u(t,\Psi(t))) & = & h(t) & t \geq t_o \end{array}\right. \tag{1.1}$$

in the non–characteristic case. Here $\bar{u}$, $h$ are $\boldsymbol{L}^1$ functions with small total variation, $b$ is smooth, $t_o \in \mathbb{R}$ is the initial time, the conserved variables vary in $\Omega = \{(t,x) \in \mathbb{R}^2 : t \geq t_o,\, x \leq \Psi(t)\}$ for a suitable Lipschitz map $\Psi : [t_o, +\infty[ \mapsto \mathbb{R}$, $u \in \mathcal{U}$ denotes the unknown vector of the conserved quantities. The present result, related to [2, 12, 16], extends those in [9, 10].

In the sequel we consider separately the convective part

$$\left\{\begin{array}{rcll} \partial_t u + \partial_x f(u) & = & 0 & (t,x) \in \Omega \\ u(t_o,x) & = & \bar{u}(x) & x \leq \Psi(t_o) \\ b(u(t,\Psi(t))) & = & h(t) & t \geq t_o \end{array}\right. \tag{1.2}$$

and the source part

$$\left\{\begin{array}{rcll} \partial_t u & = & g(t,x,u) & (t,x) \in \Omega \\ u(t_o,x) & = & \bar{u}(x) & x \leq \Psi(t_o) \\ b(u(t,\Psi(t))) & = & h(t) & t \geq t_o \end{array}\right. \tag{1.3}$$

of (1.1). Indeed, the well posedness of (1.1) follows from the well posedness of (1.2), (1.3) and their compatibility. Therefore, we require those assumptions on $f$ and $g$ that make (1.2) and (1.3) well posed, and ask that there exists a domain which is invariant for both (1.2) and (1.3). This geometric assumption replaces other compatibility conditions between the convective part and the source term, such as those in [3, 15, 20], see [14, Section 13.8]. Finally, to obtain the well posedness globally in time, we assume that (1.2) is a Temple system. In fact, we need to require on (1.2) hypotheses that ensure the well posedness for large data because the total variation and the $\boldsymbol{L}^\infty$ norm of the solution may well grow exponentially with time, see 5. in Theorem 2.6.

Below, we follow the definition [2, Definition NC] of solution to the boundary value problem (1.2). This framework is suitable, for instance, in applications to traffic modeling, where no physical viscosity is present.

## 2. – Preliminaries and Main Result.

As general references on the 1D theory of hyperbolic systems of conservation laws, we refer to [6] or [14].

Concerning the convective part (1.2), following [5], we denote with $\sigma \mapsto \mathcal{L}_i(\sigma)(u)$ the $i$-th generalized Lax curve exiting $u$ and parametrized through the signed arc length $\sigma$. Let $r_i(u) = \partial_\sigma(\mathcal{L}_i(\sigma)(u))_{|_{\sigma=0}}$ be the $i$-th right eigenvector of $Df(u)$ corresponding to the eigenvalue $\lambda_i(u)$, for $i = 1, \ldots, n$. On (1.2) we assume:

(**F**) Let $\mathcal{U}$ be the closure of an open subset of $\mathbb{R}^n$, with $0 \in \mathcal{U}$, $f : \mathcal{U} \mapsto \mathbb{R}^n$ be smooth, such that its Jacobian $Df$ has $n$ real eigenvalues $\lambda_1, \ldots, \lambda_n$ with $\max_{i=1,\ldots,n} \sup_{u \in \mathcal{U}} |\lambda_i(u)| \leq \hat{\lambda}$ for a constant $\hat{\lambda} > 0$ and such that $\partial_t u + \partial_x f(u) = 0$ is a Temple system, i.e.
- ($F_1$) The system is strictly hyperbolic in $\mathcal{U}$, i.e. for all $i = 1, \ldots, n-1$ holds $\sup_{u \in \mathcal{U}} \lambda_i(u) < \inf_{u \in \mathcal{U}} \lambda_{i+1}(u)$.
- ($F_2$) For $i = 1, \ldots, n$, the $i$-shock curve and the $i$-rarefaction curve coincide.
- ($F_3$) In $\mathcal{U}$, there exists a system of Riemann coordinates $\{w_1, \ldots, w_n\}$, such that $\partial_{w_i} u$ is parallel to $r_i$. Moreover, the coordinate change $w$ is in $\boldsymbol{C}^2(\mathcal{U}, w(\mathcal{U}))$ as also its inverse: $w^{-1} = u \in \boldsymbol{C}^2(w(\mathcal{U}), \mathcal{U})$.

On the source part (1.3), essentially an ordinary differential system, we assume (here, $|\cdot|$ denotes the norm (2.5) in $\mathbb{R}^n$):

(**G**) $g : [t_o, +\infty[ \times \mathbb{R} \times \mathcal{U} \mapsto \mathbb{R}^n$ is such that
- ($G_1$) For a.e. $t \in [t_o, +\infty[$ and all $x \in \mathbb{R}$, $g(t, x, 0) = 0$.
- ($G_2$) For all $(x, u) \in \mathbb{R} \times \mathcal{U}$ the map $t \mapsto g(t, x, u)$ is measurable.

($G_3$) The map $x \mapsto g(t,x,u)$ is uniformly in $\boldsymbol{BV}(\mathbb{R},\mathbb{R}^n)$, i.e. there exists a finite positive measure $\mu$ such that for a.e. $t \in [t_o,+\infty[$, for all $x_1, x_2 \in \mathbb{R}$ with $x_1 \leq x_2$ and for all $u \in \mathcal{U}$,

$$|g(t,x_2+,u) - g(t,x_1-,u)| \leq \mu([x_1,x_2]).$$

($G_4$) For a.e. $t \in [t_o,+\infty[$ and $x \in \mathbb{R}$, the map $u \mapsto g(t,x,u)$ is locally Lipschitz and sublinear in $\mathcal{U}$, i.e. for every compact subset $K$ of $\mathcal{U}$, there exists a function $l_K \in \boldsymbol{L^\infty_{loc}}([t_o,+\infty[,\mathbb{R})$ such that for a.e. $t \in [t_o,+\infty[$, all $x \in \mathbb{R}$ and all $u_1, u_2 \in K$,

$$|g(t,x,u_2) - g(t,x,u_1)| \leq l_K(t) \cdot |u_2 - u_1|$$

and there exists a function $l \in \boldsymbol{L^1_{loc}}([t_o,+\infty[,\mathbb{R})$ such that for a.e. $t \in [t_o,+\infty[$, all $x \in \mathbb{R}$ and all $u \in \mathcal{U}$,

$$|g(t,x,u)| \leq l(t) \cdot |u| .$$

On the domain $\Omega$ we assume that

(**Ω**) The function $\Psi$ describing the boundary of $\Omega$ is Lipschitz and non–characteristic, i.e. there exists a fixed $m \in \{1,\ldots,n-1\}$ such that for a.e. $t \in [t_o,+\infty[$

$$\sup_{u\in\mathcal{U}} \lambda_m(u) \quad < \quad \dot{\Psi}(t) \quad < \quad \inf_{u\in\mathcal{U}} \lambda_{m+1}(u) . \tag{2.1}$$

The case $\dot{\Psi} > \sup \lambda_n(u)$ is essentially equivalent to a Cauchy problem. On the contrary, if $\dot{\Psi} < \inf \lambda_1(u)$, then the boundary condition has no effect on the solution.

Finally, on the boundary condition we assume that

(**B**) The map $h$: $[t_o,+\infty[ \mapsto \mathbb{R}^m$ is in $\boldsymbol{L}^1$ and has bounded variation. $b$: $\mathcal{U} \mapsto \mathbb{R}^m$ is smooth and there exists a Lipschitz map $\sigma_b$: $\mathcal{U} \times \mathbb{R}^m \mapsto \mathbb{R}^m$, such that for all $\bar{u} \in \mathcal{U}$ and all $h \in b(\mathcal{U})$, the state

$$\tilde{u} = (\mathcal{L}_m(\sigma_m) \circ \mathcal{L}_{m-1}(\sigma_{m-1}) \circ \cdots \circ \mathcal{L}_2(\sigma_2) \circ \mathcal{L}_1(\sigma_1))(\bar{u})$$

satisfies $b(\tilde{u}) = h$ if and only if $(\sigma_1,\ldots,\sigma_m) = \sigma_b(\bar{u},h)$.

Clearly, if $b(\bar{u}) = h$, then $\sigma_b(\bar{u},h) = 0$. Moreover, if the $m \times m$ matrix $D_u b(0) \cdot [r_1(0),\ldots,r_m(0)]$ is non singular and $|b(0)|$, TV$(h)$ are both sufficiently small, then **(B)** holds in a neighborhood of $u = 0$, see [2, Formula (3.2)]. Using Riemann coordinates, boundary conditions may take a simpler form, see for instance [1, Remark 2.3].

