On quaternion algebras that split over specific quadratic number fields

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Abstract. Let d and m be two distinct squarefree integers and $\mathcal{O}_K$ the ring of integers of the quadratic field $K = \mathbb{Q}(\sqrt{d})$. Denote by $H_K(\alpha, m)$ a quaternion algebra over K, where $\alpha \in \mathcal{O}_K$. In this paper we give necessary and sufficient conditions for $H_K(\alpha, m)$ to split over K for some values of α, and we obtain a complete characterization of division quaternion algebras $H_K(\alpha, m)$ over K whenever α and m are two distinct positive prime integers. Examples are given involving prime Fibonacci numbers.
Keywords: quaternion algebras, Hilbert symbol, quadratic fields, Fibonacci numbers

1. Introduction

Let K be a field of characteristic $\neq 2$. If $\alpha, \beta \in K^* = K\backslash\{0\}$, then there exists a unique unital associative K-algebra of dimension 4 with K-basis $\{1, i, j, k\}$

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such that $i^2 = \alpha$, $j^2 = \beta$ and $ij = -ji = k$. This K-algebra will be denoted by $H_K(\alpha, \beta)$. Its presentation, as a K-algebra, is given by $K\{i, j\}/(i^2 - \alpha, j^2 - \beta, ij = -ji)$. A quaternion algebra over K is a K-algebra isomorphic to such an algebra for some $\alpha, \beta \in K^*$. The classical instance, where $K = \mathbb{R}$, is $H_K(-1, -1) = \mathbb{H}$, the Hamilton's quaternions ($\alpha = \beta = -1$).

The classification of quaternion algebras over K can be rephrased in terms of quadratic forms, and a more detailed description depends on the field K. In this vein, the most important question one may ask about a quaternion algebra $H_K(\alpha, \beta)$ is whether it is isomorphic to the matrix ring $\mathrm{M}_2(K)$; if so, we say that $H_K(\alpha, \beta)$ splits over K. For example, every quaternion algebra over $\mathbb{C}$ (or an algebraically closed field) splits, and a quaternion algebra $H_{\mathbb{R}}(\alpha, \beta)$ over $\mathbb{R}$ splits if and only if $\alpha > 0$ or $\beta > 0$ (see [14]).

Quaternion algebras are central simple algebras over K (i.e. associative and non-commutative algebras without two sided ideals whose center is precisely K, see [14]) of dimension 4 over K. Recall that the dimension d of a central simple algebra A over a field K is always a perfect square, and its square root n is defined to be the degree of A.

The theory of central simple algebras (and thus in particular quaternion algebras and cyclic algebras) has strong connections with algebraic number theory, combinatorics, algebraic geometry, coding theory, computer science and signal theory (see [6, 9, 14]). Quaternion algebras have been studied in many papers that deal with conditions for an algebra to be split or a division algebra (e.g. [15, 16, 17]).

To decide whether a quaternion algebra is a division algebra or it splits, different approaches are used: quadratic forms, the associated conics (which are projective plane curves defined by the homogeneous equation $\alpha x^2 + \beta y^2 = z^2$), cyclotomic fields, p-adic fields and some other properties of associative algebras (e.g. [15, 16, 17]).

In this paper we adopt two distinct approaches. The first consists of studying the ramification of certain integral primes, and we obtain a nice characterization of quaternion division algebras $H_K(p, q)$ solely in terms of quadratic residues, assuming that p and q are positive primes and K is quadratic number field. The second one, makes use of the Hilbert symbol in order to obtain necessary and sufficient conditions for a quaternion algebra $H_K(\alpha, m)$ to split over a quadratic field $K = \mathbb{Q}(\sqrt{d})$, for some integers α of $\mathcal{O}_K$, the ring of integers of the quadratic field K, where m and d are squarefree integers.

2. Preliminaries

Let us collect some results that we will used in the sequel. We begin by recalling the definition of the ring of integers of a number field.

