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Conditional aggregation operators defined by the Choquet integral and the Sugeno integral with respect to general fractal measures

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ABSTRACT

Conditional aggregation operators are defined by the Choquet integral and the Sugeno integral with respect to a monotone set function that assesses positive measure of the conditioning set. General Hausdorff and packing measures are introduced and examples of infinite s-sets with positive and finite generalized Hausdorff and packing measures are constructed and their fractal dimensions are compared. Coherent upper conditional previsions on the linear space of all Choquet integrable random variables are defined by the Choquet integral with respect to the general Hausdorff and packing measures when the conditioning event has positive and finite generalized Hausdorff and packing measures in its respective fractal dimensions. Conditional aggregation operators are defined by the Sugeno integral with respect to general Hausdorff and packing measures on the class of all Sugeno integrable random variables. Actually, the general Hausdorff and packing dimensions are proven to be the Sugeno integral with respect to the Lebesgue measure of the general Hausdorff and packing measures respectively.

1. Introduction and statements of results

The paper focuses on the integral representation by the Choquet integral [7] and by the Sugeno integral [38] of conditional aggregation operators defined in [15]. These two non-additive integrals have been widely studied in the literature ([11], [18], [25], [29], [30], [32], [33], [42]).

In this paper the Choquet integral is defined for bounded and unbounded random variables and the Sugeno integral for bounded and unbounded positive random variables; for each integral respectively the class of all Choquet integrable and the class of all Sugeno integrable random variables are considered to be the domain of the conditional aggregation operator.

The definition of conditional aggregation operator considered in this paper is more restrictive than that one proposed in [2] and [3] since it is required to satisfy the condition that for each conditioning set B the conditional aggregation operator of the indicator function of B, conditioned to B is equal to 1. This condition is a necessary condition to assure that aggregating any random variable on the partition of singletons is equivalent to knowing the random variable. In fact when the partition is the partition of singletons then we have complete information about the random variable, so knowing the conditional aggregation operator is equivalent to knowing the random variable. When data obtained by different sources are aggregated it is important that the contribution coming from each source is considered so that information is not lost in the aggregation process (see Example 1). A necessary condition to

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define such conditional aggregation operator, by the two non-additive integrals with respect to a monotone set function m is that the conditioning set B, belonging to the partition that represents partial information, has a positive and finite measure m. For this reason, different general Hausdorff and packing outer measures have been introduced, and the corresponding fractal dimensions have been introduced in a metric space. They are obtained considering the definition of the classical Hausdorff and packing outer measures and the different functions of the diameter of each set in the coverings. Conditional aggregation operators defined by the Choquet integral with respect to the general Hausdorff or packing outer measure are proven to be coherent upper conditional prevision [41] if the conditioning event has a positive and finite fractal measure in its fractal dimension. The necessity to introduce a new tool to define coherent conditional previsions occurs because the axiomatic definition of conditional expectation [1] may contradict a necessary condition of coherence (see [12], [13])). A new model of coherent upper conditional previsions defined on the linear space of all Chouqet integrable random variables by Hausdorff outer measures [36] has been proposed in [14]. When the conditioning event has a Hausdorff measure equal to zero we can compute the general Hausdorff outer measure or the general packing outer measure and if one of them assesses positive and finite measure to the conditioning event then the model proposed in this paper allows us to define a coherent conditional prevision continuous from below and not a 0-1 valued finitely, but not countably, additive probability. In a recent paper [16] a model of coherent upper conditional previsions based on Hausdorff outer measures has been proposed to solve some bias of human reasoning. The contribution put in evidence how the proposed model describes one of the capacity of unconscious human brain activity, which is to manage unexpected events, as it occurs in the selective attention. In the quoted paper different metric spaces are introduced to represent different reactions of people to unexpected events. The result proposed in this paper deals with a new model of coherent upper conditional previsions based on different fractal outer measures that assign different outer measures to the events. The model can be applied to represent in the same metric space different reactions of people to unexpected events. Another aspect investigated in this paper is the possibility to represent the fractal dimension of a set as the Sugeno integral with respect to the Lebesgue measure. Following [6] the Sugeno integral of the general outer Hausdorff measure or of the packing outer measure of a set with respect to the Lebesgue measure of the Borel sets of $[0, +\infty[$, is proven to be the Hausdorff or the packing dimension of the set. These results put in evidence the connection between general fractal measures and their fractal dimension with the Sugeno integral.

Examples of sets are constructed to investigate the relations between the two general outer measures and to compute the Sugeno integral of the two general measures of these sets as a function of s, with respect to the Lebesgue measure; in Example 2, an infinite s-set with a positive and finite general Hausdorff measure is constructed. In Example 3 and Example 4, we construct compact sets with the same general Hausdorff and packing dimensions so that the Sugeno integral of the general measures of these sets with respect to the Lebesgue measure coincides. In Example 5 a set with a positive and finite general Hausdorff measure is constructed. In Example 12, it is proven that the Sugeno integrals with respect to the Lebesgue measure, the general Hausorff measure, and the general packing measure of the homogeneous Moran set relating to the Fibonacci sequence coincide.

2. Conditional aggregation operators

Let (Ω, d) be a metric space and let $\mathbf{B}$ be a partition of Ω. A random variable is a function $X : \Omega \to \Re^* = \Re \cup \{-\infty; +\infty\}$ and $L^*(\Omega)$ is the class of all random variables defined on Ω, which is not a linear space; let $L^+(\Omega)$ be the class of positive random variables contained in $L^*(\Omega)$; for every $B \in \mathbf{B}$ denote by $X|B$ the restriction of X to B and by $\sup(X|B)$ the supremum value that X assumes on B. Denote by I_A the indicator function of any event $A \in \wp(B)$, i.e. $I_A(\omega) = 1$ if $\omega \in A$ and $I_A(\omega) = 0$ if $\omega \in A^c$. A monotone set function $\mu : \wp(\Omega) \to \mathbb{R}_+$ is such that $\mu(\emptyset) = 0$ and if $A, B \in \wp(\Omega)$ with $A \subset B$, then $\mu(A) \leq \mu(B)$.

Definition 1. Let $\mathbf{B}$ be a partition of Ω. For any set $B \in \mathbf{B}$ let K be class of random variables contained in $L^+(B)$; a *conditional aggregation operator* given B is any mapping $A(\cdot|B) : K \to [0, \infty[$ such that

1) $A(\cdot|B)$ is non-decreasing, i.e., $\mathbf{0} \leq X|B \leq Y|B$ implies $A(X|B) \leq A(Y|B)$,
2a) $A(\mathbf{1}_{B^c}|B) = 0$,
2b) $A(\mathbf{1}_B|B) = 1$

If $B = \Omega$ and K is the class of all and bounded random variables $L^+(B)$, then Definition 1 is the classical definition of the aggregation operator [4,21]. If $K = L^+(B)$ then the mapping $A(X|B)$ can assume values $+\infty$ and so condition 2b) of Definition 1 is not satisfied.

