

FINDING DOMINATING INDUCED MATCHINGS IN P_9 -FREE GRAPHS IN POLYNOMIAL TIME

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Abstract

Let $G = (V, E)$ be a finite undirected graph. An edge subset $E' \subseteq E$ is a *dominating induced matching (d.i.m.)* in G if every edge in E is intersected by exactly one edge of E' . The *Dominating Induced Matching (DIM)* problem asks for the existence of a d.i.m. in G . The DIM problem is NP -complete even for very restricted graph classes such as planar bipartite graphs with maximum degree 3 but was solved in linear time for P_7 -free graphs and in polynomial time for P_8 -free graphs. In this paper, we solve it in polynomial time for P_9 -free graphs.

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1. INTRODUCTION

Let $G = (V, E)$ be a finite simple undirected graph, i.e., an undirected graph without loops and multiple edges. Given an edge $e \in E$, we say that e *dominates* itself and every edge sharing a vertex with e . An edge subset $M \subseteq E$ is an *induced matching* if the pairwise distance between its members is at least 2 (i.e.,

the *distance property*), that is, M is isomorphic to kP_2 for $k = |M|$. A subset $M \subseteq E$ is a *dominating induced matching* (*d.i.m.*, for short) of G if M is an induced matching in G such that every edge in E is dominated by exactly one edge in M . Clearly, not every graph G has a d.i.m.; the DOMINATING INDUCED MATCHING (DIM) problem asks for the existence of a d.i.m. in G .

The DIM problem is also called EFFICIENT EDGE DOMINATION (EED) in various papers. Recall that a vertex $v \in V$ *dominates* itself and its neighbors. A vertex subset $D \subseteq V$ is an *efficient dominating set* (*e.d.s.*, for short) of G if every vertex of G is dominated by exactly one vertex in D . The notion of efficient domination was introduced by Biggs [2] under the name *perfect code*. The EFFICIENT DOMINATION (ED) problem asks for the existence of an e.d.s. in a given graph G (note that not every graph has an e.d.s.). A set M of edges in a graph G is an *efficient edge dominating set* (*e.e.d.s.*, for short) of G if and only if it is an e.d.s. in its line graph $L(G)$. The EFFICIENT EDGE DOMINATION (EED) problem asks for the existence of an e.e.d.s. in a given graph G . Thus, the EED problem for a graph G corresponds to the ED problem for its line graph $L(G)$. Note that not every graph has an e.e.d.s.

In [10], it was shown that the DIM problem is NP-complete; see also [3, 9, 13, 14]. However, for various graph classes, DIM is solvable in polynomial time. For mentioning some examples, we need the following notions.

Let P_k denote the chordless path P with k vertices, say a_1, \dots, a_k , and $k - 1$ edges $a_i a_{i+1}$, $1 \leq i \leq k - 1$; we also denote it as $P = (a_1, \dots, a_k)$.

For indices $i, j, k \geq 0$, let $S_{i,j,k}$ denote the graph H with vertices $u, x_1, \dots, x_i, y_1, \dots, y_j, z_1, \dots, z_k$ such that the subgraph induced by u, x_1, \dots, x_i forms a P_{i+1} (u, x_1, \dots, x_i), the subgraph induced by u, y_1, \dots, y_j forms a P_{j+1} (u, y_1, \dots, y_j), and the subgraph induced by u, z_1, \dots, z_k forms a P_{k+1} (u, z_1, \dots, z_k), and there are no other edges in $S_{i,j,k}$; u is called the *center* of H . Thus, *claw* is $S_{1,1,1}$, and P_k is isomorphic to $S_{k-1,0,0}$.

For a set \mathcal{F} of graphs, a graph G is called \mathcal{F} -free if no induced subgraph of G is contained in \mathcal{F} . If $|\mathcal{F}| = 1$, say $\mathcal{F} = \{H\}$, then instead of $\{H\}$ -free, G is called H -free.

The following results are known.

Theorem 1. *DIM is solvable in polynomial time for*

- (i) $S_{1,1,1}$ -free graphs [9],
- (ii) $S_{1,2,3}$ -free graphs [12],
- (iii) $S_{2,2,2}$ -free graphs [11],
- (iv) $S_{1,2,4}$ -free graphs [6],
- (v) $S_{2,2,3}$ -free graphs [7],
- (vi) $S_{1,1,5}$ -free graphs [8],

- (vii) P_7 -free graphs [4] (in this case even in linear time),
- (viii) P_8 -free graphs [5].

In [11], it is conjectured that for every fixed i, j, k , DIM is solvable in polynomial time for $S_{i,j,k}$ -free graphs (actually, an even stronger conjecture is mentioned in [11]); this includes P_k -free graphs for $k \geq 9$. In this paper we show that DIM can be solved in polynomial time for P_9 -free graphs (generalizing the corresponding results for P_7 -free and for P_8 -free graphs).

2. DEFINITIONS AND BASIC PROPERTIES

2.1. Basic notions

Let G be a finite undirected graph without loops and multiple edges. Let $V(G)$ or V denote its vertex set and $E(G)$ or E its edge set (say $G = (V, E)$); let $n = |V|$ and $m = |E|$. For $v \in V$, let $N(v) = \{u \in V : uv \in E\}$ denote the *open neighborhood of v* , and let $N[v] = N(v) \cup \{v\}$ denote the *closed neighborhood of v* . If $xy \in E$, we also say that x and y *see each other*, and if $xy \notin E$, we say that x and y *miss each other*. A vertex set S is *independent* in G if for every pair of vertices $x, y \in S$, $xy \notin E$. A vertex set Q is a *clique* in G if for every pair of vertices $x, y \in Q$, $x \neq y$, $xy \in E$. For $uv \in E$ let $N(uv) = N(u) \cup N(v) \setminus \{u, v\}$ and $N[uv] = N[u] \cup N[v]$.

For $U \subseteq V$, let $G[U]$ denote the subgraph of G induced by vertex set U . Clearly $xy \in E$ is an edge in $G[U]$ exactly when $x \in U$ and $y \in U$; thus, $G[U]$ can simply be denoted by U (if understandable).

For $A \subseteq V$ and $B \subseteq V$, $A \cap B = \emptyset$, we say that $A \textcircled{0} B$ (A and B *miss each other*) if there is no edge between A and B , and A and B *see each other* if there is at least one edge between A and B . If a vertex $u \notin B$ has a neighbor $v \in B$ then u *contacts* B . If every vertex in A sees every vertex in B , we denote it by $A \textcircled{1} B$. For $A = \{a\}$, we simply denote $A \textcircled{1} B$ by $a \textcircled{1} B$, and correspondingly $A \textcircled{0} B$ by $a \textcircled{0} B$. If for $A' \subseteq A$, $A' \textcircled{0} (A \setminus A')$, we say that A' is *isolated* in $G[A]$. For graphs H_1, H_2 with disjoint vertex sets, $H_1 + H_2$ denotes the disjoint union of H_1, H_2 , and for $k \geq 2$, kH denotes the disjoint union of k copies of H . For example, $2P_2$ is the disjoint union of two edges.

As already mentioned, a *chordless path* P_k , $k \geq 2$, has k vertices, say v_1, \dots, v_k , and $k - 1$ edges $v_i v_{i+1}$, $1 \leq i \leq k - 1$; the *length of P_k* is $k - 1$. We also denote it as $P = (v_1, \dots, v_k)$.

A *chordless cycle* C_k , $k \geq 3$, has k vertices, say v_1, \dots, v_k , and k edges $v_i v_{i+1}$, $1 \leq i \leq k - 1$, and $v_k v_1$; the *length of C_k* is k .

Let K_i , $i \geq 1$, denote the clique with i vertices. Let $K_4 - e$ or *diamond* be the graph with four vertices, say v_1, v_2, v_3, u , such that (v_1, v_2, v_3) forms a P_3 and

$u \textcircled{1} \{v_1, v_2, v_3\}$; its *mid-edge* is the edge uv_2 .

A *butterfly* has five vertices, say, v_1, v_2, v_3, v_4, u , such that v_1, v_2, v_3, v_4 induce a $2P_2$ with edges v_1v_2 and v_3v_4 (the *peripheral edges* of the butterfly), and $u \textcircled{1} \{v_1, v_2, v_3, v_4\}$.

We often consider an edge $e = uv$ to be a set of two vertices; then it makes sense to say, for example, $u \in e$ and $e \cap e' \neq \emptyset$, for an edge e' . For two vertices $x, y \in V$, let $\text{dist}_G(x, y)$ denote the *distance between x and y in G* , i.e., the length of a shortest path between x and y in G . The *distance between a vertex z and an edge xy* is the length of a shortest path between z and x, y , i.e., $\text{dist}_G(z, xy) = \min\{\text{dist}_G(z, v) : v \in \{x, y\}\}$. The *distance between two edges $e, e' \in E$* is the length of a shortest path between e and e' , i.e., $\text{dist}_G(e, e') = \min\{\text{dist}_G(u, v) : u \in e, v \in e'\}$. In particular, this means that $\text{dist}_G(e, e') = 0$ if and only if $e \cap e' \neq \emptyset$.

Clearly, G has a d.i.m. if and only if every connected component of G has a d.i.m.; from now on, we consider that G is connected, and connected components of induced subgraphs of G are mentioned as *components*.