Definition 2.1. *Let K be a number field.*

1. *The ring of integers $\mathcal{O}_K$ of K is defined as follows*

$$\mathcal{O}_K = \{\alpha \in K : \ P(\alpha) = 0 \text{ for some monic polynomial } P(X) \in \mathbb{Z}[X]\}.$$

2. *$\mathcal{O}_K^\times$, the unit group of K, is the set of invertible elements of $\mathcal{O}_K$.*

3. *If $K = \mathbb{Q}(\sqrt{d})$ is a quadratic field, where $d > 0$ is a squarefree integer, then there exists a unit ε_d, called the fundamental unit of K, and it is a generator (modulo the roots of unity) of the unit group $\mathcal{O}_K^\times$.*

Let K be a number field and q an element of the quaternion algebra $H_K(\alpha, \beta)$ over K; then we can write $q = a + bi + cj + ek$, where a, b, c and e are in K. The conjugate $\overline{q}$ of q is defined as $\overline{q} = a - bi - cj - ek$. The norm map is defined by N: $q \mapsto N(q) = q\overline{q} = a^2 - \alpha b^2 - \beta c^2 + \alpha\beta e^2 \in K$. So N may be regarded as a quadratic form in the four variables a, b, c and e.

Definition 2.2. *A non-zero vector $q \in H_K(\alpha, \beta)$ is said to be isotropic if $N(q) = 0$.*

Definition 2.3. *An associative algebra A over a field is called a division algebra if it has a multiplicative identity element $1 \neq 0$ and every non-zero element in A has a left and a right multiplicative inverses. If A is a finite-dimensional algebra, then A is a division algebra if and only if A has no nontrivial zero divisors.*

Definition 2.4. *A central simple algebra A over a field K is called split by K if A is isomorphic to the full matrix algebra $M_n(K)$ over K.*

Theorem 2.1 ([6, 20, 21])**.** *Let K be a field with char $K \neq 2$ and let $\alpha, \beta \in K\backslash\{0\}$. Then the quaternion algebra $H = H_K(\alpha, \beta)$ is either split or a division algebra. Furthermore, the following statements are equivalent:*

1. *$H \cong H_K(1, 1) \cong M_2(K)$;*

2. *H is not a division algebra;*

3. *H has an isotropic element;*

4. *the equation $\alpha x^2 + \beta y^2 = 1$ has a solution $(x, y) \in K \times K$;*

5. *α (resp. β) is a norm from $K(\sqrt{\beta})$ (resp. from $K(\sqrt{\alpha})$).*

The above statements can be checked by using local techniques. For a number field K, it is well known that its ring of integers $\mathcal{O}_K$ is a Dedekind ring. If $\alpha \in K$ and $\mathcal{P}$ is a prime ideal of $\mathcal{O}_K$, we have $\alpha\mathcal{O}_K = \mathcal{P}^{v_\mathcal{P}(\alpha)}I$, where $v_\mathcal{P}(\alpha)$ is the highest power of $\mathcal{P}$ dividing $\alpha\mathcal{O}_K$ and I is an ideal of $\mathcal{O}_K$. The map $\alpha \mapsto N(\mathcal{P})^{-v_\mathcal{P}(\alpha)}$ is the non-Archimedean absolute value, where N denotes the absolute norm map. We denote by $K_\mathcal{P}$, the completion of K with respect to this absolute value. We conclude this section by defining the concept of ramification of a quaternion algebra at a prime ideal.