Definition 2. $A(X|\mathbf{B})$ is the random variable defined on Ω such that to each $\omega \in \Omega$ associates $A(X|B)$ if ω belongs to B.

Partial information about a random variable can be represented by a partition $\mathbf{B}$ of Ω in the sense that we do not know the exact value assumed by the random variable, but we know if the value belongs to B for each $B \in \mathbf{B}$. If $\mathbf{B}$ is the partition of singletons, information about the random variable is complete because we know the exact values it assumes. The following condition assures that to aggregate a random variable conditioned to the partition of singletons is equivalent to knowing the random variable itself.

Definition 3. Let $\mathbf{B}$ be the partition of singletons of Ω. A conditional aggregation operator $A(\cdot|B) : \mathbf{L}^+(B) \to [0, \infty[$ is coherent if $A(X|\mathbf{B}) = X$, $\forall X \in L^+(B)$.

Condition 2b) of Definition 1 is a necessary condition for a conditional aggregation operator to satisfy Definition 3. The following example shows that in general $A(X|\mathbf{B}) \neq X$.

Example 1. Let $\Omega = [0, T]$ be a time-lapse and let $X(\omega)$ the amount, in l/m^2 of rainfall recorded in a city, at the time ω. Let $\mathbf{B}$ be the partition of singletons of $[0, T]$. If

$$X(\omega) = \begin{cases} 0 & \text{if} \quad \omega \neq \overline{\omega}, \\ 500 & \text{if} \quad \omega = \overline{\omega}, \end{cases}$$

to have information about the function X we have to require that the conditional aggregation operator $A(X|\mathbf{B}) = X$ if X is constant on the atoms of the partition.

Let μ be a monotone set function and let $B \in \mathbf{B}$ such that $\mu(B) \neq 0$, then examples of conditional aggregation operators can be given by the Choquet integral and the Sugeno integral.

2.1. Conditional aggregation operator defined by the Choquet integral

The Choquet integral with respect to a monotone set function μ is defined by

$$\int X dh^s = \int_{-\infty}^{0} \left(\mu\left(\left\{\omega \in \Omega : X(\omega) \geq x\right\}\right) - \mu(\Omega)\right) dx + \int_{0}^{+\infty} \mu\left(\left\{\omega \in \Omega : X(\omega) \geq x\right\}\right) dx.$$

The integral is in $\Re$, or is equal to $-\infty$ or $+\infty$ or it does not exist.

Let μ be a monotone set function and let $X \in L^+(B)$, then the Choquet integral is defined by

$$\int^{Cho} X d\mu = \int_{0}^{+\infty} \mu\left\{\omega \in B : X(\omega) > x\right\} d\mu.$$

If $\int^{Cho} X d\mu < +\infty$, then $X|B$ is called Choquet integrable. Let K be the class of all Choquet integrable positive random variables. The mapping $A^{Ch}(X|B)$ defined on K by

$$A^{Ch}(X|B) = \frac{1}{\mu(B)} \int^{Cho} X d\mu$$

is a conditional aggregation operator.

Remark 1. We can observe that if in Example 1 the conditional aggregation operator is defined by the Choquet integral with respect to the Lebesgue measure then $A(X|\mathbf{B}) = 0$ and so the result of the aggregation process is a no correct knowledge of the random variable X. It occurs because the singletons of the partition $\mathbf{B}$, have Hausdorff dimension 0 that is less than the Hausdorff dimension of the interval $[0, T]$, which has Hausdorff dimension 1. Example 1 shows that to avoid that in the aggregation process information is lost it is important to consider different fractal measures for pieces of information having different fractal dimensions.

2.2. Conditional aggregation operator defined by the Sugeno integral

Let μ be a monotone set function and let $X \in L^+(B)$, then the Sugeno integral is defined by

$$Su(X|B, \mu) = \sup_{x \geq 0} \left\{ x \wedge \mu\left\{\omega \in B : X(\omega) \geq x\right\} \right\}.$$

The integral is in $\Re$ or it is equal to $+\infty$. If $Su(X|B, \mu) < +\infty$, then $X|B$ is called Sugeno integrable. Let K be the class of all Sugeno integrable random variables $X|B \in L^+(B)$.

A random variable X is trivial if $X(\omega) = 0$, $\forall \omega \in (0, +\infty)$ or $X(\omega) = \infty$, $\forall\ \omega \in (0, +\infty)$.

Let m be the Lebesgue measure on the class of Borel sets of $[0, +\infty)$. If X is a non-trivial decreasing positive random variable by Theorem 3.2 of [6] we have that

$$Su(X|B, m) = \sup_{x \geq 0} \left\{ x \wedge X|B \right\} = x_0$$

where x_0 is the unique point, called midpoint in [6], such that

$$X|B < x \text{ for any } x_0 < x < +\infty \text{ and } X|B > x \text{ for any } 0 < x < x_0$$

If K is the class of all positive and bounded random variables and μ is a probability measure on B, i.e. $\mu(B)=1$, then the midpoint x_0 is the fixed point of the function $G_\mu(x)=\mu\{\omega\in B : X(\omega)\geq x\}$ and the Sugeno integral is the intersection of the first bisectrix and the function $G_\mu(x)$. The mapping $A^{Su}(X|B)$ defined on K by

$$A^{Su}(X|B)=\frac{1}{m(B)}Su(X|B,m) \text{ if } m(B)\neq 0,$$

is a conditional aggregation operator. In Section 6 we prove that the fractal dimension of a set can be represented as the Sugeno integral with respect to the Lebesgue measure of the corresponding fractal measure; it puts in evidence that the Hausdorff dimension of a set is an aggregation operator of the data represented by the set.

3. General fractal measures and dimensions

In this section, we will be defining various notions of fractal measures and dimensions in a general setting.

Let f and g be two functions on $(0,a), 0<a\leq+\infty$ with the following properties:

(H1) f and g are continuous, positive and strictly decreasing on $(0,a)$.
(H2) $\lim_{r\to 0} f(r)=\lim_{r\to 0} g(r)=+\infty$ and $\lim_{x\to a} f(r)=0$.
(H3) $\lim_{r\to\infty}\dfrac{f^{-1}(kr+c)}{f^{-1}(r)}=0$ for any $k>1$ and $c\in\mathbb{R}$.
(H4) $\limsup_{r\to 0}\Big(g(r)-g(kr)\Big)<+\infty$ for $k>1$.

It is clear that (H1) and (H2) guarantee that the function f^{-1} is defined on $(0,\infty)$. For $s\geq 0$, we consider now the family of functions $h_s(r)$ which are defined as follows

$$h_s(r)=\begin{cases} f^{-1}(sg(r)), & \text{for } r>0 \\ 0, & \text{for } r=0. \end{cases}$$

It is easy to see that the functions h_s are continuous and increasing on $[0,\infty)$, thus are a family of *Hausdorff functions*. Also let us formally set $f^{-1}(0)=a$, including the case when $a=+\infty$.