Note that if $G = (V, E)$ has a d.i.m. M , and $V(M)$ denotes the vertex set of M then $V \setminus V(M)$ is an independent set, say I , i.e.,

$$(1) \quad V \text{ has the partition } V = V(M) \cup I.$$

From now on, all vertices in I are colored white and all vertices in $V(M)$ are colored black. According to [11], we also use the following notions: A partial black-white coloring of V is *feasible* if the set of white vertices is an independent set in G and every black vertex has at most one black neighbor. A complete black-white coloring of V is *feasible* if the set of white vertices is an independent set in G and every black vertex has exactly one black neighbor. Clearly, M is a d.i.m. of G if and only if the black vertices $V(M)$ and the white vertices $V \setminus V(M)$ form a complete feasible coloring of V .

2.2. Reduction steps, forbidden subgraphs, forced edges, and excluded edges

Various papers on this topic introduced and applied some *forcing rules* for reducing the graph G to a subgraph G' such that G has a d.i.m. if and only if G' has a d.i.m., based on the condition that for a d.i.m. M , V has the partition $V = V(M) \cup I$ such that all vertices in $V(M)$ are black and all vertices in I are white (recall (1)).

A vertex $v \in V$ is *forced to be black* if for every d.i.m. M of G , $v \in V(M)$. Analogously, a vertex $v \in V$ is *forced to be white* if for every d.i.m. M of G , $v \notin V(M)$.

Clearly, if $uv \in E$ and if u, v are forced to be black, then uv is contained in every (possible) d.i.m. of G .

An edge $e \in E$ is a *forced* edge of G if for every d.i.m. M of G , $e \in M$. Analogously, an edge $e \in E$ is an *excluded* edge of G if for every d.i.m. M of G , $e \notin M$.

For the correctness of the reduction steps, we have to argue that G has a d.i.m. if and only if the reduced graph G' has one (provided that no contradiction arises in the vertex coloring, i.e., it is feasible).

Then let us introduce two reduction steps which will be applied later.

Vertex Reduction. Let $u \in V(G)$. If u is forced to be white, then

- (i) color black all neighbors of u , and
- (ii) remove u from G .

Let G' be the reduced subgraph. Clearly, Vertex Reduction is correct, i.e., G has a d.i.m. if and only if G' has a d.i.m.

Edge Reduction. Let $uv \in E(G)$. If u and v are forced to be black, then

- (i) color white all neighbors of u and of v (other than u and v), and
- (ii) remove u and v (and the edges containing u or v) from G .

Again, clearly, Edge Reduction is correct, i.e., G has a d.i.m. if and only if the reduced subgraph G' has a d.i.m.

The subsequent notions and observations lead to some possible reductions (some of them are mentioned e.g. in [3–5]).

Observation 1 [3–5]. *Let M be a d.i.m. of G .*

- (i) M contains at least one edge of every odd cycle C_{2k+1} in G , $k \geq 1$, and exactly one edge of every odd cycle C_3, C_5, C_7 in G .
- (ii) No edge of any C_4 can be in M .
- (iii) For each C_6 either exactly two or none of its edges are in M .

Proof. See e.g. Observation 2 in [4].

In what follows, we will also refer to Observation 1(i) (with respect to C_3) as to the *triangle-property*, and to Observation 1(ii) as to the *C_4 -property*.

Since by Observation 1(i), every triangle contains exactly one M -edge, and the pairwise distance of M -edges is at least 2, we have:

Corollary 1. *If G has a d.i.m. then G is K_4 -free.*

Assumption 1. From now on, by Corollary 1, we assume that the input graph is K_4 -free (else it has no d.i.m.).

Clearly, it can be checked (directly) in polynomial time whether the input graph is K_4 -free.

By Observation 1(i) with respect to C_3 and the distance property, we have the following.

Observation 2. *The mid-edge of any diamond in G and the two peripheral edges of any induced butterfly are forced edges of G .*

Assumption 2. From now on, by Observation 2, we assume that the input graph is (diamond,butterfly)-free.

In particular, we can apply the Edge Reduction to each mid-edge of any induced diamond and to each peripheral edge of any induced butterfly; that can be done in polynomial time.

Here is an example for excluded edges. By Observation 1(i), there is exactly one M -edge in the $C_3(v_1, v_2, v_3)$. Since G is K_4 - and diamond-free, every vertex $v \notin \{v_1, v_2, v_3\}$ which contacts the $C_3(v_1, v_2, v_3)$ has exactly one neighbor in (v_1, v_2, v_3) .

A *paw* has four vertices, say v_1, v_2, v_3, v_4 such that v_1, v_2, v_3 induce a C_3 and v_4 contacts exactly one vertex in v_1, v_2, v_3 , say $v_3v_4 \in E$. Thus, the edge $v_3v_4 \in E$ is excluded.

2.3. The distance levels of an M -edge xy in a P_3

Based on [5], we first describe some general structure properties for the distance levels of an edge in a d.i.m. M of G . Since G is $(K_4, \text{diamond}, \text{butterfly})$ -free, we have:

Observation 3. *For every vertex v of G , $N(v)$ is the disjoint union of isolated vertices and at most one edge. Moreover, for every edge $uv \in E$, there is at most one common neighbor of u and v .*

Since it is trivial to check whether G has a d.i.m. M with exactly one edge, from now on we can assume that $|M| \geq 2$.

Recall that the distance $dist_G(a, b)$ between two vertices a, b in graph G is the number of edges in a shortest path in G between a and b .

Theorem 2 [1]. *Every connected P_t -free graph $G = (V, E)$ admits a vertex $v \in V$ such that $dist_G(v, w) \leq \lfloor t/2 \rfloor$ for every $w \in V$.*

We call such a vertex v a *central* vertex; more exactly, a central vertex in G has shortest distance to every other vertex in G . Theorem 2 implies that every connected P_9 -free graph G admits a central vertex $v \in V$ such that $dist_G(v, w) \leq 4$ for every $w \in V$. For a central vertex v and a neighbor u of v , i.e., $uv \in E$, let

$$N_i(uv) = \{z \in V : dist_G(z, uv) = i\}$$

denote the *distance levels* of uv , $i \geq 1$. Then by Theorem 2, for every edge $uv \in E$, we have

$$(2) \quad N_k(uv) = \emptyset \text{ for every } k \geq 5.$$

Observation 4. *For every central vertex v in G , every edge $uv \in E$ is part of a P_3 of G .*

Proof. Let v be a central vertex in G , and suppose to the contrary that not every edge $uv \in E$ is part of a P_3 of G , say (u, v, w) induce a C_3 in G such that uv is not part of a P_3 , i.e., $N[u] = N[v] = \{u, v, w\}$. Clearly, in this case, w has more neighbors than v in G since G itself is no C_3 , i.e., w has a neighbor x with $xu \notin E$ and $xv \notin E$ (else there is a diamond or K_4). Moreover, w is not part of a triangle with x and y (else there is a butterfly in G). Then for every vertex $y \notin \{u, v, w\}$, $dist_G(w, y) < dist_G(v, y)$, which is a contradiction.

Thus, Observation 4 is shown. ■

Now assume that v is a central vertex in G such that every edge $uv \in E$ is part of a P_3 of G . Then one could check for any edge $uv \in E$ (with central vertex v), whether there is a d.i.m. M of G with $uv \in M$, and one could conclude: Either G has a d.i.m. M with $v \in V(M)$, or G has no d.i.m. M with $v \in V(M)$; in particular, in the latter case, if none of the edges uv is in a d.i.m. then v is white and one can apply the Vertex Reduction to v and in particular remove v .

Now assume that x is a central vertex (as in Observation 4), and let $xy \in M$ be an M -edge for which there is a vertex r such that $\{r, x, y\}$ induce a P_3 with edge $rx \in E$. By the assumption that $xy \in M$, we have that x and y are black, and it could lead to a feasible xy -coloring (if no contradiction arises).

Let $N_0(xy) = \{x, y\}$ and for $i \geq 1$, let

$$N_i(xy) = \{z \in V : dist_G(z, xy) = i\}$$

denote the *distance levels* of xy . Recall (2) which also shows that $N_5(xy) = \emptyset$. We consider a partition of V into $N_i = N_i(xy)$, $0 \leq i \leq 4$, with respect to the edge xy (under the assumption that $xy \in M$).

Observation 5. *If $v \in N_i$ for $i \geq 4$ then v is an endpoint of an induced P_6 , say with vertices $v, v_1, v_2, v_3, v_4, v_5$ such that $v_1, v_2, v_3, v_4, v_5 \in \{x, y\} \cup N_1 \cup \dots \cup N_{i-1}$ and with edges $vv_1 \in E, v_1v_2 \in E, v_2v_3 \in E, v_3v_4 \in E, v_4v_5 \in E$. Analogously, if $v \in N_3$ then v is an endpoint of a corresponding induced P_5 .*

Proof. If $i \geq 5$ then clearly there is such a P_6 . Thus, assume that $v \in N_4$. Then $v_1 \in N_3$ and $v_2 \in N_2$. Recall that y, x, r induce a P_3 . If $v_2r \in E$ then v, v_1, v_2, r, x, y induce a P_6 . Thus assume that $v_2r \notin E$. Let $v_3 \in N_1$ be a neighbor of v_2 . Now, if $v_3x \in E$ then v, v_1, v_2, v_3, x, r induce a P_6 , and if $v_3x \notin E$ but $v_3y \in E$ then v, v_1, v_2, v_3, y, x induce a P_6 . Analogously, if $v \in N_3$ then v is an endpoint of an induced P_5 (which could be part of the P_6 above). Thus, Observation 5 is shown. ■

Recall that by (1), $V = V(M) \cup I$ is a partition of V where $V(M)$ is the set of black vertices and I is the set of white vertices which is independent.