Introduce the definition of weak entropic solution to (1.1), along the same lines of [1, Definition 2.2], [2, Definition NC] and [6, Definition 4.2]:

DEFINITION 2.1. – *$u: \Omega \mapsto \mathcal{U}$ is a weak solution of the problem* (1.1) *if*

*(i) for any function* $\varphi \in \boldsymbol{C}_c^\infty(\{(t,x) \in \mathbb{R}^2 : t < t_o \text{ or } x < \Psi(t)\}, \mathbb{R})$

$$\int_{t_o}^{+\infty} \int_{-\infty}^{\Psi(t)} [\partial_t \varphi(t,x)\, u(t,x) + \partial_x \varphi(t,x) f(u(t,x))]\, dx\, dt$$

$$+ \int_{t_o}^{+\infty} \int_{-\infty}^{\Psi(t)} \varphi(t,x)\, g(t,x,u(t,x))\, dx\, dt$$

$$+ \int_{-\infty}^{\Psi(t_o)} \varphi(t_o,x)\, \bar{u}(x)\, dx = 0\,,$$

*(ii) u satisfies the boundary condition, i.e. for a.e.* $\tau \in [t_o, +\infty[$

$$\lim_{(t,x) \to (\tau, \Psi(\tau)),\, (t,x) \in \Omega} b(u(t,x)) = h(\tau)\,.$$

*Given an entropy-entropy flux pair* $(\eta, q)$ *(see* [14, Section 3.2]*), the weak solution u is* entropic *if for any* $\varphi \in \boldsymbol{C}_c^\infty(\{(t,x) \in \mathbb{R}^2 : t < t_o \text{ or } x < \Psi(t)\}, [0, +\infty[\,)$

$$\int_{t_o}^{+\infty} \int_{-\infty}^{\Psi(t)} [\partial_t \varphi(t,x)\, \eta(u(t,x)) + \partial_x \varphi(t,x)\, q(u(t,x))]\, dx\, dt$$

$$+ \int_{t_o}^{+\infty} \int_{-\infty}^{\Psi(t)} \varphi(t,x)\, D\eta(u(t,x)) g(t,x,u(t,x))\, dx\, dt$$

$$+ \int_{-\infty}^{\Psi(t_o)} \eta(t_o,x) \varphi(\bar{u}(x))\, dx \geq 0\,.$$

The following definition of solution to (non–characteristic) Riemann problems with boundary is a slight generalization of the analogous definition in [2, Section 3], see also [18, Chapter 1].

DEFINITION 2.2. – *Fix $\omega$ in $\mathbb{R}$ with* $\sup_{u \in \mathcal{U}} \lambda_m(u) < \omega < \inf_{u \in \mathcal{U}} \lambda_{m+1}(u)$ *for some* $m \in \{1, \ldots, n-1\}$. *Let* $\bar{u} \in \mathcal{U}$ *and* $h \in \mathbb{R}^m$ *be fixed. The solution to the Riemann problem with boundary*

$$\left\{ \begin{array}{rcll} \partial_t u + \partial_x f(u) & = & 0 & t > 0, \quad x < \omega t \\ u(0,x) & = & \bar{u} & x < 0 \\ b(u(t,\omega t)) & = & h & t \geq 0 \end{array} \right. \tag{2.2}$$

*is the restriction to* $\{(t,x) : t \geq 0,\ x \leq \omega t\}$ *of the Lax solution to the standard*

*Riemann problem*

$$(2.3)\qquad \left\{\begin{array}{rcl} \partial_t u + \partial_x f(u) &=& 0 \qquad (t,x) \in [0,+\infty[\, \times \mathbb{R} \\ u(0,x) &=& \left\{\begin{array}{ll} \bar{u} & \text{if } x<0 \\ \tilde{u} & \text{if } x>0 \end{array}\right. \end{array}\right.$$

*where $\tilde{u}$ satisfies $b(\tilde{u}) = h$ and the solution to* (2.3) *contains waves only of the families* $1,\ldots,m$.

Under assumptions **(F)**, **(Ω)** and **(B)**, (2.2) admits a unique solution. Indeed, **(B)** ensures that there exists a unique such $\tilde{u}$. In the case of the Cauchy problem, we generalize [2, Definition NC] as follows, see also [1, Definition 2.2]:

DEFINITION 2.3. – *Let $u: \Omega \mapsto \mathcal{U}$ be such that for a.e. $t \in [t_o, +\infty[$, $x \mapsto u(t,x)$ is in $\boldsymbol{BV}(]-\infty, \Psi(t)], \mathbb{R}^n)$. $u$ solves the convective problem* (1.2) *if*

(i) *for any function $\varphi \in \boldsymbol{C}_c^\infty(\{(t,x) \in \mathbb{R}^2 : t < t_o$ or $x < \Psi(t)\}, \mathbb{R})$*

$$\int_{t_o}^{+\infty} \int_{-\infty}^{\Psi(t)} \left[\partial_t \varphi(t,x)\, u(t,x) + \partial_x \varphi(t,x) f(u(t,x))\right] dx\, dt + \int_{-\infty}^{\Psi(t_o)} \varphi(t_o,x)\, \bar{u}(x)\, dx = 0\,,$$

(ii) *$u$ satisfies the boundary condition, i.e. for a.e. $\tau \in [t_o, +\infty[$*

$$\lim_{(t,x)\to(\tau,\Psi(\tau)),\, (t,x)\in\Omega} b(u(t,x)) = h(\tau)\,.$$

Recall the following definition of solution to (1.3), see [12, Definition 2.2].

DEFINITION 2.4. – *$u: \Omega \mapsto \mathcal{U}$ is a solution to* (1.3) *if for all $(\tau,x) \in \Omega$ the map $t \mapsto u(t,x)$ is an absolutely continuous Carathéodory solution* [17, Chapter 1] *of*

$$\left\{\begin{array}{rcll} \partial_t u &=& g(t,x,u) & t \in ]\beta(\tau,x), \tau[ \\ u(\beta(\tau,x),x) &=& \bar{u}(x) & \text{if } \beta(\tau,x) = t_o \\ b(u(\beta(\tau,x),x)) &=& h(\beta(\tau,x)) & \text{if } \beta(\tau,x) > t_o,\ x = \Psi(\beta(\tau,x)) \end{array}\right.$$

*where for any $(t,x) \in \Omega$*

$$(2.4)\qquad \beta(t,x) = \inf\{s \in ]t_o, t[ : (\theta s + (1-\theta)t, x) \in \Omega\,,\ \forall \theta \in [0,1]\}\,.$$

In the following, we express vectors and functions in their Riemann coordinates $w$. Therefore we introduce

$$(2.5)\qquad \begin{array}{rcll} |v| &=& \max_{i=1,\ldots,n} |v_i| & \text{for } v \in \mathbb{R}^n \\ \|u\| &=& |w(u)| & \text{for } u \in \mathcal{U} \\ \mathrm{TV}_w(u) &=& \sum_{i=1}^n \mathrm{TV}(w_i(u)) & \text{for } u: \mathbb{R} \mapsto \mathcal{U}\,. \end{array}$$

Remark that on any compact subset of $\mathcal{U}$, $\|\cdot\|$ (resp. $\mathrm{TV}_w(\,\cdot\,)$) is equivalent to the usual Euclidean norm (resp. total variation) because of $(F_3)$.

For $t \geq t_o$, let $\mathcal{D}_t$ be the set of triples $\boldsymbol{p} = (\bar{u}, h, \Psi)$, where

$$\begin{array}{rcll} \bar{u} & \in & \boldsymbol{L}^1 \cap \boldsymbol{BV}(\mathbb{R}, \mathcal{U}) & \text{with } \bar{u}(x) = 0 \text{ for } x > \Psi(t) \\ h & \in & \boldsymbol{L}^1 \cap \boldsymbol{BV}([t, +\infty[, \mathbb{R}^m) & \\ \Psi & \in & \boldsymbol{C}^0([t, +\infty[, \mathbb{R} \times \mathbb{R}), & \end{array}$$

and (2.1) holds, see also [2, Formula (2.4)]. Introduce in $\mathcal{D}_t$

$$\begin{array}{rcl} \mathrm{TV}(\boldsymbol{p}) & = & \mathrm{TV}_w(\bar{u}) + \mathrm{TV}(h) + |b(\bar{u}(\Psi(t))) - h(t)| \\ d(\boldsymbol{p}, \boldsymbol{p}') & = & \|\bar{u} - \bar{u}'\|_{\boldsymbol{L}^1} + \|h - h'\|_{\boldsymbol{L}^1} + \|\Psi - \Psi'\|_{\boldsymbol{C}^0} . \end{array}$$

For $t \geq t_o$ and $M > 0$, introduce also $\mathcal{D}_{t,M} = \{\boldsymbol{p} \in \mathcal{D}_t : \mathrm{TV}(\boldsymbol{p}) \leq M\}$.

The invariance assumption ensures the *compatibility* between (1.2) and (1.3).

(**U**) The set $\mathcal{U}$ is invariant with respect to both (1.2) and (1.3).

Here, invariance is understood as follows.

DEFINITION 2.5. – *$\mathcal{U}$ is invariant for* (1.2)*, resp.* (1.3)*, if any admissible data $(\bar{u}, h, \Psi)$ attaining values in $\mathcal{U}$, i.e. $\bar{u}(]-\infty, \Psi(t_o)]) \subseteq \mathcal{U}$, leads to a solution $u$ to* (1.2)*, resp.* (1.3)*, attaining values in $\mathcal{U}$, i.e. $u(\Omega) \subseteq \mathcal{U}$.*

By *admissible* data $(\bar{u}, h, \Psi) \in \mathcal{D}_{t_o}$ we mean that (1.2), resp. (1.3), with data $(\bar{u}, h, \Psi)$ admits a solution in the sense of Definition 2.3, resp. Definition 2.4, for all times $t \geq t_o$. For a treatment of invariant domains for conservation laws, see [19]. Recall that a closed set $\mathcal{U}$ is invariant with respect to (1.2) if and only if any Riemann problem with data in $\mathcal{U}$ yields a solution attaining values in $\mathcal{U}$. In the case of (1.3), a condition for invariance is provided, for instance, by the classical Nagumo condition [21]. Remark that, in both cases, $\mathcal{U}$ needs neither be convex nor compact in the $u$ coordinates.

Below we show that (1.1) generates a process $F$

$$\begin{array}{rcl} F : \{(\boldsymbol{p}, t_1, t_2) : \boldsymbol{p} \in \mathcal{D}_{t_1}, t_2 \geq t_1 \geq t_o\} & \mapsto & \bigcup_{t \geq t_o} \mathcal{D}_t \\ ((\bar{u}, h, \Psi), t_1, t_2) & \mapsto & (u(t_2), \mathcal{T}_{t_2 - t_1} h, \mathcal{T}_{t_2 - t_1} \Psi), \end{array}$$

where $u(t_2)$ is the solution to (1.1) at time $t_2$ with data $(\bar{u}, h, \Psi)$ and initial time $t_1$, while $\mathcal{T}_t$ is the translation operator, i.e. $(\mathcal{T}_t h)(s) = h(t+s)$ and $(\mathcal{T}_t \Psi)(s) = \Psi(t+s)$.