Let (Ω,ρ) be a metric space. A δ-cover of a set $E\subseteq\Omega$ is a finite or countable collection of sets $E_i\subseteq\Omega$ such that $E\subseteq\cup_i E_i$ and $\operatorname{diam}\left(E_i\right)\leq\delta$ for all i. The general Hausdorff measure on Ω corresponding to h_s is defined as follows:

$$\mathscr{H}_\delta^{h_s}(E)=\inf\left\{\sum_i h_s\left(\operatorname{diam}\left(E_i\right)\right) \mid \left\{E_i\right\}_i \text{ is a } \delta\text{-cover of } E\right\}$$

and

$$\mathscr{H}^{h_s}(E)=\lim_{\delta\to 0}\mathscr{H}_\delta^{h_s}(E).$$

Remark 2. There is no countable cover by sets of diameter less than δ if Ω is not separable for small enough $\delta>0$. As a result, the empty set is the infinimum in the definition of Hausdorff's outer measure, and after that, it is $+\infty$. Hence, $+\infty$ is the upper limit for δ reaching zero. Then, the Hausdorff measure and Hausdorff dimension for a non-separable set Ω are $\mathscr{H}^{h_s}(\Omega)=+\infty$ and $+\infty$, respectively.

A δ-packing of the set $E\subset\Omega$ is a finite or countable collection of disjoint closed balls $\left\{B\left(x_i,r_i\right)\right\}_i$ with centers in E and such that $2r_i\leq\delta$ for every i. We define the pre-packing measure as follows

$$\overline{\mathscr{P}}_\delta^{h_s}(E)=\sup\left\{\sum_i h_s\left(2r_i\right) \mid \left\{B\left(x_i,r_i\right)\right\}_i \text{ is a } \delta\text{-packing of } E\right\},$$

$$\overline{\mathscr{P}}^{h_s}(E)=\lim_{\delta\to 0}\overline{\mathscr{P}}_\delta^{h_s}(E).$$

It follows from [20] that $\overline{\mathscr{P}}^{h_s}$ is not in general countably sub-additive, so there is one more step necessary to obtain an outer measure, that is, defining

$$\mathscr{P}^{h_s}(E)=\inf\left\{\sum_i\overline{\mathscr{P}}^{h_s}\left(E_i\right) \mid E\subseteq\cup_i E_i\right\}.$$

Initially, rather than utilizing their radii, Taylor and Tricot [39] developed packing measurements using the diameters of balls. The same outcome is obtained in R^d but not in all generic metric spaces. According to Cutler [5], the radius definition is more practical because the packing measure's regularity characteristics are maintained. Generally speaking, metric space is used, but the diameter specification is not always the case. This also applies to the following significant relationship, for all $s \geq 0$,

$$\mathscr{H}^{h_s}(E) \leq \mathscr{P}^{h_s}(E), \text{ for all } E \subseteq \Omega.$$

They are metric outer measures, as shown by almost the same proof as in the typical case. It is clear that (H3) and (H4) imply the following:

$$\lim_{r\to 0} \frac{h_{s_1}(kr)}{h_{s_2}(r)} = 0 \text{ for } s_1 > s_2 \text{ and any } k \geq 1.$$

Thus we can define generalized Hausdorff and packing dimensions based on the family of the functions h_s, respectively, as follows

$$\dim_{f,g}(E) = \sup\left\{s > 0 \mid \mathscr{H}^{h_s}(E) = +\infty\right\} = \inf\left\{s > 0 \mid \mathscr{H}^{h_s}(E) = 0\right\}$$

and

$$\operatorname{Dim}_{f,g}(E) = \sup\left\{s > 0 \mid \mathscr{P}^{h_s}(E) = +\infty\right\} = \inf\left\{s > 0 \mid \mathscr{P}^{h_s}(E) = 0\right\}.$$

It is clear that

$$\dim_{f,g}(E) \leq \operatorname{Dim}_{f,g}(E).$$

This generality can be used, for example, to the possibility of fine-tuning the Hausdorff function by adding logarithmic factors. Families of functions that are frequently seen include:

(I) $h_s(r) = r^s$ corresponding to $f(r) = g(r) = -\log r$ gives the usual Hausdorff and packing measures $\mathscr{H}^s, \mathscr{P}^s$.
(II) $h_s(r) = 2^{-r^{-s}}$ corresponding to $f(r) = \log_2\left(\log_2\left(\frac{1}{r}\right)\right), g(r) = -\log_2(r)$.
(III) $h_s(r) = 2^{-\left(\log_2\left(\frac{1}{r}\right)\right)^s}$ corresponding to $f(r) = g(r) = \log_2\left(\log_2\left(\frac{1}{r}\right)\right)$.
(IV) $h_s(r) = 2^{-M\,(\frac{1}{r})^s}$ corresponding to $f(r) = \log_2\left(\frac{1}{M}\log_2\left(\frac{1}{r}\right)\right), g(r) = -\log_2(r)$, for all $M > 0$.
(V) $h_s(r) = 2^{-M\left(\frac{1}{M}\log_2\left(\frac{1}{r}\right)\right)^s}$ corresponding to $f(r) = g(r) = \log_2\left(\frac{1}{M}\log_2\left(\frac{1}{r}\right)\right)$, for all $M > 0$.

Let's mention that certain functions, like those in (II) or (IV), can be used to determine the dimension of infinite-dimensional sets. These Hausdorff functions may be valuable because they vanish more quickly than any power (see for example [8,24,26–28,40]) which gives a great interest in these general fractal measures.

4. Some examples

In the following example, an infinite s-set with positive and finite generalized Hausdorff measure is constructed.

Example 2. We will start by introducing a group of compact, completely disconnected metric spaces that are manageable and useful for further study. For $k \in \mathbb{N}$ we take $a_k \in \mathbb{N}$, and $A_k = \left\{1, \dots, a_k\right\}$ be a discrete set. Let $\Omega = \prod_{j=1}^{\infty} A_j$. Clearly $\sigma = \left(\sigma_j\right)_{j=1}^{\infty} \in \Omega$ when $\sigma_j \in A_j$ for every j. In order to prove that the product topology on n is a finite sequence, we shall now establish a metric ρ on Ω for $x = \left(x_j\right)_{j=1}^{n}$ with $x_j \in A_j$ for every $j = 1, \dots, n$. The empty segment A is the only starting segment with length zero by definition. Write $|x|$ to denote the length of x for x is an initial segment. For $n \in \mathbb{N}$, take Ω^n stand for the collection of all n-length beginning segments. Let $\Omega^* = \cup_{j=0}^{\infty}\Omega^j$ be the set of all initial segments. Let $\sigma \in \Omega$ write $\sigma|_n$ for the initial segment $\left(\sigma_1, \dots, \sigma_n\right) \in \Omega^n$. We could place the following partial order on Ω^*: For $x, y \in \Omega^*$ such that $x = \left(x_1, \dots, x_j\right)$ and $y = \left(y_1, \dots, y_k\right)$, we write $x < y$ if $j < k$ and $x_i = y_i$ for all $i = 1, \dots, j$. If $x < y$, then y is said to be a descendant of x. Let now $x \in \Omega^n$, we take x' denote the unique element of Ω^{n-1} such that $x' < x$, x' is called the parent and x the child. Also if $x \in \Omega^n$ is an initial segment, then we can define the cylinder as follows

$$[x] = \left\{\sigma \in \Omega \mid \sigma_i = x_i \text{ for all } i = 1, \dots, n\right\}.$$