Since we assume that $xy \in M$ (and is an edge in a P_3), clearly, $N_1 \subseteq I$ and thus:

(3) N_1 is an independent set of white vertices.

Moreover, no edge between N_1 and N_2 is in M . Since $N_1 \subseteq I$ and all neighbors of vertices in I are in $V(M)$, we have

(4) $G[N_2]$ is the disjoint union of edges and isolated vertices.

Let M_2 denote the set of edges $uv \in E$ with $u, v \in N_2$ and let $S_2 = \{u_1, \dots, u_k\}$ denote the set of isolated vertices in N_2 ; $N_2 = V(M_2) \cup S_2$ is a partition of N_2 . Obviously

(5) $M_2 \subseteq M$ and $S_2 \subseteq V(M)$.

If for $xy \in E$, an edge $e \in E$ is contained in *every* d.i.m. M of G with $xy \in M$, we say that e is an *xy-forced* M -edge, and analogously, if an edge $e \in E$ is contained in *no* d.i.m. M of G with $xy \in M$, we say that e is *xy-excluded*. The Edge Reduction for forced edges can also be applied for *xy-forced* edges (then, in the unsuccessful case, G has no d.i.m. containing xy), and correspondingly for *xy-forced* white vertices (resulting from the black color of x and y), the Vertex Reduction can be applied.

Obviously, by (5), we have

(6) Every edge in M_2 is an *xy-forced* M -edge.

Thus, from now on, after applying the Edge Reduction for M_2 -edges, we can assume that $V(M_2) = \emptyset$, i.e., $N_2 = S_2 = \{u_1, \dots, u_k\}$. For every $i \in \{1, \dots, k\}$, let $u'_i \in N_3$ denote the M -mate of u_i (i.e., $u_i u'_i \in M$). Let $M_3 = \{u_i u'_i : 1 \leq i \leq k\}$ denote the set of M -edges with one endpoint in S_2 (and the other endpoint in N_3). Obviously, by (5) and the distance condition for a d.i.m. M , the following holds:

(7) No edge with both ends in N_3 and no edge between N_3 and N_4 is in M .

As a consequence of (7) and the fact that every triangle contains exactly one M -edge (recall Observation 1(i)), we have

(8) For every $C_3 abc$ with $a \in N_3$, and $b, c \in N_4$, $bc \in M$ is an *xy-forced* M -edge.

This means that for the edge bc , the Edge Reduction can be applied, and from now on, we can assume that there is no such triangle abc with $a \in N_3$ and $b, c \in N_4$, i.e., for every edge $uv \in E$ in N_4 :

$$(9) \quad N(u) \cap N(v) \cap N_3 = \emptyset.$$

According to (5) and the assumption that $V(M_2) = \emptyset$ (recall $N_2 = \{u_1, \dots, u_k\}$), let:

$$\begin{aligned} T_{one} &= \{t \in N_3 : |N(t) \cap N_2| = 1\}, \\ T_i &= T_{one} \cap N(u_i), \quad 1 \leq i \leq k, \text{ and} \\ S_3 &= N_3 \setminus T_{one}. \end{aligned}$$

By definition, T_i is the set of *private* neighbors of $u_i \in N_2$ in N_3 (note that $u'_i \in T_i$), $T_1 \cup \dots \cup T_k$ is a partition of T_{one} , and $T_{one} \cup S_3$ is a partition of N_3 .

Lemma 1 [5]. *The following statements hold.*

- (i) For all $i \in \{1, \dots, k\}$, $T_i \cap V(M) = \{u'_i\}$.
- (ii) For all $i \in \{1, \dots, k\}$, T_i is the disjoint union of vertices and at most one edge.
- (iii) $G[N_3]$ is bipartite.
- (iv) $S_3 \subseteq I$, i.e., S_3 is an independent subset of white vertices.
- (v) If a vertex $t_i \in T_i$ sees two vertices in T_j , $i \neq j$, $i, j \in \{1, \dots, k\}$, then $u_i t_i \in M$ is an *xy*-forced M -edge.

Proof. (i) Holds by definition of T_i and by the distance condition of a d.i.m. M .
 (ii) Holds by Observation 3.
 (iii) Follows by Observation 1(i) since every odd cycle in G must contain at least one M -edge, and by (7).
 (iv) If $v \in S_3 = N_3 \setminus T_{one}$, i.e., v sees at least two M -vertices then clearly, $v \in I$, and thus, $S_3 \subseteq I$ is an independent subset (recall that I is an independent set).

(v) Suppose that $t_1 \in T_1$ sees a and b in T_2 . If $ab \in E$ then u_2, a, b, t_1 would induce a diamond in G . Thus, $ab \notin E$ and now, u_2, a, b, t_1 induce a C_4 in G ; by Observation 1(ii), no edge in the C_4 is in M , and by (7), the only possible M -edge for dominating $t_1 a, t_1 b$ is $u_1 t_1$, i.e., $t_1 = u'_1$. ■

By Lemma 1(iv) and the Vertex Reduction for the white vertices of S_3 , we can assume:

$$(A1) \quad S_3 = \emptyset, \text{ i.e., } N_3 = T_1 \cup \dots \cup T_k.$$

By Lemma 1(v), we can assume:

(A2) For $i, j \in \{1, \dots, k\}$, $i \neq j$, every vertex $t_i \in T_i$ has at most one neighbor in T_j .

In particular, if for some $i \in \{1, \dots, k\}$, $T_i = \emptyset$, then there is no d.i.m. M of G with $xy \in M$, and if $|T_i| = 1$, say $T_i = \{t_i\}$, then u_it_i is an xy -forced M -edge. Thus, we can assume:

(A3) For every $i \in \{1, \dots, k\}$, $|T_i| \geq 2$.

Let us say that a vertex $t \in T_i$, $1 \leq i \leq k$, is an *out-vertex* of T_i if it is adjacent to some vertex of T_j with $j \neq i$, or it is adjacent to some vertex of N_4 , and t is an *in-vertex* of T_i otherwise.

For finding a d.i.m. M with $xy \in M$, one can remove all but one in-vertices; that can be done in polynomial time. In particular, if there is an edge between two in-vertices $t_1t_2 \in E$, $t_1, t_2 \in T_i$, then either t_1 or t_2 is black, and thus, T_i is completely colored. Thus, let us assume:

(A4) For every $i \in \{1, \dots, k\}$, T_i has at most one in-vertex.

Lemma 2. *Assume that G has a d.i.m. M with $xy \in M$. Then:*

- (i) *For every $i \neq j$, there are at most two edges between T_i and T_j .*
- (ii) *If there are two edges between T_i and T_j , say $t_it_j \in E$ and $t'_it'_j \in E$ for $t_i, t'_i \in T_i$ and $t_j, t'_j \in T_j$, $t_i \neq t'_i$, $t_j \neq t'_j$, then every vertex in $(T_i \cup T_j) \setminus \{t_i, t_j, t'_i, t'_j\}$ is white.*

Proof. (i) Suppose to the contrary that there are three edges between T_1 and T_2 , say $t_1t_2 \in E$, $t'_1t'_2 \in E$, and $t''_1t''_2 \in E$ for $t_i, t'_i, t''_i \in T_i$, $i = 1, 2$. By (A2), t_i, t'_i, t''_i are distinct. Then t_1 is black if and only if t_2 is white, t'_1 is black if and only if t'_2 is white, and t''_1 is black if and only if t''_2 is white. Without loss of generality, assume that t_1 is black, and t_2 is white. Then t'_1 is white, and t'_2 is black, but now, t''_1 and t''_2 are white, which is a contradiction.

(ii) Let $t_1t_2 \in E$, $t'_1t'_2 \in E$, be two such edges between T_1 and T_2 . By (A2), $t_1 \neq t'_1$, and $t_2 \neq t'_2$. Then again, t_1 or t'_1 is black as well as t_2 or t'_2 is black, and thus, every other vertex in T_1 or T_2 is white.

Thus Lemma 2 is shown. ■

By Lemma 2(i), we can assume:

(A5) For $i, j \in \{1, \dots, k\}$, $i \neq j$, there are at most two edges between T_i and T_j .

Recall that $|T_i| \geq 2$. If there is an edge in T_i , say $ab \in E$ with $a, b \in T_i$ and there is a third vertex $c \in T_i$ then either a or b is black, and thus, by Lemma 1(i), c is forced to be white, and by the Vertex Reduction and by Lemma 2 (ii), we can assume:

(A6) If there is an edge in T_i then $|T_i| = 2$. Analogously, if there are two edges between T_i and T_j then $|T_i| = 2$ and $|T_j| = 2$.