The main result of this paper is

THEOREM 2.6. – *Let* (1.1) *satisfy assumptions* (**F**)*,* (**G**)*,* (**B**)*,* (***Ω***) *and* (**U**)*. Then, there exists a unique process $F$ with the properties:*

1. *For all $t \in [t_o, +\infty[$ and $(\bar{u}, h, \Psi) \in \mathcal{D}_{t_o}$, the function $u : \Omega \mapsto \mathcal{U}$ defined by $(u(t, \cdot), \mathcal{T}_{t-t_o} h, \mathcal{T}_{t-t_o} \Psi) = F((\bar{u}, h, \Psi), t_o, t)$ is a weak entropic solution to* (1.1).

2. *For all* $t_1, t_2, t_3$ *with* $t_3 \geq t_2 \geq t_1 \geq t_o$, $F(F(\boldsymbol{p}, t_1, t_2), t_2, t_3) = F(\boldsymbol{p}, t_1, t_3)$ *for all* $\boldsymbol{p} \in \mathcal{D}_{t_1}$, *while for all* $t \geq t_o$, $F(\boldsymbol{p}, t, t) = \boldsymbol{p}$ *for all* $\boldsymbol{p} \in \mathcal{D}_t$.
3. *If* $(\bar{u}, h, \Psi) \in \mathcal{D}_{t_o}$, *with* $\bar{u}$, $h$ *piecewise constant and* $\Psi$ *is piecewise linear and continuous, then the corresponding solution* $u$ *for small times is the gluing of the solutions to the Riemann problems on the points of jump of* $\bar{u}$ *and at* $(t_o, \Psi(t_o))$.

*Moreover, for every* $T, M > 0$, *there exist constants* $L, C$ *such that:*

4. *Fix* $(\bar{u}, h, \Psi)$, $(\bar{u}', h', \Psi')$ *in* $\mathcal{D}_{t_o,M}$ *and call* $u$, $u'$ *the corresponding solutions to* (1.1) *yielded by* $F$. *Then, if* $t, t' \in [t_o, T]$,
$$\|u(t) - u'(t')\|_{\boldsymbol{L}^1} \leq L\big(\|\bar{u} - \bar{u}'\|_{\boldsymbol{L}^1} + \|\Psi - \Psi'\|_{\boldsymbol{C}^0} + \|h - h'\|_{\boldsymbol{L}^1} + |t - t'|\big).$$
5. *For any data* $(\bar{u}, h, \Psi) \in \mathcal{D}_{t_o,M}$, *the solution yielded by* $F$ *satisfies*
$$\|u(t)\|_{\boldsymbol{L}^\infty} \leq e^{C\int_{t_o}^{t} l(\tau)d\tau} \left(\|\bar{u}\|_{\boldsymbol{L}^\infty} + \|h\|_{\boldsymbol{L}^\infty}\right),$$
$$\mathrm{TV}(u(t)) \leq e^{C(t-t_o)}(\mathrm{TV}_w(\bar{u}) + \mathrm{TV}(h) + |b(\bar{u}(\Psi(t_o))) - h(t_o)|)$$
$$+ e^{C(t-t_o)} C\mu(]-\infty, \Psi(t_o) - \lambda_m(t - t_o)])(t - t_o).$$
6. *If* $\mathcal{U}$ *is compact, then* $C$ *does not depend on* $T$.

The proof is based on the main results in [2, 5, 9] and is deferred to Section 3.

## 3. – Technical Proofs.

Throughout this section, $\varepsilon > 0$ is sufficiently small and fixed. All estimates are uniform in $\varepsilon$. The limit $\varepsilon \to 0$ will be considered only in the final part of the section.

### 3.1 – *The Convective Part.*

Throughout this paragraph we let $t_o = 0$.

We apply the approximation algorithm introduced in [5], [9], which specializes the algorithm given in [6] to possibly non convex Temple systems.

By **(U)**, we can write $w(\mathcal{U})$ as the Cartesian product of closed intervals $\mathcal{I}_i$, i.e. $w(\mathcal{U}) = \prod_{i=1}^{n} \mathcal{I}_i$, see [19]. For all $i$, let $\mathcal{I}_i^\varepsilon$ be a finite subset of $\mathcal{I}_i$ with the properties

*(i)* $\bigcup_{w \in \mathcal{I}_i^\varepsilon} ]w - \varepsilon, w + \varepsilon[ \supseteq \mathcal{I}_i$ and $0 \in \mathcal{I}_i$;

*(ii)* there exists a positive $\delta^\varepsilon$ such that $\min_{w_i', w_i'' \in \mathcal{I}_i^\varepsilon, w_i' \neq w_i''} |w_i' - w_i''| > \delta^\varepsilon$;

(iii) $\min \mathcal{I}_i^\varepsilon = \begin{cases} -1/\varepsilon & \text{if } \inf \mathcal{I}_i = -\infty \\ \min \mathcal{I}_i & \text{if } \inf \mathcal{I}_i > -\infty, \end{cases}$ $\max \mathcal{I}_i^\varepsilon = \begin{cases} 1/\varepsilon & \text{if } \sup \mathcal{I}_i = \infty \\ \max \mathcal{I}_i & \text{if } \sup \mathcal{I}_i < \infty. \end{cases}$

Then the set $\mathcal{G}^\varepsilon = \prod_{i=1}^n \mathcal{I}_i^\varepsilon$ is an $\varepsilon$*–grid* in $w(\mathcal{U})$, see [9].

As in [5, 9], fix an $\varepsilon$–grid $\mathcal{G}^\varepsilon$ and consider the Riemann problems

$$\left\{ \begin{array}{rcl} \partial_t u + \partial_x f(u) & = & 0 \\ u(0,x) & = & \begin{cases} u^l & \text{if } x < 0 \\ u^r & \text{if } x > 0, \end{cases} \end{array} \right. \tag{3.1}$$

with initial data $u^l, u^r$ such that $w^l = w(u^l), w^r = w(u^r) \in \mathcal{G}^\varepsilon$. Let the states $u^0, \ldots, u^n$ be given through their Riemann coordinates $w^0, \ldots, w^n$ as follows:

$$\begin{array}{rcll} w^0 & = & w(u^l) = (w_1^l, w_2^l, \ldots, w_{n-1}^l, w_n^l) & \\ w^i & = & (w_1^r, \ldots, w_i^r, w_{i+1}^l \ldots, w_n^l) & i = 1, \ldots, n-1 \\ w^n & = & w(u^r) = (w_1^r, w_2^r, \ldots, w_{n-1}^r, w_n^r) & \end{array} \tag{3.2}$$

Clearly $w^i \in \mathcal{G}^\varepsilon$, $i = 0, \ldots, n$. Introduce $f_i(u^{i-1}; s) = \int_0^s \lambda_i(\mathcal{L}_i(u^{i-1}, s))\, ds$ and let $\sigma_i$ be the solution of

$$\mathcal{L}_i(u^{i-1})(\sigma_i) = u^i \quad \text{for } i = 1, \ldots, n\,. \tag{3.3}$$

The exact weak entropic solution to (3.1) is the juxtaposition of $u_i(t,x) = u^i + s_i(t,x)\, r_i(u^i)$, $i = 1, \ldots, n$, where $u^i = w^{-1}(w^i)$ and $s_i$ is the solution to

$$\left\{ \begin{array}{rcl} \partial_t s_i + \partial_x f_i(u^{i-1}; s_i) & = & 0 \\ s_i(0,x) & = & \begin{cases} 0 & \text{if } x < 0 \\ \sigma_i & \text{if } x > 0\,. \end{cases} \end{array} \right.$$

Let now $s \mapsto f_i^\varepsilon(u^{i-1}; s)$ be the piecewise linear function that coincides with $s \mapsto f_i(u^{i-1}; s)$ on $w^{-1}(\mathcal{G}^\varepsilon)$. A piecewise constant weak solution to (3.1), with entropy defect $\mathcal{O}(\varepsilon)$, is obtained gluing $u_i^\varepsilon(t,x) = u^i + s_i^\varepsilon(t,x)\, r_i(u^i)$, $i = 1, \ldots, n$, with $s_i^\varepsilon$ the exact solution to the (approximate) Riemann problems

$$\left\{ \begin{array}{rcl} \partial_t s_i + \partial_x f_i^\varepsilon(u^{i-1}; s_i) & = & 0 \\ s_i(0,x) & = & \begin{cases} 0 & \text{if } x < 0 \\ \sigma_i & \text{if } x > 0, \end{cases} \end{array} \right. \tag{3.4}$$

where $\sigma_i$ is given by (3.3), see [5]. Let

$$\mathcal{D}_0(\mathcal{G}^\varepsilon) = \left\{ (\bar u, h, \Psi) \in \mathcal{D}_0 \colon \begin{cases} (\bar u, h, \Psi) \in \boldsymbol{PC} \times \boldsymbol{PC} \times \boldsymbol{PLC} \\ w(\bar u)(\mathbb{R}) \subseteq \mathcal{G}^\varepsilon,\ w(\tilde u(\bar u, h))(\mathbb{R}) \subseteq \mathcal{G}^\varepsilon \\ \sup\limits_{u \in \mathcal{U}} \lambda_m(u) < \dot\Psi < \inf\limits_{u \in \mathcal{U}} \lambda_{m+1}(u) \end{cases} \right\} \tag{3.5}$$

and

$$\mathcal{D}_{0,M}(\mathcal{G}^\varepsilon) = \mathcal{D}_0(\mathcal{G}^\varepsilon) \cap \mathcal{D}_{0,M}\,. \tag{3.6}$$

Above, $\boldsymbol{PC}$ is the set of piecewise constant functions $\mathbb{R} \mapsto \mathbb{R}^n$ with finitely many jumps. Observe that if $u \in \boldsymbol{PC} \cap \boldsymbol{L}^1(\mathbb{R})$ then $u$ has compact support. $\boldsymbol{PLC}$ is the set of piecewise linear and continuous functions $[0, +\infty[ \mapsto \mathbb{R}$ with finitely many corners on any compact interval.