Definition 2.5. *Let $\mathcal{P}$ be a prime ideal of $\mathcal{O}_K$. We say that $H_K(\alpha,\beta)$ is ramified at $\mathcal{P}$ if $H_{K_\mathcal{P}}(\alpha,\beta)$ is a division ring. The set of ramified primes of $H_K(\alpha,\beta)$ will be denoted by* $\mathrm{Ram}(H_K(\alpha,\beta))$. *The reduced discriminant $D_{H_K(\alpha,\beta)}$ of the quaternion algebra $H_K(\alpha,\beta)$ is defined as the product of those prime ideals of $\mathcal{O}_K$ that ramify in $H_K(\alpha,\beta)$.*

3. Symbols over number fields

If K is a number field of degree n, with signature (r,s) and L is a quadratic extension of K, then $L=K(\sqrt{\alpha})$ for some $\alpha\in\mathcal{O}_K$. If $\mathcal{P}$ is a prime ideal of $\mathcal{O}_K$, then the behavior of its extension $\mathcal{P}\mathcal{O}_L$ to L is as follows:

$$\mathcal{P}\mathcal{O}_L=\begin{cases}\mathcal{B}_1\mathcal{B}_2, & \mathcal{P} \text{ splits into 2 different prime ideals of } \mathcal{O}_L;\\ \mathcal{P}, & \mathcal{P} \text{ remains prime } (\mathcal{P} \text{ is called inert in } L);\\ \mathcal{B}^2, & \mathcal{P} \text{ ramifies in } L.\end{cases}$$

Note that $\mathcal{B}_i\cap K=\mathcal{B}\cap K=\mathcal{P}$ and $\mathcal{P}\cap\mathbb{Q}=p\mathbb{Z}$, where p is a prime number. We say that $\mathcal{B}$ (resp. $\mathcal{B}_i$) lies above $\mathcal{P}$. The following properties hold:

- $\mathcal{O}_K/\mathcal{P}$ is a finite field and $N(\mathcal{P}):=\mid \mathcal{O}_K/\mathcal{P}\mid = p^{f_\mathcal{P}}$ with $f_\mathcal{P}\in\mathbb{N}$. $N(\mathcal{P})$ is called the absolute norm of $\mathcal{P}$ and $f_\mathcal{P}$ is its inertia degree;
- if $v_\mathcal{P}(\alpha)$ is odd, then $\mathcal{P}$ is ramified in $L=K(\sqrt{\alpha})$;
- if $v_\mathcal{P}(\alpha)$ is even and $\mathcal{P}$ does not divide 2, then $\mathcal{P}$ is unramified in $L=K(\sqrt{\alpha})$;
- the infinite primes are just the $r+s$ non equivalent archimedean absolute values coming from the r real and s pairs of complex emebeddings.

I. The quadratic residue symbol. The quadratic residue symbol $(\frac{\alpha}{\mathcal{P}})$ is classically defined as follows. Let $\mathcal{P}$ be a prime ideal of $\mathcal{O}_K$. If α is a square in K, we let $(\frac{\alpha}{\mathcal{P}})=1$. Otherwise, we let

$$\left(\frac{\alpha}{\mathcal{P}}\right)=\begin{cases}1, & \text{if } \mathcal{P} \text{ splits in } K(\sqrt{\alpha});\\ -1, & \text{if } \mathcal{P} \text{ is inert in } K(\sqrt{\alpha});\\ 0, & \text{if } \mathcal{P} \text{ ramifies in } K(\sqrt{\alpha}).\end{cases}$$

Theorem 3.1 ([7, 12])**.** *If $\mathcal{P}$ is prime to the ideal generated by α and to the ideal generated by* 2*, then*

$$\left(\frac{\alpha}{\mathcal{P}}\right)\equiv\alpha^{\frac{N(\mathcal{P})-1}{2}}\pmod{\mathcal{P}}.$$

Remark 3.1. 1. If $K=\mathbb{Q}$, then $\mathcal{P}=p\mathbb{Z}$ where p is a prime integer. In this case the symbol $(\frac{\alpha}{\mathcal{P}})$ is denoted by $(\frac{\alpha}{p})$ and called the Legendre symbol. If $(\frac{\alpha}{p})=1$ (in this case, we say that α is a quadratic residue modulo p) and $p\equiv 1 \pmod 4$, then $(\frac{\alpha}{p})_4$ will denote the rational biquadratic symbol which is equal to 1 or -1, according as $(\alpha)^{\frac{p-1}{4}}\equiv 1$ or $-1 \pmod p$.