Then let us introduce the following forcing rules (which are correct). Since no edge in N_3 is in M (recall (7)), we have:

- (R1) All N_3 -neighbors of a black vertex in N_3 must be colored white, and all N_3 -neighbors of a white vertex in N_3 must be colored black.

Moreover, we have:

- (R2) Every T_i , $i \in \{1, \dots, k\}$, should contain exactly one vertex which is black. Thus, if $t \in T_i$ is black then all the remaining vertices in $T_i \setminus \{t\}$ must be colored white.
- (R3) If all but one vertices of T_i , $1 \leq i \leq k$, are white and the final vertex $t \in T_i$ is not yet colored, then t must be colored black.

Since no edge between N_3 and N_4 is in M (recall (7)), we have:

- (R4) For every edge $st \in E$ with $t \in N_3$ and $s \in N_4$, s is white if and only if t is black and vice versa.

Subsequently, for checking if G has a d.i.m. M with $xy \in M$, we consider the cases $N_4 = \emptyset$ and $N_4 \neq \emptyset$.

Then let us introduce the following recursive algorithm which formalizes the approach we will adopt to check if G has a d.i.m.

Algorithm DIM(G)

Input. A connected P_9 - and $(K_4, \text{diamond}, \text{butterfly})$ -free graph $G = (V, E)$.

Output. A d.i.m. of G or the proof that G has no d.i.m.

- (A) Compute a central vertex, say x , of G such that $dist_G(x, u) \leq 4$ for every $u \in V$ and every edge $xy \in E$ is part of a P_3 of G .
- (B) For each edge $xy \in E$ of G [contained in a P_3 of G] do:
 - (B.1) compute the distance levels N_i with respect to xy and apply the reduction steps as shown above: if no contradiction arose and if assumptions (A1)–(A6) hold, then go to Step (B.2), else take another edge with x ;
 - (B.2) check if G has a d.i.m. M with $xy \in M$; if *yes*, then return it, and STOP.
- (C) Apply the Vertex Reduction to x [and in particular remove x]; let G' denote the resulting graph, where the neighbors of x in G are colored by black; if G' is disconnected, then execute Algorithm DIM(H) for each connected component H of G' ; otherwise, go to Step (B), with $G = G'$.
- (D) Return “ G has no d.i.m.” and STOP.

Then, by the above, Algorithm DIM(G) is correct and can be executed in polynomial time as soon as Step (B.2) can be so.

Then in what follows let us try to show that Step (B.2) can be solved in polynomial time, with the agreement that G is $(K_4, \text{diamond}, \text{butterfly})$ -free and enjoys assumptions (A1)–(A6): in particular recall (2) that $N_k = \emptyset$ for $k \geq 5$.

Thus we consider the cases $N_4 = \emptyset$ and $N_4 \neq \emptyset$. Let $A_{xy} = \{x, y\} \cup N_1 \cup N_2 \cup N_3$, and recall $N_4 = V \setminus A_{xy}$.

3. THE CASE $N_4 = \emptyset$

In this section, we show that for the case $N_4 = \emptyset$, one can check in polynomial time whether G has a d.i.m. M with $xy \in M$; we consider the feasible xy -colorings for $G[A_{xy}]$. Recall that for every edge $uv \in M$, u and v are black, for $I = V(G) \setminus V(M)$, every vertex in I is white, $N_2 = S_2 = \{u_1, \dots, u_k\}$ and all u_i , $1 \leq i \leq k$, are black, $T_i = N(u_i) \cap N_3$, and recall assumptions (A1)–(A6) and rules (R1)–(R4). In particular, by (A1), $S_3 = \emptyset$, i.e., $N_3 = T_1 \cup \dots \cup T_k$.

Clearly, in the case $N_4 = \emptyset$, all the components of $G[S_2 \cup N_3]$ can be independently colored. Every component with at most three S_2 -vertices has a polynomial number of feasible xy -colorings. Thus, we can focus on components K with at least four S_2 -vertices.

A $P_2(u, v)$ in $G[N_3]$ is *isolated* in $G[N_3]$ if it is not part of a P_3 in $G[N_3]$.

Claim 1. *If every P_2 in component K in $G[S_2 \cup N_3]$ is isolated then K has at most three S_2 -vertices.*

Proof. Suppose to the contrary that K has at least four S_2 -vertices, say u_1, u_2, u_3, u_4 , and without loss of generality, assume that T_2 contacts T_1 and T_3 , say $t'_1 t_2 \in E$ and $t'_2 t_3 \in E$ for $t'_1 \in T_1$, $t_2, t'_2 \in T_2$, and $t_3 \in T_3$. By the isolated edges, $(u_1, t'_1, t_2, u_2, t'_2, t_3, u_3)$ induce a P_7 .

Case 1. T_4 contacts T_1 or T_3 . Without loss of generality, assume that T_4 contacts T_3 , i.e., there is a $t_4 \in T_4$ which contacts a vertex in T_3 . Clearly, $t_3 t_4 \notin E$ since $t'_2 t_3 \in E$ is isolated. Then $t'_3 t_4 \notin E$ for a second vertex $t'_3 \in T_3$, and clearly, t'_3 and t_4 do not contact the edges $t'_1 t_2$ and $t'_2 t_3$ but then $(u_1, t'_1, t_2, u_2, t'_2, t_3, u_3, t'_3, t_4)$ induce a P_9 , which is a contradiction.

Case 2. T_4 contacts T_2 but does not contact T_1 and T_3 . Let $t_4 \in T_4$ contact T_2 . Clearly, by the isolated edges, t_4 does not contact $t_2, t'_2 \in T_2$. Thus assume that $t''_2 t_4 \in E$ for a third vertex $t''_2 \in T_2$. By (A3), there is a second vertex $t'_4 \in T_4$ and a second vertex $t'_3 \in T_3$, and clearly, $t_3 t'_3 \notin E$, $t_4 t'_4 \notin E$ and t'_4 does not contact T_3 and t'_4 does not contact $t'_2, t''_2 \in T_2$. But then $(t'_4, u_4, t_4, t''_2, u_2, t'_2, t_3, u_3, t'_3)$ induce a P_9 , which is a contradiction.

Thus, Claim 1 is shown. □

From now on, we can assume that there is at least one P_3 with contact between T_i and T_{i+1} in K .

Claim 2. For any P_3 's (a, b, c) and (d, e, f) in $G[N_3]$ such that d, e, f are not in the T_i 's of a, b, c , there is an edge between $\{a, b, c\}$ and $\{d, e, f\}$.

Proof. Suppose to the contrary that there is no such edge between the P_3 's (a, b, c) and (d, e, f) in $G[N_3]$. From Lemma 1(ii), a, b, c are in at least two T_i 's; assume that $a \in T_1$. Then, by Lemma 1(v) and since $ab \in E, bc \in E$, we have $c \notin T_1$; let $c \in T_2$, i.e., $u_1c \notin E$. Then either $b \notin T_1$ or $b \notin T_2$; without loss of generality, let $b \notin T_1$. Analogously, since d, e, f are not in $T_1 \cup T_2$, assume that $d \in T_3$ and $e, f \notin T_3$.

Let P be any induced path in G between u_1 and u_3 through $N_1 \cup \{x, y\}$. Then the subgraph of G induced by $(c, b, a, u_1), P$, and (u_3, d, e, f) contains an induced P_9 , which is a contradiction. Thus, Claim 2 is shown. \square

For a P_5 $P = (a, b, c, d, e)$ in $G[N_3]$ with $a \in T_i$ and $b, c, d, e \notin T_i$, vertex a is a special P_5 -endpoint of P in $G[N_3]$.

Claim 3. There is no P_5 (a, b, c, d, e) in $G[N_3]$ with special P_5 -endpoint a .

Proof. Suppose to the contrary that (a, b, c, d, e) is a P_5 in $G[N_3]$ with special P_5 -endpoint $a \in T_i$ and $b, c, d, e \notin T_i$. But then by Observation 5, vertex a is the midpoint of a P_9 , which is a contradiction. Thus, Claim 3 is shown. \square

Claim 4. If $C = (t_i, u_i, t'_i, t_j, t_h, t_\ell)$ is a C_6 in $G[S_2 \cup N_3]$ with exactly one vertex $u_i \in S_2$ and $t_i, t'_i \in T_i, t_j \in T_j, t_h \in T_h, t_\ell \in T_\ell$ (possibly $j = h$ or $h = \ell$) then t_j and t_ℓ are xy -forced to be black, i.e., $u_j t_j$ and $u_\ell t_\ell$ are xy -forced M -edges, and thus, T_j and T_ℓ are completely colored.

Proof. By (7), no edge in N_3 is in M . By Observation 1(iii), either exactly two or none of the edges in C are in M . Since C has exactly one vertex $u_i \in S_2$, $u_i t_i$ and $u_i t'_i$ are the only edges of C which are not in N_3 , and clearly, either $u_i t_i \notin M$ or $u_i t'_i \notin M$. Thus, by Observation 1(iii), no edge in C is in M , i.e., t_i and t'_i are white, and t_j as well as t_ℓ are xy -forced to be black. Thus, Claim 4 is shown. \square

After the Edge Reduction step, we can assume that there is no such C_6 in $G[S_2 \cup N_3]$, i.e., every C_6 in $G[S_2 \cup N_3]$ has either two vertices of S_2 or none of it.