Now we can start the standard wave front tracking procedure [5, 9, 10], see also [2, 4, 6, 7, 8, 11, 13], to construct an approximate solution to (1.2). First, fix an $\varepsilon$–grid $\mathcal{G}^\varepsilon$ and approximate the data $(\bar{u}, h, \Psi)$ in (1.1) through $(\bar{u}^\varepsilon, h^\varepsilon, \Psi^\varepsilon)$ in $\mathcal{D}_{0,M}(\mathcal{G}^\varepsilon)$.

At time $t = 0$, at every point $x \in ]-\infty, \Psi^\varepsilon(0)[$ where $\bar{u}^\varepsilon$ has a jump, we approximately solve the Riemann problem (3.1) with $u^l = \bar{u}^\varepsilon(x-)$ and $u^r = \bar{u}^\varepsilon(x+)$ by means of the exact solutions to the $n$ Riemann problems (3.4). At $(0, \Psi^\varepsilon(0))$ we approximately solve the Riemann problem with boundary restricting to $\Omega^\varepsilon$ the juxtaposition of the solutions to (3.4) with $u^l = \bar{u}^\varepsilon(\Psi^\varepsilon(0)-)$ and $u^r$ given by the condition **(B)**, i.e.

$$u^r = (\mathcal{L}_m(\sigma_m) \circ \mathcal{L}_{m-1}(\sigma_{m-1}) \circ \cdots \circ \mathcal{L}_2(\sigma_2) \circ \mathcal{L}_1(\sigma_1))(\bar{u}^\varepsilon(\Psi^\varepsilon(0)-)),$$

with $(\sigma_1, \ldots, \sigma_m) = \sigma_b(\bar{u}^\varepsilon(\Psi^\varepsilon(0)-), h^\varepsilon(0))$.

Gluing the local approximate solutions above, we obtain a piecewise constant approximate solution of (1.2) on $\Omega^\varepsilon$ defined up to the first time $t_1$, at which one of the following interactions take place:

(I) two or more waves collide in the interior of $\Omega^\varepsilon$;
(II) one or more waves hits the boundary;
(III) the boundary condition $h$ changes.

In case (I), we extend the approximate solution beyond $t_1$ by solving the corresponding Riemann problem, while in cases (II) and (III) by solving the corresponding Riemann problem with boundary. Observe that in the case (II) no wave comes out from the boundary. We need to prove that this procedure gives an approximate solution $u^\varepsilon$ defined on all $\Omega^\varepsilon$. To this aim, we prove that the total variation of the approximate solution and the number of interaction points are bounded.

First we prove that the total variation of the approximate solution $u^\varepsilon$ is bounded for all $t$ uniformly in $\varepsilon$. Fix a positive time $\bar{t}$; then $u^\varepsilon$ at time $\bar{t}$ and the approximate boundary conditions $h^\varepsilon$ can be written as

$$u^\varepsilon = \sum_{a=1}^{N_\varepsilon} u_a \, \chi_{[x_a, x_{a-1}[} \quad \text{and} \quad h^\varepsilon = \sum_{a=1}^{N_\varepsilon} h_a \, \chi_{[t_{a-1}, t_a[}\,,$$

where $x_0 = \Psi^\varepsilon(\bar{t})$ and $t_0 = \bar{t}$. For $a = 1, \ldots, N_\varepsilon$, call $\sigma_{i,a}$ the total size of the $i$–waves in the Riemann problem between $u_a$ and $u_{a+1}$ at $x_a$ as defined by (3.3). Clearly, $\sigma_{i,0}$, $i \in \{1, \ldots, m\}$, is a wave starting from the boundary.

In the sequel we omit $\varepsilon$ to simplify the notation. Following [5], we introduce for later use the quantity $\tau_{i,a}$ as the signed length of the wave $\sigma_{i,a}$ measured in the space of the Riemann coordinates. More precisely, set $u^l = u_{a-1}$ and $u^r = u_a$, then $\tau_{i,a} = w^i - w^{i-1}$, where $w^{i-1}$ and $w^i$ are defined in (3.2). Introduce the following functionals

$$
\begin{aligned}
V &= \sum_{i=1}^{m}\sum_{a=1}^{N}\left|\tau_{i,a}\right| + K\sum_{i=m+1}^{n}\sum_{a=1}^{N}\left|\tau_{i,a}\right|, & V_h &= \sum_{a\geq 0}\left|h_{a+1}-h_a\right| \\
Q &= \sum_{i_1>i_2,\,a_1<a_2}\left|\tau_{i_1,a_1}\tau_{i_2,a_2}\right|, & \Upsilon &= V + K_h V_h + Q
\end{aligned}
$$

where $K$ and $K_h$ are suitable positive constants. $Q$ is essentially the Glimm interaction potential, see [6]. We omit the dependencies on $\boldsymbol{p}$, $\bar{t}$ and $\varepsilon$.

Proposition 3.1. – *Fix the total variation of the initial data. Then, if the constants $K, K_h$ are sufficiently large, along any approximate solution $u$, the map $t \mapsto (V + K_h V_h)(t)$ is non increasing.*

Proof. – The map $t \mapsto (V + K_h V_h)(t)$ can change is value only after an interaction. Thus, fix a time $\bar{t} > 0$ at which an interaction takes place, and let $\Delta V = V(\bar{t}+) - V(\bar{t}-)$ and $\Delta V_h = V_h(\bar{t}+) - V_h(\bar{t}-)$. Consider the cases (I), (II) and (III) separately:

(I) Clearly $\Delta V_h = 0$. By [5, Paragraph 2], we have that $\Delta V \leq 0$.

(II) Again $\Delta V_h = 0$. Furthermore $\Delta V < 0$ because no wave start from the boundary.

(III) By [2, Paragraph 6], $\sum_{i=1}^{m}\left|\tau_{i,0}^{+}\right| \leq \mathcal{L}ip\{w\}\left\|D_h\sigma_b\right\|_{\boldsymbol{C}^0}\left|h_+ - h_-\right|$ and thus $\Delta V + K_h \Delta V_h \leq 0$ for $K_h \geq \mathcal{L}ip\{w\}\left\|D_h\sigma_b\right\|_{\boldsymbol{C}^0}$.

The proof is complete. □

Proposition 3.2. – *The total number of interactions is finite.*

Proof. – To bound the number of interaction points, we prove that at any interaction either the number of waves decreases, or $\Upsilon$ diminishes at least by a fixed quantity, provided $K_h$ is sufficiently large. Consider the cases (I), (II) and (III) separately:

(I) The number of waves can increase only if the waves belong to different families, but in this case, by [5, Paragraph 2], $\Delta Q < -\delta_\varepsilon{}^2$.

(II) In this case, the number of waves decreases because all the waves hitting the boundary are absorbed by it.

(III) In this case $\Delta Q = \sum_{i=1}^{m}\sum_{k>i}\sum_{a\geq 1}\left|\tau_{i,0}^{+}\tau_{k,a}\right| \leq V\sum_{i=1}^{m}\left|\tau_{i,0}^{+}\right|$. Therefore, by the proof of Proposition 3.1,

$$
\Delta\Upsilon \leq \left(\mathcal{L}ip(w)\left\|D_h\sigma_b\right\|_{\boldsymbol{C}^0}(1+V) - K_h\right)\left|h_+ - h_-\right| < -\delta_\varepsilon
$$

for $K_h > \mathcal{L}ip(w)(\left\|D_h\sigma_b\right\|_{\boldsymbol{C}^0})(1+V) + 1$.

The proof is complete. □

By the same argument used in [5, 9], the above algorithm yields a semigroup $S^\varepsilon\colon [0,+\infty[\,\times\mathcal{D}_{0,M}(\mathcal{G}^\varepsilon)\mapsto\mathcal{D}_{0,M}(\mathcal{G}^\varepsilon)$, for every $\varepsilon>0$, whose orbits approximately solve (1.2). Recall the definition (3.5) and introduce the three canonical projections $\pi_i$, $i=1,2,3$, defined in $\mathcal{D}_{0,M}$. With a slight modification of the construction in [5], one can prove the following proposition, where we write the $x$-jump of a function $h\colon[0,+\infty[\,\mapsto\mathbb{R}$ at $(t,x_a)$ as $\Delta h_a(t)=h(t,x_a+)-h(t,x_a-)$.