2. If $K=\mathbb{Q}(\sqrt{d})$ is a quadratic field, $\mathcal{P}$ is a prime ideal of $\mathcal{O}_K$ above a prime number p and $\alpha\in\mathcal{O}_K$, then according to [12], we have :

$$\left(\frac{\alpha}{\mathcal{P}}\right)=\begin{cases}\left(\dfrac{N(\alpha)}{p}\right), & \text{if } p \text{ is inert in } K;\\ \left(\dfrac{\alpha}{p}\right)^{f_{\mathcal{P}}}, & \text{if } \alpha\in\mathbb{Q};\\ \left(\dfrac{\alpha}{\mathcal{P}'}\right)\left(\dfrac{N(\alpha)}{p}\right), & \text{if } p\mathcal{O}_K=\mathcal{P}\mathcal{P}', \mathcal{P}\neq\mathcal{P}'.\end{cases}$$

II. The Hilbert symbol. Let $\mathcal{P}$ be a prime of K (finite or infinite) and let $K_{\mathcal{P}}$ be the completion of K at $\mathcal{P}$. For $\alpha,\beta\in K_{\mathcal{P}}$, we define the local Hilbert Symbol as

$$(\alpha,\beta)_{\mathcal{P}}=\begin{cases}1, & \text{if } H_{K_{\mathcal{P}}}(\alpha,\beta) \text{ splits;}\\ -1, & \text{if not.}\end{cases}$$

Recall that the Hilbert symbol $(a,b)_{\mathcal{P}}$ is equal to -1 if and only if the equation $ax^2+by^2=1$ has no solutions in $K_{\mathcal{P}}$.

If $i_{\mathcal{P}}$ is the natural injection from K to $K_{\mathcal{P}}$, then for $\alpha,\beta\in K$ we define the global Hilbert Symbol as

$$\left(\frac{\alpha,\beta}{\mathcal{P}}\right)=i_{\mathcal{P}}^{-1}((i_{\mathcal{P}}(\alpha),i_{\mathcal{P}}(\beta))_{\mathcal{P}}).$$

Theorem 3.2 ([7])**.** *The Hilbert symbol satisfies the following conditions:*

i. $(\frac{\alpha,\beta}{\mathcal{P}})=(\frac{\beta,\alpha}{\mathcal{P}})$;

ii. *If $\mathcal{P}$ is a prime ideal of K unramified in $K(\sqrt{\alpha})$, then* $(\frac{\alpha,\beta}{\mathcal{P}})=(\frac{\alpha}{\mathcal{P}})^{v_{\mathcal{P}}(\beta)}$;

iii. *if $\mathcal{P}_\infty$ denotes an infinite prime, then* $(\frac{\alpha,\beta}{\mathcal{P}_\infty})=-1$ *if and only if* $i_{\mathcal{P}}(\alpha)<0$ *and* $i_{\mathcal{P}}(\beta)<0$;

iv. *β is a norm in $K(\sqrt{\alpha})$ if and only if* $(\frac{\alpha,\beta}{\mathcal{P}})=1$ *for all prime $\mathcal{P}$;*

v. $\prod_{\mathcal{P}}(\frac{\alpha,\beta}{\mathcal{P}})=1$ (*the product formula*).

Remark 3.2. Let K be a number field and $\alpha,\beta\in K^*=K\backslash\{0\}$. Then $H_K(\alpha,\beta)$ splits if and only if $\mathrm{Ram}(H_K(\alpha,\beta))=\emptyset$. More generally, we have

$$\mathrm{Ram}(H_K(\alpha,\beta))=\left\{\mathcal{P} \text{ the prime ideal of } K \ : \ \left(\frac{\alpha,\beta}{\mathcal{P}}\right)=-1\right\}.$$

Note also that the following splitting criterion for a quaternion algebras is well known [1, Corollary 1.10]: the quaternion algebra $H_K(\alpha, \beta)$ is split if and only if its discriminant $D_{H_K(\alpha,\beta)}$ is equal to the ring of integers $\mathcal{O}_K$ of K.