Claim 5. If C is a C_7 in $G[S_2 \cup N_3]$ then C has exactly two vertices in S_2 , say $C = (t_i, u_i, t'_i, t_j, u_j, t'_j, t_h)$, and then t_h is xy -forced to be black, i.e., $u_h t_h$ is an xy -forced M -edge.

Proof. Let C be a C_7 in $G[S_2 \cup N_3]$. Recall that by Lemma 1(iii), there is no C_7 in $G[N_3]$. Thus, $|V(C) \cap S_2| \geq 1$, and clearly, by (A1), no vertex in $V(C) \cap N_3$ contacts two vertices in $V(C) \cap S_2$, i.e., $|V(C) \cap S_2| \leq 2$.

If there is exactly one S_2 -vertex in a C_7 in $G[N_3]$, say $C = (t_1, u_1, t'_1, t_2, t', t'', t''')$ with $t_1, t'_1 \in T_1$ then $t_2, t', t'', t''' \notin T_1$, but now, $(t'_1, t_2, t', t'', t''')$ induce

a P_5 with special P_5 -endpoint $t'_1 \in T_1$ such that $t_2, t', t'', t''' \notin T_1$, which is a contradiction to Claim 3.

Now assume that $C = (t_1, u_1, t'_1, t_2, u_2, t'_2, t_3)$ is a C_7 in $G[S_2 \cup N_3]$. Suppose to the contrary that t_3 is white. Then t_1 and t'_2 are black which implies that t'_1 and t_2 are white, which is a contradiction since $t'_1 t_2 \in E$. Thus, t_3 is xy -forced to be black, i.e., $u_3 t_3$ is an xy -forced M -edge, and Claim 5 is shown. \square

After the Edge Reduction step, we can assume that there is no C_7 in $G[S_2 \cup N_3]$.

Claim 6. *If there is a C_9 C in $G[S_2 \cup N_3]$ then $|V(C) \cap S_2| = 3$, say $V(C) \cap S_2 = \{u_1, u_2, u_3\}$, and for the component K in $G[S_2 \cup N_3]$ containing C , we have $K = G[\{u_1, u_2, u_3\} \cup T_1 \cup T_2 \cup T_3]$.*

Proof. Let C be a C_9 in $G[S_2 \cup N_3]$. Recall that by Lemma 1(iii), there is no C_9 in $G[N_3]$, i.e., $|V(C) \cap S_2| \geq 1$, and clearly, $|V(C) \cap S_2| \leq 3$.

If C contains only one S_2 -vertex then, as in the proof of Claim 5, it leads to a P_5 in N_3 with corresponding special P_5 -endpoint, which is a contradiction to Claim 3. Thus, $|V(C) \cap S_2| \geq 2$.

First assume that $|V(C) \cap S_2| = 2$. If $C = (t_1, u_1, t'_1, t_2, u_2, t'_2, t_3, t_4, t_5)$ (possibly $t_3, t_4 \in T_3$ or $t_4, t_5 \in T_4$) then this leads to a P_5 $(t'_2, t_3, t_4, t_5, t_1)$ with special P_5 -endpoint t_1 , which is a contradiction to Claim 3. If $C = (t_1, u_1, t'_1, t_2, t_3, u_3, t'_3, t_4, t_5)$ then t_2 is xy -forced to be black: Suppose to the contrary that t_2 is white. Then t'_1 and t_3 are black, which implies that t_1 and t'_3 are white, but now, t_4 and t_5 are black, which is a contradiction since there is no M -edge in N_3 . Thus, $u_2 t_2$ is an xy -forced M -edge, and after the Edge Reduction, $|V(C) \cap S_2| = 2$ is impossible.

Thus, $|V(C) \cap N_2| = 3$; let $C = (t_1, u_1, t'_1, t_2, u_2, t'_2, t_3, u_3, t'_3)$ be a C_9 with three such S_2 -vertices u_1, u_2, u_3 . Suppose to the contrary that there is a vertex $t_4 \in T_4$ which contacts C , say $t'_3 t_4 \in E$. Clearly, by Lemma 1(v), $t_4 t_3 \notin E$. Since $(u_1, t'_1, t_2, u_2, t'_2, t_3, u_3, t'_3, t_4)$ do not induce a P_9 , we have $t_4 t'_1 \in E$ or $t_4 t_2 \in E$ or $t_4 t'_2 \in E$.

If $t_4 t'_1 \in E$ then $(t_2, t'_1, t_4, t'_3, t_1)$ would induce a P_5 in N_3 with special P_5 -endpoint t_2 , which is impossible by Claim 3. Similarly, if $t_4 t'_2 \in E$ then $(t_3, t'_2, t_4, t'_3, t_1)$ would induce a P_5 in N_3 with special P_5 -endpoint t_1 , which is a contradiction to Claim 3.

Thus, $t_4 t_2 \in E$ which leads to a C_7 $(t_2, u_2, t'_2, t_3, u_3, t'_3, t_4)$. But then, by Claim 5, t_4 is xy -forced to be black, i.e., $u_4 t_4$ is an xy -forced M -edge, and after the Edge Reduction, there is no such C_7 . Thus, Claim 6 is shown. \square

Corollary 2. *Every component in $G[S_2 \cup N_3]$ with at least four S_2 -vertices is C_9 -free.*

Lemma 3. *In the case $N_4 = \emptyset$, for every component K in $G[S_2 \cup N_3]$, a complete coloring of K (if there is no contradiction) can be done in polynomial time.*

Proof. For finding a complete feasible xy -coloring of component K (or a contradiction), we first use Vertex Reduction and Edge Reduction as in the previous results.

Let $V(K) \cap N_3 = T_1 \cup \dots \cup T_h$ (recall that $h \geq 4$ since otherwise, a complete feasible xy -coloring of K can be done in polynomial time). Clearly, for every i , $1 \leq i \leq h$, we have $|T_i| \geq 2$.

If every T_i in K would have only one out-vertex $t_i \in T_i$, then the procedure starts by fixing a coloring of T_1 ; for every i , $1 \leq i \leq h$, there are only two possible colorings of T_i since by (A4), every T_i has at most one in-vertex. If the already colored out-vertex $t_i \in T_i$ with contact to $t_{i+1} \in T_{i+1}$ is white then t_{i+1} is black, the in-vertex of T_{i+1} is white, and T_{i+1} is completely colored. Analogously, if t_i is black then t_{i+1} is white, the in-vertex of T_{i+1} is black, and T_{i+1} is completely colored.

Thus, we can assume that there is a T_i with at least two out-vertices (such that at least one of them is white). In this case, the procedure starts by fixing a coloring of T_1 with at least two out-vertices (this can be repeated for all $|T_1|$ colorings of T_1) and applies the forcing rules, and then the next step of the procedure is using a white out-vertex $t_1 \in T_1$, say with contact to T_2 , such that the neighbor $t_2 \in T_2$ of t_1 is black. If t_2 contacts only T_1 then t_2 does not play any role for the procedure. If t_2 contacts some T_j , $j \neq 1, 2$, the problem is how T_j can be completely colored.

Now we can assume that every black out-vertex (which was already colored by a white neighbor in the previous step) contacts at least two T_i 's, say, $t_2 \in T_2$ was colored black by a white vertex $t'_1 \in T_1$ with $t'_1 t_2 \in E$ (i.e., T_1 was already colored), and $t_2 t_3 \in E$ for $t_3 \in T_3$ but T_3 is not yet completely colored. Then t_3 is white, and if $t_3 t'_3 \in E$ for another $t'_3 \in T_3$ then t'_3 is black and T_3 is completely colored. Thus assume that $t_3 t'_3 \notin E$ for any $t'_3 \in T_3$. If t_3 is the only out-vertex in T_3 then the in-vertex is black (recall that by (A4), every T_i has at most one in-vertex) and T_3 is completely colored. Thus assume that t'_3 is an out-vertex, say $t'_3 t_4 \in E$ for $t_4 \in T_4$. We first show:

Claim 7. *If there is any contact between the P_3 (t'_1, t_2, t_3) and the P_2 (t'_3, t_4) then T_3 is completely colored.*

Proof. Clearly, $t_3 t'_3 \notin E$, $t_3 t_4 \notin E$, and $t_2 t'_3 \notin E$. If $t'_1 t'_3 \in E$ then t'_3 is black and T_3 is completely colored. Thus assume that $t'_1 t'_3 \notin E$. If $t_2 t_4 \in E$ then t_4 is white and thus, t'_3 is black and T_3 is completely colored. Thus assume that $t_2 t_4 \notin E$.

Finally, if $t'_1 t_4 \in E$ then $(t'_1, t_2, t_3, t_4, t'_3, t_4)$ induce a C_6 , which is impossible by Claim 4 and the Edge Reduction.

Thus, T_3 is completely colored. □

Now we assume

$$(t'_1, t_2, t_3) \textcircled{\circ} (t'_3, t_4).$$

Moreover, $(u_1, t'_1, t_2, t_3, u_3, t'_3, t_4, u_4)$ induce a P_8 in G . Clearly, $|T_1| \geq 2$ and $|T_4| \geq 2$; let $t_1 \in T_1$ be a second vertex in T_1 and $t'_4 \in T_4$ be a second vertex in T_4 .