Proposition 3.3. – *Under* **(F)***, for any* $M,\varepsilon>0$ *and for any* $\varepsilon$*-grid* $\mathcal{G}^\varepsilon$*, the system* (1.2) *generates an operator*

$$\begin{array}{rcccc} S^\varepsilon & : & [0,+\infty[\,\times\mathcal{D}_{0,M}(\mathcal{G}^\varepsilon) & \mapsto & \mathcal{D}_{0,M}(\mathcal{G}^\varepsilon) \\ & & (t,\boldsymbol{p}) & \mapsto & S^\varepsilon_t\boldsymbol{p} \end{array}$$

*such that the map* $t\mapsto\pi_1\circ S^\varepsilon_t\boldsymbol{p}$ *is a weak solution to* (1.2) *with data* $\boldsymbol{p}$*, for all* $\boldsymbol{p}\in\mathcal{D}_{0,M}(\mathcal{G}^\varepsilon)$*. Moreover* $S^\varepsilon$ *has the following properties.*

1. $S^\varepsilon$ *is a semigroup, i.e.* $S^\varepsilon_0=\mathrm{Id}$ *and* $S^\varepsilon_{t_1}\circ S^\varepsilon_{t_2}=S^\varepsilon_{t_1+t_2}$.
2. *The map* $(t,x)\mapsto\pi_1\circ S^\varepsilon_t\boldsymbol{p}(x)$ *is piecewise constant with discontinuities along finitely many polygonal lines and with finitely many interaction points.*
3. *For all* $\boldsymbol{p}\in\mathcal{D}_{0,M}(\mathcal{G}^\varepsilon)$*, both maps* $t\mapsto\left\|S^\varepsilon_t\boldsymbol{p}\right\|_\infty$ *and* $t\mapsto\mathrm{TV}(S^\varepsilon_t\boldsymbol{p})$ *are non increasing.*
4. *Let* $\eta$ *be any convex entropy for* (1.2) *with entropy flux* $q$*. For any* $M>0$ *there exists a positive constant* $C$ *independent from* $\varepsilon$ *such that for all* $\boldsymbol{p}\in\mathcal{D}_{0,M}(\mathcal{G}^\varepsilon)$ *and* $t\in[0,+\infty[$

$$\sum_a\left(\dot{x}_a\cdot\Delta\eta^\varepsilon_a-\Delta q^\varepsilon_a\right)\geq-C\cdot\varepsilon\,, \tag{3.7}$$

*where* $x=x_a(t)$ *is the support of the* $a$*-th discontinuity in* $\pi_1\circ S^\varepsilon_t\boldsymbol{p}$*; here* $\eta^\varepsilon=\eta\big(\pi_1\circ S^\varepsilon_t\boldsymbol{p}\big)$*,* $q^\varepsilon=q\big(\pi_1\circ S^\varepsilon_t\boldsymbol{p}\big)$*.*

5. *The second and third components of* $S^\varepsilon_t$ *are the right* $t$ *translations:* $\pi_2(S^\varepsilon_t\boldsymbol{p})=\mathcal{T}_t(\pi_2\boldsymbol{p})$ *and* $\pi_3(S^\varepsilon_t\boldsymbol{p})=\mathcal{T}_t(\pi_3\boldsymbol{p})$*.*

We now prove the uniform in $\varepsilon$ Lipschitz continuous dependence of the approximate solutions on the initial data and in the boundary condition by means of the now classical technique based on *pseudopolygonals*, see [2, 4, 5, 7, 8, 9, 10, 11, 12]. The underlying idea is that of shifting the location of each jump in the initial data and boundary condition at constant rates.

Definition 3.4. – *Let* $a<b$*. An elementary path in* $\boldsymbol{PC}$ *is a map*

$$\begin{array}{rccc} \gamma\colon & ]a,b[ & \mapsto & \boldsymbol{PC} \\ & \theta & \mapsto & \displaystyle\sum_{a=1}^{N(\theta)} u_a\chi_{[x_a(\theta),x_{a-1}(\theta)[} \end{array} \quad \textit{with} \quad \begin{array}{l} x_a(\theta)=\bar{x}_a+\theta\xi_a \\ x_a(\theta)<x_{a-1}(\theta)\,,\ \theta\in\,]a,b[\,. \end{array}$$

*Fix* $T>0$ *and assume that* $\Psi',\Psi''\in\boldsymbol{PLC}$ *do not coincide on* $[0,T]$*. The ele-*

*mentary path in* ***PLC*** *joining* $\Psi'$ *and* $\Psi''$ *on* $[0,T]$ *is the curve*

$$\gamma(\theta)(t) = \begin{cases} \Psi'(t) + [\![\, [\![\, \Psi''(t) - \Psi'(t) ]\!]_+ + \theta \,]\!]_+ & \theta < 0 \\ \Psi''(t) + [\![\, [\![\, \Psi'(t) - \Psi''(t) \,]\!]_+ - \theta \,]\!]_+ & \theta > 0, \end{cases}$$

*defined for* $|\theta| \le \|\Psi' - \Psi''\|_{C^0([0,T])}$, *where* $[\![\, x \,]\!]_+ = \max\{x, 0\}$. *If* $\Psi' = \Psi''$, *the map* $\gamma$ *defined by* $\gamma(\theta) = \Psi'$ *for all* $\theta$ *is also an elementary path in* ***PLC***. *An elementary path in* $\mathcal{D}_{0,M}(\mathcal{G}^\varepsilon)$ *is a map* $\gamma\colon ]a,b[ \mapsto \mathcal{D}_{0,M}(\mathcal{G}^\varepsilon)$ *such that* $\pi_i \circ \gamma$ *is a* ***PC****-elementary path for* $i = 1, 2$, *and a* ***PLC****-elementary path for* $i = 3$. *A continuous map* $\gamma\colon [a,b] \mapsto \mathcal{D}_{0,M}(\mathcal{G}^\varepsilon)$ *is a pseudopolygonal in* $\mathcal{D}_{0,M}(\mathcal{G}^\varepsilon)$ *if there exist countably many disjoint open intervals* $J_k \subseteq ]a,b[$ *such that* $]a,b[ \setminus \bigcup_k J_k$ *is countable and the restriction of* $\gamma$ *to each* $J_k$ *is an elementary path in* $\mathcal{D}_{0,M}(\mathcal{G}^\varepsilon)$.

Exactly as [2, Proposition 3], any two triples in $\mathcal{D}_{0,M}(\mathcal{G}^\varepsilon)$ can be joined by a pseudopolygonal contained in $\mathcal{D}_{0,M}(\mathcal{G}^\varepsilon)$. Furthermore, $S^\varepsilon$ preserves pseudopolygonals: if $\gamma$ is a pseudopolygonal, then so is $S^\varepsilon_t \circ \gamma$, for all $t \ge 0$.

Consider a pseudopolygonal $\gamma$ joining two triples in $\mathcal{D}_{0,M}(\mathcal{G}^\varepsilon)$. Introduce the shift speed of the boundary

$$\kappa(\gamma) = \begin{cases} 0 & \text{if } \theta \mapsto (\pi_3 \circ \gamma)(\theta) \text{ is constant} \\ 1 & \text{otherwise.} \end{cases} \tag{3.8}$$

Define the *generalized shift speeds*

$$\eta_{i,a} = \max\{\kappa, |\xi_{i,a}|\}, \quad \eta_{i,0} = \kappa, \quad \tilde\eta_a = \kappa + \left|\tilde\xi_a\right| \inf_{u \in \mathcal{U}} \lambda_{m+1}(u) \tag{3.9}$$

where $\xi_{i,a}$ is the horizontal shift speed of the $i$-th wave $\sigma_{i,a}$ at $x_a$ and $\tilde\xi_a$ is the vertical shift speed of the jump at $t_a$ in the boundary condition.

Along a pseudopolygonal, through

$$\bar\Upsilon_\eta(\gamma) = \sum_{i,a} |\sigma_{i,a}| \eta_{i,a} W_{i,a} \quad \text{and} \quad \tilde\Upsilon_\eta(\gamma) = \sum_{i,\tilde a} \|h_{\tilde a} - h_{\tilde a - 1}\| \tilde\eta_{\tilde a} \tilde W_{i,\tilde a}$$

define the functionals

$$\Upsilon_\eta(\gamma) = \bar\Upsilon_\eta(\gamma) + \tilde\Upsilon_\eta(\gamma) \tag{3.10}$$

$$\Xi_\varepsilon(\gamma) = \int_a^b \Upsilon_\eta(\gamma(\theta))\, d\theta$$

$$\|\gamma\|_\varepsilon = \int_a^b \left( \Upsilon_\eta(\gamma(\theta)) + \kappa(\gamma(\theta)) \right) d\theta\,, \tag{3.11}$$

$W_{i,a}$, $\tilde{W}_{i,\tilde{a}} \geq 1$ being weights bounded uniformly in $\varepsilon$, see (3.15). Call

$$\ell_X(\gamma) = \sup\left\{\sum_{j=1}^{N} d(\gamma(\theta_j), \gamma(\theta_{j-1})) : N \in \mathbb{N}, a = \theta_0 < \theta_1 < \ldots < \theta_N = b\right\}$$

the length of the curve $\gamma$ with respect to the distance $d$ in the metric space $X$. For instance, in $\mathcal{D}$, we consider the metric

$$d(\boldsymbol{p}', \boldsymbol{p}'') = \|\bar{u}'' - \bar{u}'\|_{\boldsymbol{L}^1} + \|h'' - h'\|_{\boldsymbol{L}^1} + \|\Psi'' - \Psi'\|_{\boldsymbol{C}^0}. \tag{3.12}$$

Referring to the choice (2.5) of the norms, we denote

$$\mathrm{TV}\big(\boldsymbol{p}_{|[0,T]}\big) = \mathrm{TV}(\bar{u}) + \|b(\bar{u}(\Psi(0+))) - h(0+)\| + \mathrm{TV}\big(h_{|[0,T]}\big).$$

LEMMA 3.5. – *Fix a positive $M$. Then, there exists a positive constant $C$ such that for all $\boldsymbol{p}_1, \boldsymbol{p}_2 \in \mathcal{D}_{0,M}$ with $\pi_3(\boldsymbol{p}_i)$ having Lipschitz constant $L_i$, for all pseudopolygonal $\gamma: [a,b] \mapsto \mathcal{D}_{0,M}$ joining $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ and for all small $\varepsilon$, setting $\gamma_i = \pi_i \circ \gamma$, the following estimates hold:*

$$\begin{aligned}
\|\gamma\|_\varepsilon &\geq C^{-1}\ell_{\mathcal{D}}(\gamma)\\
\|\gamma\|_\varepsilon &\leq C(\ell_{\boldsymbol{L}^1}(\gamma_1) + \ell_{\boldsymbol{L}^1}(\gamma_2) + (1 + \mathrm{TV}(\boldsymbol{p}_{1|[0,T]}) + \mathrm{TV}(\boldsymbol{p}_{2|[0,T]}))\ell_{\boldsymbol{C}^0}(\gamma_3))\\
\Xi_\varepsilon(\gamma) &\geq C^{-1}(\ell_{\boldsymbol{L}^1}(\gamma_1) + \ell_{\boldsymbol{L}^1}(\gamma_2))\\
\Xi_\varepsilon(\gamma) &\leq C(\ell_{\boldsymbol{L}^1}(\gamma_1) + \ell_{\boldsymbol{L}^1}(\gamma_2) + (\mathrm{TV}(\boldsymbol{p}_{1|[0,T]}) + \mathrm{TV}(\boldsymbol{p}_{2|[0,T]}))\ell_{\boldsymbol{C}^0}(\gamma_3))
\end{aligned}$$