4. Main results

Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic number field and $\mathcal{O}_K$ its ring of integers, where $d \neq 1$ is a squarefree integer.

4.1 First case: $\alpha \in \mathcal{O}_K$ and $m \equiv 1 \pmod 4$ a squarefree integer

Let us begin by defining an algebraic integer $\alpha \in \mathcal{O}_K$ to be odd if any prime ideal appearing in its factorization into prime ideals in $\mathcal{O}_K$ does not lie above 2.

The numbers m and α are said to satisfy hypotheses (H) if they satisfy the following two conditions:

1. α is an odd integer of $\mathcal{O}_K$, $m \equiv 1 \pmod 4$ and d are two distinct squarefree integers with $m > 0$;
2. α and m are relatively prime.

Moreover, we adopt, in the sequel, the following notation:

1. $\frac{m}{\gcd(m,d)} = (q_1 q_2 \cdots q_t)^\lambda (r_1 r_2 \cdots r_s)^\gamma$, where $\lambda,\ \gamma \in \{0, 1\}$ and q_i, r_i are prime numbers satisfying $(\frac{d}{q_i}) = -1$ if $\lambda = 1$, and $(\frac{d}{r_i}) = 1$ if $\gamma = 1$ for all i;
2. $\mathcal{Q}_i$ (resp. $\mathcal{R}_i$, $\mathcal{R}_i'$) denotes the prime ideal (resp. ideals) of $\mathcal{O}_K$ above q_i (resp. r_i);
3. $\mathcal{I}$ denotes the set of all prime ideals $\mathcal{P}$ of $\mathcal{O}_K$ such that $v_{\mathcal{P}}(\alpha)$ is odd and whose absolute norm is a rational prime.

We can now establish our first main result.

Theorem 4.1. *If m and α satisfy the hypotheses* (H)*, then $H_K(\alpha, m)$ splits if and only if the following conditions are satisfied:*

1. $(\frac{N(\alpha)}{q_i}) = (\frac{\alpha}{\mathcal{Q}_i}) = 1$, *if* $\lambda = 1$*;*
2. $(\frac{N(\alpha)}{r_j}) = (\frac{\alpha}{\mathcal{R}_j}) = 1$, *if* $\gamma = 1$*;*
3. *for any prime $\mathcal{P} \in \mathcal{I}$ we have* $(\frac{m}{N(\mathcal{P})}) = 1$.

Proof. Let $\mathcal{P}$ be an odd prime ideal of K. Lemma 1 of [3] implies that the discriminant $D(L/K) = (\mathcal{Q}_1 \mathcal{Q}_2 \cdots \mathcal{Q}_t)^\lambda (\mathcal{R}_1 \mathcal{R}_1' \mathcal{R}_2 \mathcal{R}_2' \cdots \mathcal{R}_s \mathcal{R}_s')^\gamma$, where $L = K(\sqrt{m})$.

If $\mathcal{P} \neq \mathcal{Q}_i$, $\mathcal{P} \neq \mathcal{R}_j$ and $\mathcal{P} \neq \mathcal{R}'_j$, then $\mathcal{P}$ is unramified in $K(\sqrt{m})$. Hence by putting $N(\mathcal{P}) = p^{f_{\mathcal{P}}}$, Remark 3.1 implies that

$$\left(\frac{\alpha, m}{\mathcal{P}}\right) = \left(\frac{m}{\mathcal{P}}\right)^{v_{\mathcal{P}}(\alpha)} = \left(\frac{m}{p}\right)^{f_{\mathcal{P}} v_{\mathcal{P}}(\alpha)} = \begin{cases} \left(\frac{m}{p}\right), & \text{if } \mathcal{P} \in \mathcal{I}; \\ 1, & \text{if not.} \end{cases}$$