Claim 8. *If $t_4 t'_4 \notin E$, then T_3 is completely colored.*

Proof. If $t_4 t'_4 \notin E$ then clearly, $t'_3 t'_4 \notin E$. Since $(u_1, t'_1, t_2, t_3, u_3, t'_3, t_4, u_4, t'_4)$ do not induce a P_9 in G , we have $t'_1 t'_4 \in E$ or $t_2 t'_4 \in E$ or $t_3 t'_4 \in E$. If $t_3 t'_4 \in E$ then $|T_3| = 2$ (recall (A6)) and T_3 is completely colored. Thus assume that $t_3 t'_4 \notin E$. Now, if $t_2 t'_4 \in E$ then $(t_2, t_3, u_3, t'_3, t_4, u_4, t'_4)$ induce a C_7 , which is impossible by Claim 5 and the Edge Reduction. Thus, $t_2 t'_4 \notin E$ which implies that $t'_1 t'_4 \in E$. But now, since t'_1 is white, t'_4 is black, t_4 is white, t'_3 is black, and T_3 is completely colored. □

From now on, we assume

$$t_4 t'_4 \in E.$$

Case 1. $t_1 t'_1 \in E$. Then t_1 is black. By Claim 2, (t_2, t'_1, t_1) and (t'_3, t_4, t'_4) do not induce a $2P_3$. Recall that (t_2, t'_1) and (t'_3, t_4) induce a $2P_2$. Thus, t_1 should contact (t'_3, t_4, t'_4) or t'_4 should contact (t_2, t'_1, t_1) .

If $t'_1 t'_4 \in E$ then t'_4 is black, t_4 is white, t'_3 is black, and T_3 is completely colored. Thus assume $t'_1 t'_4 \notin E$. Analogously, if $t_1 t_4 \in E$ then t_4 is white, and t'_3 is black, and T_3 is completely colored. Thus assume $t_1 t_4 \notin E$.

If $t_1 t_3 \in E$ then clearly, $t_1 t'_3 \notin E$, and if $t_1 t_3 \notin E$ but $t_1 t'_3 \in E$ then $(t_1, t'_1, t_2, t_3, u_3, t'_3)$ induce a C_6 which is impossible by Claim 4 and the Edge Reduction. Thus, assume that $t_1 t'_3 \notin E$.

If $t_2 t'_4 \in E$ then $(t_2, t_3, u_3, t'_3, t_4, t'_4)$ induce a C_6 which is impossible by Claim 4 and the Edge Reduction. Thus, assume that $t_2 t'_4 \notin E$.

Now, $t_1 t'_4 \in E$ is the only possible edge between (t_2, t'_1, t_1) and (t'_3, t_4, t'_4) but now, $(t_2, t'_1, t_1, t'_4, t_4)$ induce a P_5 with special endpoint t_2 , which is impossible by Claim 3.

Thus, in Case 1, T_3 is completely colored.

Case 2. $t_1 t'_1 \notin E$. Recall that t'_1 and t_3 are white and $(t'_1, t_2, t_3) \textcircled{\circ} (t'_3, t_4)$. Clearly, $t_1 t_2 \notin E$, and since $(t_1, u_1, t'_1, t_2, t_3, u_3, t'_3, t_4, u_4)$ do not induce a P_9 in G , we have $t_1 t_3 \in E$ or $t_1 t'_3 \in E$ or $t_1 t_4 \in E$.

If $t_1 t'_3 \in E$ then clearly, $t_1 t_3 \notin E$. But then $(t_1, u_1, t'_1, t_2, t_3, u_3, t'_3)$ induce a C_7 , which is impossible by Claim 5 and the Edge Reduction. Thus, assume $t_1 t'_3 \notin E$ and either $t_1 t_3 \in E$ or $t_1 t_4 \in E$.

Case 2.1. $t_1 t_3 \notin E$. Then $t_1 t_4 \in E$. First assume that t_1 is white, which implies that t_4 is black, and there is a black vertex $t''_1 \in T_1$. Clearly,

$t''_1 t_1 \notin E$, $t''_1 t'_1 \notin E$, $t'_1 t_2 \notin E$, and since t_4 is black, we have $t'_1 t_4 \notin E$. Since $(t''_1, u_1, t'_1, t_2, t_3, u_3, t'_3, t_4, u_4)$ do not induce a P_9 in G , we have $t''_1 t_3 \in E$ or $t''_1 t'_3 \in E$.

If $t''_1 t'_3 \in E$ then $t''_1 t_3 \notin E$, but now $(t''_1, u_1, t'_1, t_2, t_3, u_3, t'_3)$ induce a C_7 , which is impossible by Claim 5 and the Edge Reduction. Thus, assume $t''_1 t'_3 \notin E$ which implies $t''_1 t_3 \in E$. But now $(t''_1, u_1, t_1, t_4, t'_3, u_3, t_3)$ induce a C_7 , which is impossible by Claim 5 and the Edge Reduction.

Thus t_1 is black which implies that t_4 is white, t'_3 is black, T_3 is completely colored, and Case 2.1 is done.

Case 2.2. $t_1 t_3 \in E$. Since t_3 is white, t_1 is black. Clearly, since $t_1 t_3 \in E$, we have $t_1 t'_3 \notin E$. If $t_1 t_4 \in E$ then t_4 is white and thus, t'_3 is black and T_3 is completely colored. Thus, assume that $t_1 t_4 \notin E$.

If $|T_1| \geq 3$ then let $t''_1 \in T_1$ be a second white vertex in T_1 . Then $t''_1 t_3 \notin E$, and Case 2.1 applies for $t''_1 \in T_1$. Thus we have

$$|T_1| = 2.$$

Next assume that the black out-vertex t_1 has a second white neighbor, say $t_0 \in T_0$ with $t_0 t_1 \in E$. Since $(t_0, t_1, t_3, t_2, t'_1)$ do not induce a P_5 with special endpoint t_0 (recall Claim 3), we have $t_0 t_2 \in E$. Recall that $t'_1 t'_4 \notin E$ (else T_3 is completely colored) and $t_2 t'_4 \notin E$ (else $(t_2, t_3, u_3, t'_3, t_4, t'_4)$ induce a C_6 which is impossible by Claim 4 and the Edge Reduction).

Since (t'_1, t_2, t_0) and (t'_3, t_4, t'_4) do not induce a $2P_3$ (recall Claim 2), we have $t_0 t'_3 \in E$ or $t_0 t_4 \in E$ or $t_0 t'_4 \in E$. If $t_0 t'_3 \in E$ or $t_0 t'_4 \in E$ then t'_3 is black and T_3 is completely colored. Thus assume that $t_0 t_4 \in E$. But now, $(t_0, t_1, t_3, u_3, t'_3, t_4)$ induce a C_6 which is impossible by Claim 4 and the Edge Reduction. Thus, t_1 has only one white neighbor, namely t_3 .

Next we show:

Claim 9. t'_4 is no out-vertex.

Proof. Suppose to the contrary that $t'_4 t_5 \in E$ for some $t_5 \in T_5$. Recall that t_4 does not contact t'_1, t_2, t_3 . Since t'_4 is white (else T_3 is completely colored), t'_4 does not contact t'_1, t_3 , and recall that $t'_4 t_2 \notin E$ (else $(t_2, t_3, u_3, t'_3, t_4, t'_4)$ induce a C_6 , which is impossible by Claim 4 and the Edge Reduction).

Recall that (t'_1, t_2, t_3) and (t_4, t'_4, t_5) do not induce a $2P_3$. Thus, since t_4 and t'_4 do not contact (t'_1, t_2, t_3) , only t_5 could contact (t'_1, t_2, t_3) . Since t_2 and t_5 are black, we have $t_5 t_2 \notin E$. If $t_5 t_3 \in E$ then clearly, $t_5 t'_3 \notin E$ but then $(t_3, u_3, t'_3, t_4, t'_4, t_5)$ induce a C_6 , which is impossible by Claim 4 and the Edge Reduction. Thus, $t_5 t_3 \notin E$. Now, if $t_5 t'_1 \in E$ then $(t_1, t_3, t_2, t'_1, t_5)$ induce a P_5 with special endpoint t_5 , which is a contradiction by Claim 3.

Thus $t_5 t'_1 \notin E$, $t_5 t_2 \notin E$, and $t_5 t_3 \notin E$. But then (t'_1, t_2, t_3) and (t_4, t'_4, t_5) induce a $2P_3$, which is a contradiction by Claim 2. Thus, Claim 9 is shown. \square

This implies that t_4 is the only out-vertex in T_4 .