Assign a number $r(x) > 0$ matching the following conditions to each $x \in \Omega^*$ in order to construct a metric on ρ on Ω:

(a) $r(A) > 0$.
(b) $r(y) < r(x)$ for all $y > x$.
(c) $\lim_{n\to\infty} r\left(\sigma|_n\right) = 0$.

If the longest common initial segment of $\sigma, \tau \in \Omega$ is α, then we can define $\rho(\sigma, \tau) = r(x)$. Furthermore, we define $\rho(\sigma, \sigma) = 0$. The product topology on Ω is then induced by a metric called ρ (see [19]). Remark that if $\sigma, \tau \in \Omega$ and $r\left(\sigma|_n\right) \le \varepsilon < r\left(\sigma|_{n-1}\right)$, then $\rho(\sigma, \tau) \le \varepsilon$ is equivalent to $\sigma_i = \tau_i$ for $i = 1, \dots, n$ which implies that $B(\sigma, r) = \left[\sigma|_n\right]$.

This will be of particular interest because of an intriguing particular form of sequence space. Take $s > 0$, $A_k = \left\{1, \dots, 2^{2^{k-1}}\right\}$ and we can choose $r \in (0,1)$ such that $\frac{1}{r^s} = 2$. Now, if $x \in \Omega^n$, we define $r(x) = r^n$. Infinite s-space will be the name of the generated sequence space. In the following, we shall find that the Hausdorff function that best describes the dimensions of this space is provided by $h_s(r) = 2^{-r^{-s}}$. More precisely, it follows from [26,27] that

$$\text{if } (\Omega, \rho) \text{ is infinite } s\text{-space, then } \mathscr{H}^{h_s}(\Omega) = \frac{1}{2}.$$

In the following example, we construct compact sets with the same generalized Hausdorff and packing dimension.

Example 3. Let (Ω, ρ) be a separable metric space and let $\mathscr{K}(\Omega)$ denote the set of non-empty compact subsets of Ω. We can define now a metric $\tilde{\rho}$ on the space $\mathscr{K}(\Omega)$ as follows: For $A, \tilde{A} \in \mathscr{K}(\Omega)$ let

$$\widetilde{\rho}(A, \tilde{A}) = \max\left\{\sup_{x\in A}\left\{\operatorname{dist}(x, \tilde{A})\right\}, \sup_{y\in\tilde{A}}\left\{\operatorname{dist}(y, A)\right\}\right\}.$$

The Hausdorff metric space, often known as the hyperspace associated with Ω, is the space $(\mathscr{K}(\Omega), \widetilde{\rho})$ and it inherits numerous attractive geometrical qualities from Ω. As an illustration, $\mathscr{K}(\Omega)$ is complete or compact depending on the value of Ω, and vice versa. The Hausdorff metric is discussed in [19, Section 2.4], along with proofs of the aforementioned statements. Tildes will be employed to indicate references to the hyperspace in order to prevent confusion between metric spaces and the related hyperspaces. For instance, the closed ball of radius ε about the set A is denoted by $\widetilde{B}_\varepsilon(A) \subset \mathscr{K}(\Omega)$. This is the case if $A \subset \Omega$ is compact and $\varepsilon > 0$.

The similarity dimension, which only applies to sets that are similar to themselves, is the following useful concept of dimension. The following is how self-similar sets are obtained: Let $m \in \mathbb{N}$ and for $i = 1, \dots, m$ let $S_i : \Omega \to \Omega$ be a similarity with ratio $r_i \in (0,1)$. This means that for every $x, y \in \Omega$ we have $\rho\left(S_i(x), S_i(y)\right) = r_i \rho(x, y)$. In this case, a singular non-empty compact set $E \subset \Omega$ exists such that $E = \cup_{i=1}^m S_i(E)$. This results in a set E that is considered to be self-similar. The singular positive number s_0 with the property that $\sum_{i=1}^m r_i^{s_0} = 1$ is the definition of the similarity dimension of the set E (see [19, Chapter 4] for further details on self-similar sets).

The standard Hausdorff dimension $\dim_H(E)$, upper box dimension $\overline{\dim}_B(E)$, and similarity dimension s_0 are related as follows:

$$\dim_H(E) \le \overline{\dim}_B(E) \le s_0.$$

If the set of contractions $\left\{S_i\right\}_{i=1}^m$ meets the open set condition, this relationship may be reinforced in Euclidean space. In other words, there is an open set O if and only if $O \supset \cup_{i=1}^m S_i(O)$ with this union disjoint. The aforementioned inequalities can be changed into equalities assuming that the open set requirement is met.

Now, let $E \subset \mathbb{R}^n$ be a self-similar set satisfying the open set condition. We assume that s_0 is the similarity dimension of E and let $h_s(r) = 2^{-r^{-s}}$ for $s \ge 0$. Then, by using [26,27], we have that

$$\mathscr{H}^{h_s}(\mathscr{K}(E)) = \begin{cases} 0 & \text{for } s > s_0 \\ +\infty & \text{for } s < s_0 \end{cases}$$

and

$$\mathscr{P}^{h_s}(\mathscr{K}(E)) = \begin{cases} 0 & \text{for } s > s_0 \\ +\infty & \text{for } s < s_0 \end{cases}$$

which implies that

$$\dim_{f,g}(\mathscr{K}(E)) = \operatorname{Dim}_{f,g}(\mathscr{K}(E)) = s_0 = \dim_H(E) = \overline{\dim}_B(E).$$

Example 4. Let (Ω, ρ) be a metric space as in Example 1. Suppose that Ω is a sequence space with $\dim_H(\Omega) = s_0 < +\infty$. Let $h_s(r) = 2^{-r^{-s}}$ for all $s \ge 0$. It follows from [26] that

$$\mathscr{H}^{h_s}(\mathscr{K}(\Omega)) = \begin{cases} 0 & \text{for } s > s_0 \\ +\infty & \text{for } s < s_0 \end{cases}$$

and we have

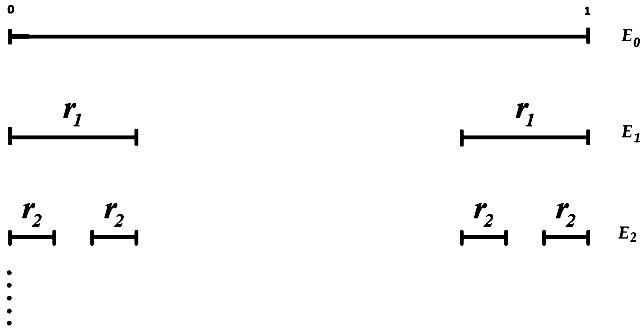

Fig. 1. The construction of the set E.

$$\dim_{f,g}(\mathscr{K}(\Omega)) = s_0 = \dim_H(\Omega).$$

The next example gives a metric space with a positive and finite general Hausdorff measure.