PROOF. – Let $\gamma = \gamma^1 \oplus \gamma^2 \oplus \ldots$ be a pseudopolygonal, where $\gamma^k = \gamma_{|J_k}$, $k \in \mathbb{N}$, are elementary paths. Then

$$\begin{aligned}
\ell_{\boldsymbol{L}^1}(\gamma_1^k) &= \sup\left\{\sum_{a,j} \left\|u_a^k - u_{a-1}^k\right\|(\theta_j - \theta_{j-1})\left|\xi_a^k\right|\right\}\\
&= \sup\left\{\sum_{i,a,j} \left|\sigma_{i,a}^k \xi_{i,a}^k\right|(\theta_j - \theta_{j-1})\right\}\\
\ell_{\boldsymbol{L}^1}(\gamma_2^k) &= \sup\left\{\sum_{a,j} \left\|h_a^k - h_{a-1}^k\right\|(\theta_j - \theta_{j-1})\left|\tilde{\xi}_a^k\right|\right\}\\
\ell_{\boldsymbol{C}^0}(\gamma_3^k) &\leq \sup\left\{\sum_{j} \kappa(\gamma^k)(\theta_j - \theta_{j-1})\right\}.
\end{aligned}$$

By (3.9), $\left|\xi_{i,a}\right| \leq \eta_{i,a} \leq \left|\xi_{i,a}\right| + \kappa(\gamma)$, and $\frac{1}{C}\left|\tilde{\xi}_{i,a}\right| \leq \tilde{\eta}_a \leq C\Big(\left|\tilde{\xi}_{i,a}\right| + \kappa(\gamma)\Big)$. Therefore, by (3.12),

$$\ell_{\mathcal{D}}(\gamma) \leq \int_a^b \left( \sum_{i,a} \left( \left|\sigma_{i,a}\xi_{i,a}\right| + \|h_a - h_{a-1}\| \left|\tilde{\xi}_a\right| \right) + \kappa(\gamma) \right) d\theta \leq C\|\gamma\|_\varepsilon$$

$$\|\gamma\|_\varepsilon \leq C \left( \ell_{\boldsymbol{L}^1}(\gamma_1) + \ell_{\boldsymbol{L}^1}(\gamma_2) + \int_a^b \left( 1 + \sum_{i,a} \left|\sigma_{i,a}\right| + \|h_a - h_{a-1}\| \right) \kappa(\gamma) d\theta \right)$$

$$\leq C \Big( \ell_{\boldsymbol{L}^1}(\gamma_1) + \ell_{\boldsymbol{L}^1}(\gamma_2) + \Big( 1 + \mathrm{TV}(\boldsymbol{p}_{1|[0,T]}) + \mathrm{TV}(\boldsymbol{p}_{2|[0,T]}) \Big) \ell_{\boldsymbol{C}^0}(\gamma_3) \Big).$$

The last two inequalities can be proved similarly. □

It immediately follows that the metric on $\mathcal{D}_{0,M}$ defined by

$$d^\varepsilon_\eta(\boldsymbol{p}_1, \boldsymbol{p}_2) = \inf \left\{ \|\gamma\|_\varepsilon \colon \gamma \text{ pseudopolygonal joining } \boldsymbol{p}_1 \text{ to } \boldsymbol{p}_2 \right\}$$

is equivalent to the distance (3.12), see also [2, 4, 5, 6, 10].

Due to the possible *"movement"* of the boundary, below it is necessary to consider one more type of interaction, namely the points where

(IV) the boundary stops shifting, i.e. where $\kappa$ passes from 1 to 0.

The following interaction estimates

$$\text{(3.13)} \qquad \begin{array}{rrcl} \text{(I):} & \left|\sum_{a>0} \sigma^+_{i,a}\right| & \leq & \left(1 + K \sum_{k \neq i} \left|\sum_{a>0} \tau^-_{k,a}\right|\right) \left|\sum_{a>0} \sigma^-_{i,a}\right| \\ \text{(III):} & \left|\sigma^+_{i,0}\right| & \leq & K\|h_+ - h_-\| \end{array}$$

hold for a suitable positive constant $K > 1$. The former estimate comes from [5, Formula (5.7)], while the latter holds because $\sigma_b$ is Lipschitz.

Proposition 3.6. – *Consider a point $P_* = (t_*, x_*)$ of interaction. Let $u(t,x)$ be the approximate solution to* (1.2) *defined for $t < t_*$ by extending backward the shocks and for $t \geq t_*$ by solving the approximate Riemann problem. Then*

$$\text{(3.14)} \qquad \begin{aligned} \text{(I)}: \quad & \sum_{a>0} \left|\sigma^+_{i,a}\eta^+_{i,a}\right| \leq \left(1 + K \sum_{k \neq i} \left|\sum_{a>0} \tau^-_{k,a}\right|\right)^2 \sum_{a>0} \left|\sigma^-_{i,a}\eta^-_{i,a}\right| \\ & \qquad + K \left|\sum_a \tau^-_{i,a}\right| \sum_{k \neq i} \sum_{a>0} \left|\sigma^-_{k,a}\eta^-_{k,a}\right| \\ \text{(III)}: \quad & \left|\sigma^+_{i,0}\eta^+_{i,0}\right| \leq K\tilde{\eta}_{\tilde{a}}\|h_+ - h_-\| \ . \end{aligned}$$

PROOF. – We consider the various cases separately.

(I) If $\eta^+_{i,a} = \left|\xi^+_{i,a}\right|$, then (3.14) follows from [5, Formula (5.8)]. If $\eta^+_{i,a} = \kappa$ we assume $\eta^-_{i,a} = \kappa$, since in the case $\eta^-_{i,a} > \kappa$ the right hand side in (3.14) (I) becomes greater. Now, (3.14) (I) follows from [5, Formula (5.8)] setting for all $i, a$, $\xi^-_{i,a} = 1$, which implies $\xi^+_{i,a} = 1$.

(III) In this case, $\sigma^+_{i,a} = 0$ for any $a > 0$. By (3.13) and (3.9)

$$\begin{aligned} \left|\sigma^+_{i,0}\eta^+_{i,0}\right| &\leq K|h_+ - h_-|\max\left\{\kappa, \left|\xi^+_{i,0}\right|\right\} \\ &\leq K|h_+ - h_-|\left(\kappa + \left|\tilde{\xi}_a\right|\left|\inf_{u\in\mathcal{U}}\lambda_1(u) - \inf_{u\in\mathcal{U}}\lambda_{m+1}(u)\right|\right) \\ &\leq K\,\tilde{\eta}_{\tilde{a}}\,|h_+ - h_-|\,. \end{aligned}$$ □

Recall that $h$ and $\bar{u}$ have bounded support, hence there exists a time $T^\varepsilon$ such that no interaction takes place for $t > T^\varepsilon$, see [22].

Following [5], assign weight 1 at all waves in $u(T^\varepsilon, \cdot)$. Next consider a point $P_*$ of interaction and suppose that the weights $W^+_{i,a}$ of the waves exiting the interaction are already assigned. The incoming waves are weighted as follows. If no $i$-wave exits the interaction, each $i$-wave that enters the interaction is assigned weight $W^-_{i,a} = 1$. In the other cases let

$$(3.15)\qquad \begin{aligned} (\text{ I })\quad W^-_{i,a} &= \left(1 + K\sum_{k\neq i}\left|\sum_{a>0}\tau^-_{k,a}\right|\right)^2\max_{a>0}W^+_{i,a} \\ &\quad + K\sum_{k\neq i}\left(\left|\sum_{a>0}\tau^-_{k,a}\right|\max_{a>0}W^+_{k,a}\right) \\ (\text{ III })\quad \tilde{W}_{i,\tilde{a}} &= KW^+_{i,0}\,. \end{aligned}$$

In case (II), $W^-_{i,a} = 1$ because no wave exits the interaction, and, in case (IV), it is not necessary to define weights because there is no interaction.

PROPOSITION 3.7. – *Fix an elementary path $\gamma$. Let an interaction take place at $P_*$. Let $\Upsilon_\eta(t) = \Upsilon_\eta(S^\varepsilon_t \circ \gamma)$, where $\Upsilon_\eta$ is defined in* (3.10), *and $\kappa(t) = \kappa(S^\varepsilon_t \circ \gamma)$, $\kappa$ being defined in* (3.8). *Then in any of the cases (I), (II), (III)*

$$\Upsilon_\eta(t_* +) \leq \Upsilon_\eta(t_* -) \quad \text{and} \quad \Upsilon_\eta(t_* +) + \kappa(t_* +) \leq \Upsilon_\eta(t_* -) + \kappa(t_* -)\,.$$

PROOF. – Since $\kappa$ can only decreases passing from 1 to 0, it is sufficient to show that $\Delta\Upsilon_\eta \leq 0$ in all cases.