If $\mathcal{P} = \mathcal{Q}_i$, $\mathcal{R}_j$ or $\mathcal{R}'_j$, then, since α and m are relatively prime, $v_{\mathcal{P}}(\alpha) = 0$ and $v_{\mathcal{P}}(m) = 1$. Hence $\mathcal{P}$ is unramified in $K(\sqrt{\alpha})$ and thus

$$\left(\frac{\alpha, m}{\mathcal{P}}\right) = \left(\frac{\alpha}{\mathcal{P}}\right)^{v_{\mathcal{P}}(m)} = \begin{cases} \left(\frac{N(\alpha)}{q_i}\right), & \text{if } \mathcal{P} = \mathcal{Q}_i; \\ \left(\frac{\alpha}{\mathcal{P}}\right), & \text{if not.} \end{cases}$$

• As $m \equiv 1 \pmod 4$, then the prime ideal $\mathcal{P}_2$ of $\mathcal{O}_K$ above 2 is unramified in $K(\sqrt{m})$, and since α is an odd integer (hence $v_{\mathcal{P}_2}(\alpha) = 0$), we have:

$$\left(\frac{\alpha, m}{\mathcal{P}_2}\right) = \left(\frac{m}{\mathcal{P}_2}\right)^0 = 1.$$

• Now, let $\mathcal{P}_\infty$ be an infinite prime ideal, then $(\frac{\alpha,m}{\mathcal{P}_\infty}) = 1$, because $i_{\mathcal{P}_\infty}(m) = m > 0$. Finally, Remarks 3.1 and 3.2 imply the assertions. □

Remark 4.1. Thanks to Hilbert's symbol product formula, we get the same results if we consider the conditions $m \equiv 1 \pmod 4$ and $\alpha \in \mathcal{O}_K$ with $d \not\equiv 1 \pmod 8$ instead of the conditions $m \equiv 1 \pmod 4$ and α is an odd integer of $\mathcal{O}_K$. In this case we will say that m and α satisfy the hypothesis $\hat{\mathrm{H}}$.

In [15, Theorem 3.6 and Proposition 3.7] the second author gave conditions for a quaternion algebra $H_K(\alpha, p)$ (where p is a prime positive integer) to split over the quadratic field $K = \mathbb{Q}(i)$. The following corollary generalizes Theorem 3.6 and Proposition 3.7 from [15]. With similar proof ideas as in Theorem 3.6 and Proposition 3.7 from [15], we obtain in the following corollary necessary and sufficient conditions for a quaternion algebra $H_K(\alpha, m)$ to split over any quadratic number field $K = \mathbb{Q}(\sqrt{d})$, where $d \neq m$ is a squarefree integer.

Corollary 4.1. *If m and α satisfy hypotheses $\hat{\mathrm{H}}$, $\alpha \in \mathbb{Z}$ and m is prime, then $H_K(\alpha, m)$ splits if and only if one of the following conditions holds:*

1. *α is a square in $\mathbb{N}$;*
2. *$-\alpha$ is a square in $\mathbb{N}$ and m divides d or d is a quadratic residue modulo m;*
3. *$\alpha = \pm\delta t^2$ with $\delta > 1$ and each prime divisor of δ, which splits in K, is a quadratic residue modulo m and*

   - either m divides d or d is not a quadratic residue modulo m;

- *or d and α are quadratic residue modulo m.*

Corollary 4.2. *If α is a unit of K and m divides d, then $H_K(\alpha, m)$ splits.*

Proof. It is enough to note that if α is a unit of K, then $\mathcal{I} = \emptyset$; and since m divides d we have $\lambda = \gamma = 0$. □

4.2 Applications of Theorem 4.1

In this subsection, we state some applications of the first main theorem. We start with a classical lemma [10]:

Lemma 4.1. *Let K be a quadratic field.*

1. *$K(\sqrt{\alpha})$ is a biquadratic number field if and only if $N(\alpha)$ is a square in $\mathbb{N}$.*

2. *$K(\sqrt{\alpha})$ is a quartic cyclic number field if and only if $dN(\alpha)$ is a square in $\mathbb{N}$.*

The first application of the main result is the following theorem.