Example 5. Let $\Omega = \{x_0, x_1, \dots, x_\infty\}$ be a countable metric space with a metric ρ fulfilling, for a sequence $a_n \searrow 0$ as $n \to +\infty$,

$$\rho(x_n, x_\infty) = a_n \text{ and } \rho(x_n, x_m) \geq a_n \text{ for all } m < n < \infty.$$

It is clear that $\mathscr{H}^{h_s}(\Omega) = 0$ for every f and g satisfying the conditions (H1)-(H4). Now, if we suppose moreover the functions h_s satisfies $h_s(a_n) = 2^{-n}$, then it follows from [26] that

$$\frac{1}{2} \leq \mathscr{H}^{h_s}(\mathscr{K}(\Omega)) \leq 1.$$

In the next three examples, we will create a set E that has both a positive and finite general Hausdorff measure as well as the zero or infinite classical Hausdorff measure. In the following example, a set E that satisfies the strong separation condition is constructed such that its Hausdorff outer measure and its packing outer measure coincide for every $s \geq 0$ since the coverings of the set E are packings of E. This condition is not sufficient to assure that the generalized Hausdorff measures and the generalized packing outer measures coincide.

Example 6. A generalization of the Cantor set construction could be conceived of as the following generic construction of a subset of $\mathbb{R}$. Let $[0,1] = E_0 \supset E_1 \supset E_2 \supset \dots$ be a decreasing sequence of sets, with each E_k a union of a finite number of disjoint closed intervals which are called kth level basic intervals, with each interval of E_k containing at least two intervals of E_{k+1}, and the length of kth level intervals is $r_k = 2^{-k^2}$. Then we consider the set

$$E = \bigcap_{k=0}^{\infty} E_k.$$

We assume that the $(k+1)$th level intervals I_1 and I_2 contained in I are of equal length and evenly spaced in the overall construction of the collection E for each kth level interval I (see Fig. 1). Write now $h_s(r) = 2^{-\left(\log_2\left(\frac{1}{r}\right)\right)^s}$. It is clear that

$$\mathscr{H}_{r_k}^{h_{\frac{1}{2}}}(E) \leq 2^k h_{\frac{1}{2}}(r_k) \leq 2^k 2^{-k} = 1$$

which implies that

$$\mathscr{H}^{h_{\frac{1}{2}}}(E) \leq 1.$$

Now, distribute a mass μ on E in such a way that $\mu(I) = 2^{-k} = h_{\frac{1}{2}}(r_k)$ whenever I is any level k interval. By using similar techniques such as in [20, Example 4.4], it follows from [26, Lemma 3.3.2] that

$$\mathscr{H}^{h_{\frac{1}{2}}}(E) \geq \mu(E) > 0.$$

This implies that

$$0 < \mathscr{H}^{h_{\frac{1}{2}}}(E) < +\infty.$$

Now, since

$$\lim_{r \to 0} \frac{h_{\frac{1}{2}}(r)}{r^s} = +\infty \quad \text{for each} \quad s > 0,$$

we also have

$$\mathscr{H}^s(E) = \mathscr{P}^s(E) = 0 \quad \text{for all} \quad s > 0.$$

Here, too, we may apply Judit's methods in [22,23] to get $\mathscr{H}^{h_{\frac{1}{2}}}(E) = 1$.

Example 7. Let $s > 0$. Write $g_s(r) = r^s$ and $h_s(r) = 2^{-\left(\log_2\left(\frac{1}{r}\right)\right)^s}$ for $r \in (0,1)$. Simple calculus is used to obtain

$$\lim_{r\to 0} \frac{g_s(r)}{h_s(r)} = \begin{cases} 0 & \text{if } s < 1, \\ +\infty & \text{if } s \geq 1. \end{cases}$$

It follows that

(1) if $s < 1$ and E satisfies $0 < \mathscr{H}^{h_s}(E) < +\infty$, then we always that $\mathscr{H}^{g_s}(E) = 0$;
(2) if $s \geq 1$ and E satisfies $0 < \mathscr{H}^{h_s}(E) < +\infty$, then we always have $\mathscr{H}^{g_s}(E) = +\infty$.

In particular, if $s \geq 1$, then we cannot have both $0 < \mathscr{H}^{h_s}(E) < +\infty$ and $\mathscr{H}^{g_s}(E) = 0$.

Example 8. Let $s > 0$. Write $g_s(r) = r^s$ and $h_s(r) = 2^{-\left(\frac{1}{r}\right)^s}$ for $r \in (0,1)$. It is clear that

$$\lim_{r\to 0} \frac{g_s(r)}{h_s(r)} = +\infty.$$

It follows that, if $s > 0$ and E satisfies $0 < \mathscr{H}^{h_s}(E) < +\infty$, then we always have $\mathscr{H}^{g_s}(E) = +\infty$.

5. Coherent upper conditional previsions defined by the Choquet integral with respect to general fractal dimensional metric outer measures

In this section, we extend the results proven in [14] for Hausdorff outer measures to the general case where the Choquet integral is defined with respect to general fractal outer measures. A particular class of conditional aggregation operators can be considered to construct coherent upper conditional expectations. For every $B \in \mathbf{B}$ coherent upper conditional expectations or previsions $\overline{P}(\cdot|B)$ are functionals defined on a linear space $L(B)$.

Definition 4. Coherent upper conditional previsions are functionals $\overline{P}(\cdot|B)$ defined on $L(B)$, such that the following axioms of coherence hold for every X and Y in $L(B)$ and every strictly positive constant λ:

(i) $\overline{P}(X|B) \leq \sup(X|B)$;
(ii) $\overline{P}(\lambda X|B) = \lambda \overline{P}(X|B)$ (positive homogeneity);
(iii) $\overline{P}(X + Y|B) \leq \overline{P}(X|B) + \overline{P}(Y|B)$ (subadditivity).