(I) In this case $\Delta\tilde{\Upsilon}^\varepsilon_\eta = 0$ and $\kappa$ remains constant. Moreover $\Delta\bar{\Upsilon}_\eta \leq 0$. Indeed, as proved in [5, Paragraph 6] and [10, Proposition 3.6], by (I) in (3.15), it holds that, with obvious notation, $\sum_a |\sigma^+_{i,a}\eta^+_{i,a}|W^+_{i,a} \leq \sum_a |\sigma^-_{i,a}\eta^-_{i,a}|W^-_{i,a}$.

(II) As before, $\Delta\tilde{\Upsilon}_\eta = 0$ and $\kappa$ remains constant. Furthermore $\Delta\bar{\Upsilon}_\eta \leq 0$.

(III) In this case $\Delta\Upsilon_\eta \leq 0$ because for (3.14) (III) and (3.15) (III) we have

$$\left|\sigma_{i,0}^{+}\eta_{i,0}^{+}\right| W_{i,0}^{+} \leq K\tilde{\eta}_{\tilde{a}} \|h_+ - h_-\| W_{i,0}^{+} = \tilde{\eta}_{\tilde{a}} \|h_+ - h_-\| W_{i,\tilde{a}}\,. \qquad \square$$

As a consequence of Proposition 3.7, the length of $S^\varepsilon \circ \gamma$ computed as in (3.11) is non increasing as a function of time.

### 3.2 – *The Source Term.*

We approximate $g$ as

$$g^\varepsilon(t,x,u) = \sum_{k\in\mathbb{Z}} \frac{1}{\varepsilon}\left(\int_{(k-1)\varepsilon}^{k\varepsilon} g(t,\xi,u)\,d\xi\right)\cdot \chi_{](k-1)\varepsilon,k\varepsilon]}(x) \tag{3.16}$$

and consider the approximate problem

$$\begin{cases} \partial_t u &= g^\varepsilon(t,x,u) & (t,x) &\in & \Omega \\ u(0,x) &= \bar{u}^\varepsilon(x) & x &\leq & \Psi^\varepsilon(t_o) \\ b(u(t,\Psi^\varepsilon(t))) &= h^\varepsilon(t) & t &\geq & t_o \end{cases} \tag{3.17}$$

where $(\bar{u}^\varepsilon, h^\varepsilon, \Psi^\varepsilon)$ are as in the previous paragraph. In [9, Lemma 4.3] the following lemma is proved.

LEMMA 3.8. – *Let $g$ be as in* (**G**). *Then $g^\varepsilon$ satisfies* (**G**) *with* $(G_3)$ *modified as follows: if $l,k \in \mathbb{Z}$ and $l \leq k$, for all $x_1 \in\, ]l\varepsilon,(l+1)\varepsilon]$ and $x_2 \in\, ]k\varepsilon,(k+1)\varepsilon]$ we have*

$$|g^\varepsilon(t,x_2,u) - g^\varepsilon(t,x_1,u)| \leq 3\mu([l\varepsilon,(k+1)\varepsilon]). \tag{3.18}$$

The following lemma can be proved as [12, Lemma 3.7].

LEMMA 3.9. – *The differential equation* (3.17) *generates the map*

$$\begin{array}{rccccl} \Sigma^\varepsilon\colon & \mathcal{I} & \times & \mathcal{D}_{t_o} & \mapsto & \boldsymbol{L}^1 \cap \boldsymbol{BV}(\mathbb{R},\mathcal{U}) \\ & (t_o,t) & , & \boldsymbol{p} & \mapsto & \Sigma^\varepsilon_{t_o,t}\boldsymbol{p} \end{array} \tag{3.19}$$

*in the sense that for all $(\bar{u}^\varepsilon, h^\varepsilon, \Psi^\varepsilon) \in \mathcal{D}_{t_o}$, the map $t \mapsto \Sigma^\varepsilon_{t_o,t}(\bar{u}^\varepsilon, h^\varepsilon, \Psi^\varepsilon)$ is the solution to* (3.17). *For all $R > 0$ and $T > t_o$, there exist a positive $\tilde{l} \in \boldsymbol{L}^1_{loc}([t_o,+\infty[)$ and constants $C, \hat{M} > 0$, both independent from $\varepsilon$, such that for all $t \in [t_o,T]$ and*

$\boldsymbol{p} = (\bar{u}, h, \Psi) \in \mathcal{D}_{t_o}$ *with* $\mathrm{TV}(\boldsymbol{p}_{|[t_o,T]}) \leq R$,

$$\left\| \Sigma^{\varepsilon}_{t_o,t} \boldsymbol{p} \right\|_{\boldsymbol{L}^\infty} \leq e^{\int_{t_o}^{t} \tilde{l}(\tau)\,d\tau} \cdot \|\bar{u}\|_{\boldsymbol{L}^\infty} + \sup{}_{\tau \in [t_o,t]} e^{\int_{\tau}^{t} \tilde{l}(s)\,ds} \cdot \|h(\tau)\| \tag{3.20}$$

$$\mathrm{spt}\left(\Sigma^{\varepsilon}_{t_o,t} \boldsymbol{p}\right) \subseteq \mathrm{spt}(\bar{u}) \cup \Psi(\mathrm{spt}(h) \cap [t_o, t]) \tag{3.21}$$

$$\begin{aligned} \mathrm{TV}\left(\Sigma^{\varepsilon}_{t_o,t} \boldsymbol{p}\right) \leq & e^{C(t-t_o)} \cdot (1 + C(t - t_o)) \cdot \mathrm{TV}\left(\boldsymbol{p}_{|[t_o,t]}\right) \\ & + e^{C(t-t_o)} \cdot 9L_w \cdot n\mu(\mathbb{R}) \cdot (t - t_o)\,. \end{aligned} \tag{3.22}$$

*Finally, there exists an* $\varepsilon$*–grid* $\bar{\mathcal{G}}^{\varepsilon}$ *such that*

$$(\bar{u}, h, \Psi) \in \mathcal{D}_{t_o,M}(\mathcal{G}^{\varepsilon}) \Rightarrow \left(\Sigma^{\varepsilon}_{t_o,t}(\bar{u}, h, \Psi), \mathcal{T}_{t-t_o} h, \mathcal{T}_{t-t_o} \Psi\right) \in \mathcal{D}_{t_o,\hat{M}}(\bar{\mathcal{G}}^{\varepsilon}). \tag{3.23}$$

Above, $\mathrm{spt}(u)$ denotes the support of the function $u$.

### 3.3 – *Operator Splitting.*

An approximate solution to (1.1) is constructed through the following operator splitting scheme. Fix positive $\varepsilon, M$ and an $\varepsilon$–grid $\mathcal{G}^{\varepsilon}$. Let $\boldsymbol{p} = (\bar{u}, h, \Psi) \in \mathcal{D}_{0,M}(\mathcal{G}^{\varepsilon})$. Let $l > k$ be in $\mathbb{N}$ and for $t_o \in [k\varepsilon, (k+1)\varepsilon[$ define recursively

$$F^{\varepsilon}_{t_o,t} \boldsymbol{p} = \begin{cases} S^{\varepsilon}_{t-t_o} \boldsymbol{p} & \text{if } t \in [t_o, (k+1)\varepsilon[, \\ \left(\Sigma^{\varepsilon}_{t_o,t}\left(S^{\varepsilon}_{t-t_o} \boldsymbol{p}\right), \mathcal{T}_{t-t_o} h, \mathcal{T}_{t-t_o} \Psi\right) & \text{if } t = (k+1)\varepsilon, \\ S^{\varepsilon}_{t-l\varepsilon} \left( \bigcirc_{i=k+1}^{l-1} F^{\varepsilon}_{i\varepsilon,(i+1)\varepsilon} \right) F^{\varepsilon}_{t_o,(k+1)\varepsilon} u & \text{if } t \in [l\varepsilon, (l+1)\varepsilon[. \end{cases} \tag{3.24}$$

Concerning the grid, refine it recursively. Indeed start with an initial datum $\boldsymbol{p} \in \mathcal{D}(\mathcal{G}^{\varepsilon})$ assigned at time $t_o$. For $t \in [t_o, (k+1)\varepsilon[$, $F^{\varepsilon}_{t_o,t} \boldsymbol{p}$ attains values in the same grid $\mathcal{G}^{\varepsilon}$. At time $(k+1)\varepsilon$ we apply the o.d.e. solver $\Sigma^{\varepsilon}_{t_o,(k+1)\varepsilon}$ and at the same time pass to another $\varepsilon$–grid $\mathcal{G}^{\varepsilon}_1 = \bar{\mathcal{G}}^{\varepsilon}$, according to (3.23).

Recursively, if $F^{\varepsilon}_{t_o,l\varepsilon} \boldsymbol{p}$ attains values in $\mathcal{G}^{\varepsilon}_m$, then $F^{\varepsilon}_{t_o,t} \boldsymbol{p}$ is valued in the same grid for all $t \in [l\varepsilon, (l+1)\varepsilon[$. Applying $\Sigma^{\varepsilon}_{l\varepsilon,(l+1)\varepsilon}$ we pass to another $\varepsilon$–grid $\mathcal{G}^{\varepsilon}_{m+1} = \bar{\mathcal{G}}^{\varepsilon}_m$, see [9, Paragraph 4].