Next we show:

Claim 10. *The black vertex t_2 contacts only one T_i (namely T_3) which is not yet completely colored.*

Proof. Suppose to the contrary that t_2 contacts a second T_i (apart from T_3) which is not yet completely colored. Then $T_i \neq T_4$ since by the above $T_4 = \{t_4, t'_4\}$ and t_2 is nonadjacent to t_4, t'_4 . Then say $T_i = T_5$, i.e., t_2 contacts T_5 with $t_2t_5 \in E$. Note that t_5 does not contact T_3 (else T_3 is completely colored) and does not contact T_4 (else, by Claim 9, $t_5t_4 \in E$ and then vertices $t_2, t_3, u_3, t'_3, t_4, t_5$ would induce a C_6 according to Claim 4). Since T_5 is not yet completely colored, T_5 is not contacted by t'_1 and t_3 , and furthermore (similarly to the above with respect to T_3) there is an out-vertex of T_5 say $t'_5 \in T_5$ (nonadjacent to t_5). Note that t'_5 does not contact T_3 (else, say $t'_5t''_3 \in E$ with $t''_3 \in T_3$, vertices $t_2, t_3, u_3, t'_3, t'_5, u_5, t_5$ induce a C_7 in $G[S_2 \cup N_3]$) and does not contact T_4 (else, by Claim 9, $t'_5t_4 \in E$ and then vertices $u_1, t'_1, t_2, t_5, u_5, t'_5, t_4, t'_3, u_3$ induce a P_9). Then vertices $t'_5, u_5, t_5, t_2, t_3, u_3, t'_3, t_4, t'_4$ induce a P_9 , which is a contradiction.

Thus, Claim 10 is shown. □

Finally we show:

Claim 11. *There is only one black vertex which contacts a T_i which is not yet completely colored.*

Proof. Suppose to the contrary that there are two such black vertices, say t_2 which contacts T_3 and t_6 which contacts T_7 such that T_3, T_7 are not yet completely colored. By Claim 10, t_2 does not contact T_7 and t_6 does not contact T_3 . If $t'_1 \neq t'_5$ then clearly, the P_3 's (t'_1, t_2, t_3) and (t'_5, t_6, t_7) do not induce a $2P_3$ but $t_2t_7 \notin E$ and $t_6t_3 \notin E$. Thus, $t'_1t_6 \in E$ or $t'_5t_2 \in E$, say without loss of generality, $t'_1t_6 \in E$ but now, $(t_3, t_2, t'_1, t_6, t_7)$ induce a P_5 with special endpoint t_3 , which is a contradiction to Claim 3. Analogously, if $t'_1 = t'_5$, it leads to the same contradiction. Thus, Claim 11 is shown. □

In general, if t_2 does not completely color T_3 then we can add a possible coloring of T_3 which leads to a complete coloring of every neighbor T_i of T_3 . Since G is P_9 -free, Case 2.2 appears only once in component K .

Thus, Lemma 3 is shown. ■

4. THE CASE $N_4 \neq \emptyset$

Recall $A_{xy} = \{x, y\} \cup N_1 \cup N_2 \cup N_3$ and $N_5 = \emptyset$. In the case $N_4 \neq \emptyset$, we show that one can check in polynomial time whether G has a d.i.m. M with $xy \in M$.

Clearly, again in the case $N_4 \neq \emptyset$, all the components of $G[S_2 \cup N_3 \cup N_4]$ can be independently colored.

Recall Observation 5; if $t \in N_3$ then t is an endpoint of a corresponding induced P_5 in $\{x, y\} \cup N_1 \cup S_2 \cup N_3$. If there is a $P_5(t, a, b, c, d)$ with endpoint t and four vertices $a, b, c, d \in N_4$ (such that only one of them, say a contacts t) then t is the midpoint of a P_9 in G , which is a contradiction. Analogously, if there is a $P_5(t, a, b, c, t')$ with $t, t' \in N_3$ such that t and t' are in distinct T_i 's then t is the midpoint of a P_9 in G , which is a contradiction. This argument is used in some of the next proofs.

Proposition 1. *If the colors of all vertices in $G[A_{xy}]$ are fixed then the colors of all vertices in N_4 are forced.*

Proof. Let $v \in N_4$ and let $w \in N_3$ be a neighbor of v . Since by (7), every edge between N_3 and N_4 is xy -excluded, we have: If w is white then v is black, and if w is black then v is white. ■

Let K be a nontrivial component of $G[S_2 \cup N_3 \cup N_4]$. Clearly, K can have several components in $G[S_2 \cup N_3]$ which are connected by some N_4 -vertices. K can be feasibly colored (if there is no contradiction) by starting with a component in $G[S_2 \cup N_3]$ or with a component in $G[N_4]$ which is part of K .

Recall (8) and (9) for the fact that after the Edge Reduction, there is no triangle between N_3 and N_4 with exactly one vertex in N_3 , and for every edge uv in $G[N_4]$, u and v have no common neighbor in N_3 . Moreover, for N_4 -vertices which are isolated in N_4 , we have

$$(10) \quad \text{If } v \in N_4 \text{ with } N(v) \cap N_4 = \emptyset \text{ then } v \text{ is white.}$$

Thus, after the Vertex Reduction, we can assume that every vertex in N_4 has a neighbor in N_4 , i.e., every component in $G[N_4]$ has at least one edge.

Similarly, we have:

Claim 12. *If $v, w \in N_4$ with $vw \in E$ is an N_4 -isolated edge in $G[N_4]$ then vw is an xy -forced M -edge, i.e., v and w are black.*

Proof. Let $v, w \in N_4$ with $vw \in E$ such that v and w do not have any other neighbors in N_4 . Clearly, since $vw \in E$, at least one of v and w is black, say v is black. If w is white then v needs a black M -mate in N_4 since by (7), there is no M -edge between N_3 and N_4 . But since vw is N_4 -isolated, there is no such M -mate of v , i.e., w is black. Thus, Claim 12 is shown. □

Thus, after the Edge Reduction, there is no such N_4 -isolated edge in N_4 .

By the way, there are possible contradictions: For instance, if for an N_4 -isolated edge vw , v or w contacts a black vertex in N_3 then there is no d.i.m.

with $xy \in M$. Analogously, if $vt \in E$ for $t \in N_3$ and $wt' \in E$ for $t' \in N_3$ and $tt' \in E$ (i.e., (t, v, w, t') induce a C_4) then there is no d.i.m. with $xy \in M$.

Claim 13. *If $t \in T_i, t' \in T_j$ (possibly $i = j$), and $a, b, c \in N_4$ induce a C_5 $C = (t, a, b, t', c)$ in $G[N_3 \cup N_4]$ then ab is an xy -forced M -edge.*

Proof. Let $C = (t, a, b, t', c)$ be a C_5 in $G[N_3 \cup N_4]$. Then the edges $ta, tc, t'b, t'c$ are edges between N_3 and N_4 . By Observation 1(i), every C_5 has exactly one M -edge, and by (7), no edge between N_3 and N_4 is in M . Thus, ab is an xy -forced M -edge, and Claim 13 is shown. \square

In general, for any C_5 in $G[N_3 \cup N_4]$ with exactly one edge in $G[N_4]$, this edge is xy -forced as an M -edge. After the Edge Reduction step, we can assume that there is no such C_5 in $G[N_3 \cup N_4]$ with exactly one edge in $G[N_4]$.

Corollary 3. *If $ab \in E$ for $a, b \in N_4$ and $at \in E, bt' \in E$ for $t, t' \in N_3, t \neq t'$, such that (t, a, b, t') induce a P_4 in G then there is no common neighbor $c \in N_4$ of t and t' .*

Proof. Suppose to the contrary that there is such a common neighbor $c \in N_4$ with $tc \in E$ and $t'c \in E$. Then $ac \notin E$ and $bc \notin E$ since there are no triangles $(t, a, c), (t', b, c)$. But then $C = (t, a, b, t', c)$ induce a C_5 , which is a contradiction by Claim 13 and the Edge Reduction. Thus, Corollary 3 is shown. \blacksquare

Claim 14. *If $t \in T_i$ and $a, b, c \in N_4$ induce a C_4 $C = (t, a, b, c)$ then t is black and u_it is an xy -forced M -edge.*

Proof. Let $C = (t, a, b, c)$ be a C_4 with exactly one N_3 -vertex t . Suppose to the contrary that t is white. Then by Observation 1(ii), a and c are black, b is white, and by (7), there are M -mates $a' \in N_4$ of a and $c' \in N_4$ of c , i.e., $aa' \in M$ and $cc' \in M$. Since G is butterfly-free, b is nonadjacent to at least one vertex of $\{a', c'\}$, say b is nonadjacent to c' without loss of generality by symmetry. By (8) and the Edge Reduction, $tc' \notin E$; let $t' \in N_3$ be a neighbor of c' , i.e., $t'c' \in E$. Then t' is white (since $cc' \in M$), i.e., $t'b \notin E$; furthermore, by (7), $t'c \notin E$. Then, since (t', c', c, b, a) do not induce a P_5 (else by Observation 5, t' is the midpoint of a P_9 in G), we have $t'a \in E$ but now, $C = (t, a, t', c', c)$ is a C_5 with exactly one edge in N_4 , namely cc' . By Claim 13 and the Edge Reduction, we have that there is no such C_5 , i.e., t is black and u_it is an xy -forced M -edge. Thus, Claim 14 is shown. \square

After the Edge Reduction step, we can assume that there is no such C_4 in $G[N_3 \cup N_4]$.