When $L(B)$ is the linear space of all bounded random variables, Definition 1 is the definition given in [41]. Suppose that $\overline{P}(X|B)$ is a coherent upper conditional expectation on $L(B)$. Then its conjugate coherent lower conditional expectation is defined by

$$\underline{P}(X|B) = -\overline{P}(-X|B).$$

If for every X belonging to a linear space $L(B)$ we have $P(X|B) = \underline{P}(X|B) = \overline{P}(X|B)$, then $P(X|B)$ is called a coherent linear conditional expectation (de Finetti (1972) [9], de Finetti (1974) [10], Dubins (1975) [17], Regazzini (1985) [34], Regazzini (1987) [35]) and it is a linear, positive and positively homogenous functional on K (Walley (1991) [41, Corollary 2.8.5]). The unconditional coherent upper expectation $\overline{P} = \overline{P}(\cdot|\Omega)$ is obtained as a particular case when the conditioning event is Ω. Coherent upper conditional probabilities are obtained when only 0-1 valued random variables are considered. From axioms i)-iii) and by the conjugacy property, we have that

$$1 \leq \underline{P}(I_B|B) \leq \overline{P}(I_B|B) \leq 1$$

so that

$$\underline{P}(I_B|B) = \overline{P}(I_B|B) = 1.$$

In [15] coherent upper conditional previsions have been defined on the linear space of all bounded random variables through conditional aggregator operators.

According to the following conclusion, coherent conditional previsions are defined on the linear space of all Choquet integral random variables $X|B$ as the Choquet integral with respect to a fractal outer measure $\mathscr{H}^{h_s}$ such that $\mathscr{H}^{h_s}(B) \neq 0$ and they are defined by a 0-1-valued finitely but not countably additive probability when the conditioning event has a fractal outer measure $\mathscr{H}^{h_s}$ zero or infinity in it dimension.

Theorem 1. *Let (Ω, d) be a metric space, $s \geq 0$, and let $\mathbf{B}$ be a partition of Ω. For every $B \in \mathbf{B}$ denote by s the dimension $\dim_{f,g}$ of the conditioning event B and by $\mathscr{H}^{h_s}$ the s-dimensional outer measure.*

(1) *Let m_B be a 0-1 valued finitely additive, but not countably additive, probability on $\wp(B)$. Then for each $B \in \mathbf{B}$ the functional $\overline{P}(X|B)$ defined on the linear space of all Choquet integral random variable $\mathbb{L}(B)$ by*

$$\overline{P}(X|B) = \begin{cases} \dfrac{1}{\mathscr{H}^{h_s}(B)} \displaystyle\int_B X d\mathscr{H}^{h_s} & \text{if } 0 < \mathscr{H}^{h_s}(B) < +\infty, \\ \displaystyle\int_B X d m_B & \text{if } \mathscr{H}^{h_s}(B) \in \{0, +\infty\} \end{cases}$$

is a coherent upper conditional prevision.

(2) *Let m_B be a 0-1 valued finitely additive, but not countably additive, probability on $\wp(B)$. Thus, for each $B \in \mathbf{B}$, the function defined on $\wp(B)$ by*

$$\overline{P}(A|B) = \begin{cases} \dfrac{\mathscr{H}^{h_s}(A \cap B)}{\mathscr{H}^{h_s}(B)} & \text{if } 0 < \mathscr{H}^{h_s}(B) < +\infty, \\ m_B & \text{if } \mathscr{H}^{h_s}(B) \in \{0, +\infty\} \end{cases}$$

is a coherent upper conditional probability.

(3) *All above results hold for the general dimensional packing measures $\mathscr{P}^{h_s}$.*

Proof. (1) Since $\mathbb{L}(B)$ is a linear space it suffices to prove that, for every $B \in \mathbf{B}$ $\overline{P}(X|B)$ satisfies conditions (i), (ii), (iii) of Definition 4. If B has finite and positive $\mathscr{H}^{h_s}$ outer measure in its dimension $\dim_{f,g}$, then

$$\overline{P}(X|B) = \frac{1}{\mathscr{H}^{h_s}(B)} \int_B X d\mathscr{H}^{h_s},$$

which implies that properties (i) and (ii) are satisfied since they hold for the Choquet integral [11, Proposition 5.1]. Now, property (iii) follows from the Subadditivity Theorem [11, Theorem 6.3] since $\mathscr{H}^{h_s}$ outer measures are monotone, submodular, and continuous from below. If B has an outer measure $\mathscr{H}^{h_s}$ in its dimension equal to zero or infinity we have that the class of all coherent (upper) previsions on $L(B)$ is equivalent to the class of 0-1 valued additive probabilities defined on $\wp(B)$, then $\overline{P}(X|B) = m(B)$. Then properties (i), (ii), (iii) are satisfied since m is a 0-1 valued finitely additive probability on $\wp(B)$.

(2) Coherent upper conditional probabilities can be generated by limiting coherent upper conditional predictions to the class of indicator functions. A coherent upper conditioning prediction model built on dimensional outer measures is the end outcome. The coherent upper conditional probability, which is often employed as a gauge of the possibility of an event occurring given that another event has already occurred (by assumption, presumption, assertion, or evidence), is defined in the model using the s-dimensional outer measure.

(3) The proof of Assertion (3) is identical to the proof of the statement in the first and the second assertion and is therefore omitted. □

Remark 3. In Theorem 1 for each conditioning set B such that the general Hausdorff or packing outer measure id equal to zero or infinity, conditional probability is defined by a 0-1 valued finitely, but not countably, additive probability. It assures that, in this case, the restriction of the conditional probability to the Borel σ-field is a full conditional probability in the sense of Dubins [17] and in particular it satisfies the general compound rule for every Borelian sets A, B, C

$$P(A \cap B|C) = P(A|B \cap C)P(B|C).$$

6. Fractal dimensions defined by the Sugeno integral of generalized fractal measures

In this section, fractal dimensions $\dim_{f,g}$ and $\text{Dim}_{f,g}$ introduced in Example 4 are proven to be respectively the Sugeno integral with respect to the Lebesgue measure of the generalized fractal measures $\mathscr{H}^{h_s}$ and $\mathscr{P}^{h_s}$.

Theorem 2. *Let (Ω, d) be a metric space, let E be a subset of Ω, and let m be the Lebesgue measure on the class of Borel sets of $[0, +\infty)$. For $s \geq 0$ let*

$$\left(\nu^s, s_0\right) \in \left\{ \left(\mathscr{H}^{h_s}, \dim_{f,g}\right), \left(\mathscr{P}^{h_s}, \text{Dim}_{f,g}\right) \right\}.$$

Then the Sugeno integral of the function $v^s(E)$, as a function of s, with respect to m is equal to fractal dimension s_0, that is

$$S(v^s(E), m) = s_0.$$

Proof. Firstly consider the case $0 < s_0 < \infty$; $v^s(E)$, as function of s is a decreasing non-trivial positive function, then by Theorem 3.2 and Theorem 3.11 of [6] we have

$$Su(v^s(E), m) = \sup_{x \geq 0} \left\{ x \wedge v^s(E) \right\} = s_0$$

where s_0 is the unique midpoint [6], i.e. it is the unique point such that

$$v^s(E) < x \text{ for any } s_0 < x < +\infty \text{ and } v^s(E) > x \text{ for any } 0 < x < s_0.$$

Let $s_0 = 0$; $v^s(E)$, as function of s, is a decreasing trivial function such that $v^s(E) \in \Re^+ \cup \{+\infty\}$ for $s = s_0 = 0$ and $v^s(E) = 0$ for $s > 0$, then

$$Su(v^s(E), m) = 0 = s_0. \quad \square$$

Remark 4. By the previous theorem, we have that fractal dimensions $\dim_{f,g}$ and $\mathrm{Dim}_{f,g}$ are examples of aggregation operators. In general, the Sugeno integral with respect to the Lebesgue measure m does not define conditional aggregation operators since condition 2b) of Definition 1 may be not satisfied. It occurs for example if E is the Cantor set because $S(\mathbf{1}_E|E|, m) = 0 \neq 1$.