LEMMA 3.10. – *Let* $T > t_o$*. The operator* $F^{\varepsilon} \colon \mathcal{I} \times \mathcal{D}_{t_o} \mapsto \mathcal{D}_{t_o}$ *is well defined and can be written as* $F^{\varepsilon}_{t_o,t}(\bar{u}, h, \Psi) = (u^{\varepsilon}(t), \mathcal{T}_{t-t_o} h, \mathcal{T}_{t-t_o} \Psi)$*. Moreover, the total number of discontinuities is finite on any strip* $[t_o, T] \times \mathbb{R}$*.*

PROOF. – For (3.24) to be well defined, it is necessary to check that all compositions are possible: by Lemma 3.9, if $\boldsymbol{p} \in \mathcal{D}_{t_o}$, then $S_t^\varepsilon \boldsymbol{p}$ is in $\mathcal{D}_{t_o}$ as well as $\Sigma^\varepsilon_{k\varepsilon,(k+1)\varepsilon}\boldsymbol{p}$. The use of a discrete grid at each convective step ensures that the number of interactions is finite over all $[0,T]$, see [9, Lemma 4.4]. □

LEMMA 3.11. – *For all $R > 0$ and $T > t_o$, there exist positive $\tilde{l} \in \boldsymbol{L}^1([t_o,T])$ and a constant $C$, both independent from $\varepsilon$, such that for $t \in [t_o,T]$ and for $\boldsymbol{p} = (\bar{u}, h, \Psi) \in \mathcal{D}_{t_o}$ with $\|h\|_{\boldsymbol{L}^\infty} + \|\bar{u}\|_{\boldsymbol{L}^\infty} \le R$, the function $u$ defined by $(u(t), \mathcal{T}_{t-t_o}h, \mathcal{T}_{t-t_o}\Psi) = F^\varepsilon_{t_o,t}\boldsymbol{p}$ satisfies*

$$\|u(t)\|_{\boldsymbol{L}^\infty} \le e^{\int_{t_o}^t \tilde{l}(\tau)d\tau} \cdot \left(\|h\|_{\boldsymbol{L}^\infty} + \|\bar{u}\|_{\boldsymbol{L}^\infty}\right), \tag{3.25}$$

$$\begin{aligned} \mathrm{TV}(u(t)) \le\, & e^{C(t-t_o)} \cdot (1 + C(t-t_o)) \cdot \mathrm{TV}\big(\boldsymbol{p}_{|[t_o,t]}\big) \\ & + e^{C(t-t_o)} \cdot 9L_w n \cdot \mu(\mathbb{R}) \cdot (t-t_o)\,. \end{aligned} \tag{3.26}$$

PROOF. – The first estimate follows from Proposition 3.1 and (3.20). Similarly, to prove (3.26) we use Proposition 3.1 and (3.22). □

In particular, the previous lemma provides an upper bound of the total variation of the approximate solution uniform in $\varepsilon$. By Helly Compactness Theorem, the above lemmas yield an existence result to (1.1). We now proceed towards an estimate of the Lipschitz constant for $F^\varepsilon$ uniform in $\varepsilon$.

LEMMA 3.12. – *Fix $M > 0$, $N \in \mathbb{N}$ and $T = t_o + N\varepsilon$. Let $\boldsymbol{p}_1, \boldsymbol{p}_2$ in $\mathcal{D}_{t_o,M}(\mathcal{G}^\varepsilon)$ with* $\max\{\mathrm{TV}(\boldsymbol{p}_{1|[t_o,T]}), \mathrm{TV}(\boldsymbol{p}_{2|[t_o,T]})\} \le R$ *and a pseudopolygonal $\gamma$ joining $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$. Then, for all $t \in [t_o,T]$, there exist weights uniformly bounded from above by a quantity dependent from $M, T$ but not $\varepsilon$, such that for $t \in [t_o,T]$*

$$\left\|F^\varepsilon_{t_o,t} \circ \gamma\right\|_\varepsilon \le e^{\int_{t_o}^t \tilde{l}(\tau)\,d\tau} \cdot \|\gamma\|_\varepsilon\,, \qquad \Xi_\varepsilon\Big(F^\varepsilon_{t_o,t} \circ \gamma\Big) \le e^{\int_{t_o}^t \tilde{l}(\tau)\,d\tau} \cdot \Xi_\varepsilon(\gamma)\,.$$

Thanks to the construction above, this proof is entirely similar to that of [9, Lemma 4.7].

PROOF OF THEOREM 2.6. – Let $\varepsilon_\upsilon = 2^{-\upsilon}$ for $\upsilon \in \mathbb{N}$. For any data construct a sequence of approximate solutions by means of (3.24). A standard argument, see [6, 7, 11, 13], shows that this is a Cauchy sequence in $\boldsymbol{L}^1$ and that it converges to a weak entropic solution of (1.1), proving points 1., 2. and 3.

Consider now point 4., with $\boldsymbol{p} = (\bar{u}, h, \Psi)$, $\boldsymbol{p}' = (\bar{u}', h', \Psi')$ and $t = t'$:

$$\begin{aligned} & \|u(t) - u'(t)\|_{\boldsymbol{L}^1} \\ \le\, & d\big(F_{t_o,t}\boldsymbol{p}, F_{t_o,t}\boldsymbol{p}'\big) \\ \le\, & C \lim_{\upsilon \to +\infty} d_{\varepsilon_\upsilon}\Big(F_{t_o,t}\boldsymbol{p}_{\varepsilon_\upsilon}, F_{t_o,t}\boldsymbol{p}'_{\varepsilon_\upsilon}\Big) \end{aligned}$$

$$\begin{aligned}
&\leq C e^{\int_{t_o}^{t} \tilde{l}(\tau)\,d\tau} \lim_{\upsilon \to +\infty} d_{\varepsilon_\upsilon}(\boldsymbol{p}_{\varepsilon_\upsilon}, \boldsymbol{p}'_{\varepsilon_\upsilon}) \\
&\leq C e^{\int_{t_o}^{t} \tilde{l}(\tau)\,d\tau} d(\boldsymbol{p}, \boldsymbol{p}') \\
&\leq C \cdot \left(1 + \hat{\lambda}\right) e^{\int_{t_o}^{t} \tilde{l}(\tau)\,d\tau} \left(\|\bar{u} - \bar{u}'\|_{\boldsymbol{L}^1} + \|\Psi - \Psi'\|_{\boldsymbol{C}^0} + \|h - h'\|_{\boldsymbol{L}^1}\right).
\end{aligned}$$

The case $t \neq t'$ follows by standard argumets, see for instance [2, 7, 12].

Finally, point 5. follows from Lemma 3.11. □

## REFERENCES


[1] F. ANCONA - G. M. COCLITE, *On the attainable set for Temple class systems with boundary controls*, SIAM J. Control Optim, **43** (6) (electronic, 2005), 2166-2190.

[2] D. AMADORI - R. M. COLOMBO, *Continuous dependence for* $2 \times 2$ *conservation laws with boundary*, J. Differential Equations, 138 (2) (1997), 229-266.

[3] D. AMADORI - G. GUERRA, *Global weak solutions for systems of balance laws*, Appl. Math. Lett., **12** (6) (1999), 123-127.

[4] P. BAITI - A. BRESSAN, *The semigroup generated by a Temple class system with large data*, Differential Integral Equations, **10** (3) (1997), 401-418.

[5] S. BIANCHINI, *The semigroup generated by a Temple class system with non-convex flux function*, Diff. Integral Equations, **13** (10-12) (2000), 1529-1550.

[6] A. BRESSAN, *Hyperbolic systems of conservation laws*, volume 20 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2000. *The one-dimensional Cauchy problem.*

[7] A. BRESSAN - R. M. COLOMBO, *The semigroup generated by* $2 \times 2$ *conservation laws*, Arch. Rational Mech. Anal., **133** (1) (1995), 1-75.

[8] R. M. COLOMBO, *Wave front tracking in systems of conservation laws*, Appl. Math., **49** (6) (2004), 501-537.

[9] R. M. COLOMBO - A. CORLI, *On a class of hyperbolic balance laws*, J. Hyperbolic Differ. Equ., **1** (4) (2004), 725-745.

[10] R. M. COLOMBO - A. GROLI, *On the initial boundary value problem for Temple systems*, Nonlinear Anal., **56** (4) (2004), 569-589.

[11] R. M. COLOMBO - N. H. RISEBRO, *Continuous dependence in the large for some equations of gas dynamics*, Comm. Partial Differential Equations, **23** (9-10) (1998), 1693-1718.

[12] R. M. COLOMBO - M. D. ROSINI, *Well posedness of balance laws with boundary*, J. Math. Anal. Appl., **311** (2) (2005), 683-702.

[13] G. CRASTA - B. PICCOLI, *Viscosity solutions and uniqueness for systems of inhomogeneous balance laws*, Discrete Contin. Dynam. Systems, **3** (4) (1997), 477-502.

[14] C. M. DAFERMOS, *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, Springer-Verlag, Berlin, 2000.

[15] C. M. DAFERMOS - L. HSIAO, *Hyperbolic systems and balance laws with inhomogeneity and dissipation*, Indiana Univ. Math. J., **31** (4) (1982), 471-491.

[16] C. DONADELLO - A. MARSON, *Stability of front tracking solutions to the initial and boundary value problem for systems of conservation laws*, No.D.E.A., to appear.

[17] A. F. FILIPPOV, *Differential equations with discontinuous righthand sides*, Kluwer Academic Publishers Group, Dordrecht, 1988.
[18] J. GOODMAN, *Initial Boundary Value Problems for Hyperbolic Systems of Conservation Laws*, PhD thesis, California University, 1982.
[19] D. HOFF, *Invariant regions for systems of conservation laws*, Trans. Amer. Math. Soc., **289** (2) (1985), 591-610.
[20] T. P. LIU, *Quasilinear hyperbolic systems*, Comm. Math. Phys., **68** (2) (1979), 141–172.
[21] M. NAGUMO, *Über die Lage der Integralkurven gewöhnlicher Differentialgleichungen*, Proc. Phys.-Math. Soc. Japan (3), **24** (1942), 551-559.
[22] D. SERRE, *Solutions à variations bornées pour certains systèmes hyperboliques de lois de conservation*, J. Differential Equations, **68** (2) (1987), 137-168.

Rinaldo M. Colombo: Department of Mathematics
Brescia University Via Branze 38, 25133 Brescia, Italy

Massimiliano D. Rosini: Dept. of Pure and Applied Mathematics
L'Aquila University, Via Vetoio, loc. Coppito, 67010 L'Aquila, Italy

*Pervenuta in Redazione*
*il 29 maggio 2006 e in forma rivista l'11 luglio 2007*