Theorem 4.2. *If m and α satisfy the hypotheses* (H) *and $K(\sqrt{\alpha}) = \mathbb{Q}(\sqrt{d}, \sqrt{d_1})$ is a biquadratic number field, then $H_K(\alpha, m)$ splits if and only if the following conditions are satisfied:*

1. *$(\frac{d_1}{r_j}) = 1$, if $\gamma = 1$;*

2. *For any prime $p|d_1$ such that $(p, d) = 1$ and $(\frac{d}{p}) = 1$, we have $(\frac{m}{p}) = 1$.*

Proof. As $K(\sqrt{\alpha}) = \mathbb{Q}(\sqrt{d}, \sqrt{d_1})$ is a biquadratic number field, then the previous Lemma imply the $N(\alpha)$ is a square; moreover $\alpha = t^2 d_1$ with $t \in \mathbb{Q}(\sqrt{d})$. So $H_K(\alpha, m)$ splits if and only if the following conditions are satisfied:

1. $(\frac{\alpha}{\mathcal{R}_j}) = (\frac{t^2 d_1}{\mathcal{R}_j}) = (\frac{d_1}{r_j}) = 1$, if $\gamma = 1$;

2. For any prime $\mathcal{P} \in \mathcal{I}$, we have $(\frac{m}{N(\mathcal{P})}) = 1$.

If a prime $\mathcal{P} \in \mathcal{I}$, then $v_{\mathcal{P}}(\alpha) = 2v_{\mathcal{P}}(t) + v_{\mathcal{P}}(d_1) = 2v_{\mathcal{P}}(t) + v_p(d_1)v_{\mathcal{P}}(p)$ is odd and $N(\mathcal{P}) = p$ is a prime integer, which is equivalent to $p|d_1$, $(p, d) = 1$, $(\frac{d}{p}) = 1$ and $(\frac{m}{p}) = 1$. □

As a second application of the main Theorem, we assume that $K(\sqrt{\alpha})$ is a quartic cyclic number field.

Theorem 4.3. *If m and α satisfy hypotheses* (H), *$d \equiv 1 \pmod 4$ and $K(\sqrt{\alpha}) = \mathbb{Q}(\sqrt{a + b\sqrt{d}})$ is quartic cyclic number field ($a^2 = d(b^2 + c^2)$), then $H_K(\alpha, m)$ splits if and only if the following conditions are satisfied:*

1. *$\lambda = 0$ and $(-1)^{(r_j-1)(d-1)/8}(\frac{2}{r_j})^b(\frac{r_j}{d})_4 = 1$, if $\gamma = 1$;*

2. *For any prime $\mathcal{P} \in \mathcal{I}$, we have $(\frac{m}{N(\mathcal{P})}) = 1$.*

Proof. It is an immediate consequence of the main Theorem (i.e. Theorem 4.1) and of the main theorem of [22]. □

As a third application of the main Theorem, we give necessary and sufficient conditions for $H_K(\alpha, m)$ to split over K whenever $\alpha = \varepsilon_d$ is a fundamental unit of $K = \mathbb{Q}(\sqrt{d})$.

Theorem 4.4. *Let us assume that the norm of ε_d is -1, and let $d = p_1 \cdots p_n$. Suppose that p_j and m are primes $\equiv 1 \pmod 4$ such that $\gamma = 1$ and $(\frac{p_j}{r_i}) = 1$ for all $1 \leq j \leq n$ and $1 \leq i \leq s$. Then $H_K(\varepsilon_d, m)$ splits if and only if*

$$r_i \equiv 1 \pmod 4 \text{ and } (\frac{r_i}{d})_4 (\frac{d}{r_i})_4 = 1,$$

for all $1 \leq i \leq s$.