In the following examples, some consequences of the results proven in the paper are shown.

Example 9. Let Ω as in Example 1 and $\mathscr{K}(\Omega)$ as in Example 3 then by Theorem 2 we have that

$$S(\mathscr{H}^{h_s}(\mathscr{K}(\Omega)), m) = S(\mathscr{H}(\Omega), m).$$

Example 10. Let E be the set as in Example 5 and $h_s(r) = 2^{-\left(\log_2\left(\frac{1}{r}\right)\right)^s}$. By using Theorem 2, we get

$$S(\mathscr{H}^{h_s}(E), m) = \frac{1}{2} = S(\mathscr{P}^{h_s}(E), m).$$

We can observe that if we chose $h_s(r) = r^s$, then we obtain $\mathscr{H}^s(E) = 0 = \mathscr{P}^s(E)$, $\forall s > 0$ and $\mathscr{H}^0(E) = +\infty = \mathscr{P}^0(E)$ so that the $\dim_H(E) = 0 = \dim_P(E)$. By using Theorem 2 we have

$$S(\mathscr{H}^s(E), m) = 0 = S(\mathscr{P}^s(E), m).$$

Example 11. Let us recall the class of homogeneous Moran sets [37]. We denote by $\{n_k\}_{k\geq 1}$ a sequence of positive integers with $n_k \geq 2$ and $\Phi = \{\Phi_k\}_{k\geq 1}$ be a sequence of vectors satisfying

$$\Phi_k = (c_{k,1}, c_{k,2}, \cdots, c_{k,n_k}), \text{ with } 0 < c_{k,j} < 1, \ \forall k \in \mathbb{N}, \ \forall 1 \leq j \leq n_k.$$

$$D_{m,k} = \left\{ \left(i_m, i_{m+1}, \ldots, i_k\right); \quad 1 \leq i_j \leq n_j, \quad m \leq j \leq k \right\} \quad \text{and} \quad D_k = D_{1,k}.$$

Define $D = \bigcup_{k\geq 1} D_k$.

Let $\sigma = \left(\sigma_1, \ldots, \sigma_k\right) \in D_k, \tau = \left(\tau_{k+1}, \ldots, \tau_m\right) \in D_{k+1,m}$, we denote $\sigma * \tau = \left(\sigma_1, \ldots, \sigma_k, \tau_{k+1}, \ldots, \tau_m\right)$.

We say that the collection $\mathscr{F} = \left\{J_\sigma, \sigma \in D\right\}$ fulfills the Moran structure if it satisfies the following conditions:

(1) For all $\sigma \in D$, J_σ is similar to J, that is there exists a similarity mapping $S_\sigma : \mathbb{R}^d \to \mathbb{R}^d$ such that $S_\sigma(J) = J_\sigma$. Here we set $J_\emptyset = J$.
(2) For all $k \geq 0$ and $\sigma \in D_k, J_{\sigma*1}, J_{\sigma*2}, \ldots, J_{\sigma*n_{k+1}}$ are subsets of J_σ, and satisfy that $J^\circ_{\sigma*i} \cap J^\circ_{\sigma*j} = \emptyset \, (i \neq j)$, where A° denotes the interior of A.
(3) For any $k \geq 1, \sigma \in D_{k-1}$ and $1 \leq j \leq n_k$, $c_{k,j} = \dfrac{\left|J_{\sigma*j}\right|}{\left|J_\sigma\right|}$, $1 \leq j \leq n_k$, where $|A|$ denotes the diameter of A.

Let $\mathscr{F} = \mathscr{F}\left(J, \left\{n_k\right\}, \left\{\Phi_k\right\}\right)$ be a collection having a Moran structure. The set

$$E(\mathscr{F}) = \bigcap_{k\geq 1} \bigcup_{\sigma \in D_k} J_\sigma$$

is called a Moran set determined by $\mathscr{F}$. It is convenient to denote $M\left(J,\{n_k\},\{\Phi_k\}\right)$ the collection of Moran sets determined by J, $\{n_k\}$ and $\{\Phi_k\}$. If we ask $c_{k,j}=c_k$ for all $1\le j\le n_k$, where $\{c_k\}_{k\ge 1}$ is a sequence of positive numbers, we can get the Moran structure and Moran sets. In this situation, we call them by homogeneous Moran structure and the collection of Moran sets and denote by $\mathscr{F}=\mathscr{F}\left(J,\{n_k\},\{c_k\}\right)$ and $\mathscr{M}=\mathscr{M}\left(J,\{n_k\},\{c_k\}\right)$. If $\lim\limits_{k\to+\infty}\sup\limits_{\sigma\in D_k}|J_\sigma|>0$, then E contains interior points. Thus the measure and dimension properties will be trivial. We assume therefore $\lim\limits_{k\to+\infty}\sup\limits_{\sigma\in D_k}|J_\sigma|=0$. Now, define

$$\underline{s}:=\liminf_{k\to+\infty}\frac{\log(n_1\cdots n_k)}{-\log(c_1\cdots c_k)}$$

and

$$\overline{s}:=\limsup_{k\to+\infty}\frac{\log(n_1\cdots n_k)}{-\log(c_1\cdots c_k)}.$$

It follows from Theorem 2 that

$$S(\mathscr{H}^s(E(\mathscr{F})),m)=\underline{s}\quad\text{and}\quad S(\mathscr{P}^s(E(\mathscr{F})),m)=\overline{s}.$$

It is clear that if $J=[0,1]$, $n_k=2$ and $c_k=\frac{1}{3}$ for all $k\ge 1$, then the set $E(\mathscr{F})$ is the middle-third Cantor set which implies that

$$S(\mathscr{H}^s(E(\mathscr{F})),m)=\frac{\log(2)}{\log(3)}=S(\mathscr{P}^s(E(\mathscr{F})),m).$$