Corollary 4. (i) *If (a, b, c, d) induce a P_4 in $G[N_4]$ with N_3 -neighbor t of a then (t, a, b, c, d) induce a C_5 in $G[N_3 \cup N_4]$.*

- (ii) If (a, b, c) induce a P_3 in $G[N_4]$ with N_3 -neighbor t of a and t' of c (clearly, $t \neq t'$) then either $tt' \in E$, i.e., (t, a, b, c, t') induce a C_5 in $G[N_3 \cup N_4]$, or $t, t' \in T_i$.

Claim 15. *There is no $P_3(a, b, c)$ in $G[N_4]$ with white end-vertices a and c .*

Proof. Suppose to the contrary that there is such a $P_3(a, b, c)$ in $G[N_4]$ with white end-vertices a and c , and thus black vertex b . Let $t_a \in T_i$ be an N_3 -neighbor of a , and let t_c be an N_3 -neighbor of c . By Claim 14 and the Edge Reduction, $t_a c \notin E$ and $t_c a \notin E$, i.e., $t_a \neq t_c$. Clearly, t_a and t_c are black, and thus, $t_c \notin T_i$ (and there is no T_j with $t_a, t_c \in T_j$). Moreover, $t_a t_c \notin E$ since both of them are black (recall that by (7), there is no M -edge in N_3). But then (t_a, a, b, c, t_c) induce a P_5 , and it leads to a P_9 in G with midpoint t_a , which is a contradiction. Thus, Claim 15 is shown. □

Corollary 5. *If vertex z in $G[N_4]$ has degree at least 3 in $G[N_4]$ then z is white.*

Proof. Suppose to the contrary that there is a black vertex z in $G[N_4]$ with degree at least 3, say $zz_i \in E$, $1 \leq i \leq 3$. Without loss of generality, assume that z_1 is black. But then z_2 and z_3 are white, and thus, (z_2, z, z_3) induce a P_3 with white end-vertices z_2, z_3 , which is a contradiction to Claim 15. Thus, Corollary 5 is shown. ■

Thus, after the Vertex Reduction, every vertex in a component of $G[N_4]$ has degree at most 2 in $G[N_4]$. For every component D of $G[N_4]$, this leads to feasible colorings of D .

Claim 16. *Every component D in $G[N_4]$ is either a P_k , $3 \leq k \leq 8$, or a C_k , $k \in \{3, 6, 9\}$, and D has at most three feasible colorings.*

Proof. Recall that after the Vertex Reduction, every vertex in a component D of $G[N_4]$ has degree at most 2 in $G[N_4]$. If D is cycle-free then, since D contains a P_3 (recall (10) and Claim 12) and G is P_9 -free, D is a P_k , $3 \leq k \leq 8$. If D contains a C_k C then, since G is P_9 -free, $k \leq 9$, and since every vertex in C has degree 2, C is no C_4, C_5, C_7, C_8 , since every black vertex in C must have an M -mate in C . Thus, C is either a C_3, C_6 , or C_9 . Clearly, for a C_k , $k \in \{3, 6, 9\}$, say $C = (z_1, \dots, z_k)$, there are three feasible colorings; for example, in a C_9 , if z_1 is white then z_4 and z_7 are white and the remaining vertices are black, and similarly if z_2 is white or z_3 is white. For induced paths P_k , $3 \leq k \leq 8$, say $P = (z_1, \dots, z_k)$, there are either one or two feasible colorings; if z_1 is white then z_2 and z_3 are black and thus z_4 is white etc. Thus, it leads to exactly one feasible coloring for P_4, P_5, P_7, P_8 , and for exactly two feasible colorings for P_3 and P_6 . Thus, Claim 16 is shown. □

Claim 17. *Let (a, b, c) be a P_3 in $G[N_4]$ for $a, b, c \in N_4$. If b is white then all N_3 -neighbors of a, b, c are in the same T_i .*

Proof. Let $t_a \in N_3$ be the neighbor of a , and analogously, let t_b, t_c be the neighbors of b, c in N_3 . Without loss of generality, let $t_a \in T_1$. Clearly, $t_ab \notin E$, $t_ac \notin E$, and $t_ca \notin E$, $t_cb \notin E$. Since b is white, a and c are black, and there are black M -mates $a' \in N_4$ of a and $c' \in N_4$ of c (recall that by (7), there is no M -edge between N_3 and N_4). By Claim 16, $a'b \notin E$ and $c'b \notin E$.

Now by Corollary 4(i), $t_a c' \in E$ and $t_c a' \in E$. Clearly, t_a and t_c are white, and thus, $t_a t_c \notin E$. Since (t_a, a, b, c, t_c) do not induce a P_5 with $t_c \notin T_1$ (else it leads to a P_9 in G), we have $t_c \in T_1$.

Suppose that $t_b \notin T_1$. Then, since (t_b, b, c, c', t_a) do not induce a P_5 (else there is a P_9 in G with midpoint t_a), we have $t_b t_a \in E$, and analogously, since (t_b, b, a, a', t_c) do not induce a P_5 , we have $t_b t_c \in E$ but now, $t_b \notin T_1$ contacts two vertices in T_1 , which is a contradiction (recall Lemma 1(v)). Thus, $t_b \in T_1$, and Claim 17 is shown. \square

Corollary 6. *For a component D in $G[N_4]$ with $P_3(a, b, c)$ such that b is white, there are three N_3 -neighbors of D in the same T_i such that every vertex of D contacts one of them.*

Proof. If D is a $P_5(a', a, b, c, c')$ as in the proof of Claim 17 then clearly, there are three N_3 -neighbors of D in the same T_i such that every vertex a', a, b, c, c' contacts one of them.

Clearly, D is P_9 -free. Now assume that there is a neighbor $d \in N_4$ of c' (recall that every vertex in D has degree at most 2). Clearly, d is white, $dc \notin E$ and $db \notin E$. Let $t_b \in N_3$ be a neighbor of vertex b . Since (t_b, b, c, c', d) do not induce a P_5 , by the discussion at the beginning of the section, we have $t_b d \in E$. Accordingly, if $e \in N_4$ is a neighbor of d and $t_c \in N_3$ is a neighbor of vertex c then, since (t_c, c, c', d, e) do not induce a P_5 , we have $t_c e \in E$ etc. Thus, Corollary 6 is shown. \blacksquare

Claim 18. *Let D_1, D_2 be two components in $G[N_4]$ and let $a, b \in V(D_1)$ with white vertex a and $ab \in E$ as well as $c, d \in V(D_2)$ with $cd \in E$. Then c and d are colored black by the white vertex a .*

Proof. Let $t_a \in T_1$ be an N_3 -neighbor of a . Then t_a is black, and all other vertices in T_1 are white. Let $t_c \in N_3$ be a neighbor of c . Clearly, $t_a b \notin E$ and $t_c d \notin E$, and ab, cd induce a $2P_2$ in $G[N_4]$. If $t_c \notin T_1$, say $t_c \in T_2$, and $t_a t_c \notin E$ then (b, a, t_a, u_1) , (d, c, t_c, u_2) , and the shortest path in $N_1 \cup \{x, y\}$ between u_1 and u_2 lead to a P_9 , which is a contradiction. Thus, either $t_c \in T_1$ or $t_c t_a \in E$ which implies that t_c is white, and thus, c is black. Analogously, d is colored black by the white vertex a , and Claim 18 is shown. \square

Corollary 7. *There is only one component in $G[N_4]$.*

Proof. Suppose to the contrary that there are two such components D_1, D_2 in $G[N_4]$. Clearly, by Claim 12 and the Edge Reduction, D_1 contains a white vertex a ; let $ab \in E$ for $a, b \in V(D_1)$. As in the proof of Claim 18, let $t_a \in T_1$ be an N_3 -neighbor of a which is black, and all other vertices in T_1 are white, and c and d are black for an edge $cd \in E$, $c, d \in V(D_2)$. Clearly, D_2 has at least three vertices; let e be a neighbor of c or d , say $de \in E$. Then e is white, and thus, an N_3 -neighbor t_e of e is black, and thus, $t_e \notin T_1$ and $t_a t_e \notin E$; let $t_e \in T_2$. But now, (b, a, t_a, u_1) , (d, e, t_e, u_2) , and the shortest path in $N_1 \cup \{x, y\}$ between u_1 and u_2 lead to a P_9 , which is a contradiction. Thus, Corollary 7 is shown. ■

Let K be a nontrivial component of $G[S_2 \cup N_3 \cup N_4]$, and let Q_1, \dots, Q_ℓ be the components of K in $G[S_2 \cup N_3]$ and let D be the component of K in $G[N_4]$. For each of the (at most three) feasible colorings of D , it leads to a partial coloring in every Q_i since there is no contact between Q_i and Q_j , $i \neq j$, and thus, there are contacts between Q_i and D . Then, as in Section 3, for every Q_i , it can be independently checked in polynomial time whether Q_i has a feasible coloring or a contradiction.

This finally shows:

Theorem 3. *DIM is solvable in polynomial time for P_9 -free graphs.*

5. CONCLUSION

In [9], it is shown that for every graph class of bounded clique-width, the DIM problem can be solved in polynomial time. However, there are many examples where the clique-width is unbounded but DIM is solvable in polynomial time; for example, the clique-width of P_9 -free graphs is unbounded. The complexity of DIM is still an open problem for many examples.

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