Example 12. Let $A=\{a,b\}$ be a two-letter alphabet, and A^* the free monoid generated by A. Let F be the homomorphism on A^*, defined by $F(a)=ab$ and $F(b)=a$. It is easy to see that $F^n(a)=F^{n-1}(a)F^{n-2}(a)$. We denote by $|F^n(a)|$ the length of the word $F^n(a)$, thus

$$F^n(a)=s_1s_2\cdots s_{|F^n(a)|},\quad s_i\in A.$$

Therefore, as $n\to\infty$, we get the infinite sequence

$$\omega=\lim_{n\to+\infty}F^n(a)=s_1s_2s_3\cdots s_n\cdots\in\{a,b\}^{\mathbb{N}}$$

which is called the Fibonacci sequence. For any $n\ge 1$, write $\omega_n=\omega|_n=s_1s_2\cdots s_n$. We denote by $|\omega_n|_a$ the number of the occurrence of the letter a in ω_n, and $|\omega_n|_b$ the number of occurrence of b. Then $|\omega_n|_a+|\omega_n|_b=n$. It follows from [37] that $\lim\limits_{n\to+\infty}\frac{|\omega_n|_a}{n}=\eta$, where $\eta^2+\eta=1$.

Let $0<r_a<\frac{1}{2}, 0<r_b<\frac{1}{3}, r_a, r_b\in\mathbb{R}$. In the above Moran construction (Example 10), let

$$|J|=1,\quad n_k=\begin{cases}2, & \text{if } s_k=a\\ 3, & \text{if } s_k=b\end{cases}$$

and

$$c_k=\begin{cases}r_a, & \text{if } s_k=a\\ r_b, & \text{if } s_k=b\end{cases},\quad 1\le j\le n_k.$$

Then we construct the homogeneous Moran set relating to the Fibonacci sequence and denote it by $E(\omega)=\left(J,\{n_k\},\{c_k\}\right)$. By the construction of $E(\omega)$, we have

$$|J_\sigma|=r_a^{|\omega_k|_a}r_b^{|\omega_k|_b},\quad\forall\sigma\in D_k.$$

For $k\in\mathbb{N}$, define

$$s_k=-\frac{|\omega_k|_a\log(2)+|\omega_k|_b\log(3)}{|\omega_k|_a\log r_a+|\omega_k|_b\log r_b}.$$

It follows from [37] that

$$s:=\lim_{k\to+\infty}s_k=-\frac{\log(2)+\eta\log(3)}{\log r_a+\eta\log r_b}$$

exists, where $\eta^2+\eta=1$. By using Theorem 2, this implies that

$$S(\mathscr{H}^s(E(\omega)), m) = s = S(\mathscr{P}^s(E(\omega)), m).$$

The previous examples can be considered to construct conditional aggregation operators defined by the Sugeno integral with respect to Generalized Haudorff and packing measures.

Theorem 3. *Let (Ω, d) be a metric space and let $\mathbf{B}$ a partition of Ω. For each $B \in \mathbf{B}$ define*

$$\left(v^s(B), s_0\right) \in \left\{ \left(\mathscr{H}^{h_s}(B), \dim_{f,g}(B)\right), \left(\mathscr{P}^{h_s}(B), \mathrm{Dim}_{f,g}(B)\right) \right\}.$$

If $v^{s_0}(B) > 0$, then a conditional aggregation operator $A^{Su}(X|B)$ can be defined by the Sugeno integral on the class of all Sugeno integral random variables

$$A^{Su}(X|B) = \frac{1}{v^{s_0}s(B)} Su(X|B, v^{s_0}) \text{ if } v^{s_0}(B) \neq 0$$

and s_0 is the Sugeno integral of $v^s(B)$ as a function of s, with respect to the Lebesgue measure on the class of the Borel sets of $[0, +\infty)$,

$$S(v^s(B), m) = s_0.$$

Proof. It follows from the property of the Sugeno integral. □

In the following example, a set is constructed such that its Hausdorff measure is equal to zero and its packing measure is positive and finite in its dimension. So we can compute the Sugeno integral of the packing measure of the set with respect to the Lebesgue measure.

Example 13. Take $h_s = r^s$ for $s \geq 0$ and let $C_p^2 = C_p \times C_p$, where C_p is the middle-α Cantor set with $\alpha = 1 - 2p$ which obtained by repeated removal of the middle proportion α of intervals. Then the middle-α Cantor set can be written as

$$C_p = \left\{ x = (1-p) \sum_{i=0}^{\infty} a_i \, p^i \, : \, a_i \in \{0, 1\} \right\},$$

for $0 < p < \frac{1}{2}$ where $\alpha = 1 - 2p$. It is well known that $\dim_H(C_p) = \frac{\log 2}{\log p^{-1}}$.

The family of projections $\left\{ \mathrm{proj}_\theta \, C_p^2 \right\}_{0 \leq \theta < \pi}$ is affine-equivalent to the self-similar sets generated by iterated function systems $\{px, px+1, px+u, px+1+u\}_{u>0}$. We want to have $\dim_{\mathrm{H}} C_p^2 < 1$, so we assume that $p < \frac{1}{4}$. We may assume also, without loss of generality, that $u \geq 1$. It follows from [31] that for every $p \in \left(\frac{1}{6}, \frac{1}{4}\right)$ and for almost every $\theta \in \left[arctan\frac{1-2p}{p}, arctan\frac{2}{1-3p}\right]$ we obtain

$$\mathscr{H}^s\left(\mathrm{proj}_\theta \, C_p^2\right) = 0 \quad \text{and} \quad 0 < \mathscr{P}^s\left(\mathrm{proj}_\theta \, C_p^2\right) < \infty,$$

where s is the similarity dimension $\frac{\log 4}{\log p^{-1}}$. So we have

$$A^{Su}(X|B) = \frac{1}{\mathscr{P}^{\frac{\log 4}{\log p^{-1}}}\left(\mathrm{proj}_\theta \, C_p^2\right)} Su\left(X|B, \mathscr{P}^{\frac{\log 4}{\log p^{-1}}}\right)$$

and

$$S\left(\mathscr{P}^s\left(\mathrm{proj}_\theta \, C_p^2\right), m\right) = \frac{\log 4}{\log p^{-1}}.$$

7. Conclusions

The results proposed in this paper permit to put in evidence the relations among important concepts such as coherent conditional previsions, Choquet integral, Sugeno integral, dimensional fractal outer measures, and fractal dimensions. We have that a coherent upper conditional prevision of a Choquet integrable random variable $X|B$ can be defined as the Choquet integral of $X|B$ with respect to the s_0-dimensional fractal measure where s_0 is the fractal dimension of set B and the s_0-dimensional fractal measure of B is positive and finite; s_0 is the Sugeno integral with respect to the Lebesgue measure on the class of the Borel sets of $[0, +\infty)$ of the s_0-dimensional fractal outer measure of B. As a consequence for all sets E such that the fractal dimension s_0 is known, the Sugeno integral with respect to the Lebegue measure of the fractal measure $v^s(E)$, as a function of s, can be determined.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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