Proof. Note that m and $\alpha = \varepsilon_d$ satisfy the hypotheses (H), $\mathcal{I} = \emptyset$. Then [5, Corollary at page 143] gives the claimed result. □

We now suppose that the norm of ε_d is 1. It is known that ε_d does not always belong to $\mathbb{Z}[\sqrt{d}]$. Actually, $\varepsilon_d \in \mathbb{Z}[\sqrt{d}]$ if $d \equiv 3, 2 \pmod 4$ or $d \equiv 1 \pmod 8$, but if $d \equiv 5 \pmod 8$, ε_d may not belong to $\mathbb{Z}[\sqrt{d}]$. To simplify, we set $\varepsilon = \varepsilon_d^3$ if $d \equiv 5 \pmod 8$ and $\varepsilon_d \notin \mathbb{Z}[\sqrt{d}]$, and $\varepsilon = \varepsilon_d$ otherwise.

Theorem 4.5. *Let m and d be squarefree integers such that $\gamma = 1$, let the norm of $\varepsilon = x + y\sqrt{d}$ be 1. Then $H_K(\varepsilon_d, m)$ splits if and only if one of the following conditions holds:*

1. *$(\frac{d_1}{r_j}) = 1$ where d_1 divides d and $2d_1(x+1)$ is a square in $\mathbb{N}$ for all $1 \leq i \leq s$;*

2. *$(\frac{2}{r_j}) = (\frac{d_1}{r_j})$ where d_1 divides d and $d_1(x+1)$ is a square in $\mathbb{N}$ for all $1 \leq i \leq s$.*

Proof. As the norm of ε_d is 1, then ε is too, i.e., $(x+1)(x-1) = dy^2$. Furthermore $K(\sqrt{\varepsilon})$ is a biquadratic field.

• If x is odd, then according to the unique prime factorization in $\mathbb{Z}$, there exist two squarefree integers d_1 and d_2 such that $d = d_1 d_2$, $(x+1) = 2d_1 y_1^2$ and $(x-1) = 2d_2 y_2^2$ with $y_1, y_2 \in \mathbb{N}$ and $2y_1 y_2 = y$. In this case, one can easily check that $2d_1(x+1)$ is a square in $\mathbb{N}$ and that $d_1\varepsilon = (y_1 d_1 + y_2\sqrt{d})^2$. With Theorem 4.2, we have $H_K(\varepsilon_d, m)$ splits if and only if $(\frac{d_1}{r_j}) = 1$.

• If x is even, then proceeding similarly, we get that $d_1(x+1)$ is a square in $\mathbb{N}$ and $2d_1\varepsilon = (y_1 d_1 + y_2\sqrt{d})^2$. As in the first case $H_K(\varepsilon_d, m)$ splits if and only if $(\frac{2}{r_j}) = (\frac{d_1}{r_j})$. □

Remark 4.2. Note that if the norm of ε_d is 1, then d_1 exists and it is unique.

4.3 Second case: m and α are prime integers

In this subsection, we replace $\{\alpha, m\}$ by $\{p, q\}$, where p and q are rational primes. The purpose is then to establish the following theorem that classify division quaternion algebras $H_{\mathbb{Q}(\sqrt{d})}(p, q)$ over quadratic fields where $d \neq 1$ is a squarefree integer:

Theorem 4.6. *Let $d \neq 1$ be a squarefree integer and let $K = \mathbb{Q}(\sqrt{d})$, with discriminant Δ_K. Let p and q be two positive primes. Then the quaternion algebra $H_K(p, q)$ is a division algebra if and only if one of the following conditions holds:*

1. *p and q are odd and distinct, p or $q \equiv 1 \pmod 4$, $(\frac{p}{q}) = -1$ and $(\frac{\Delta_K}{p}) = 1$ or $(\frac{\Delta_K}{q}) = 1$;*

2. *$q = 2$, $p \equiv 3 \pmod 8$ or $p \equiv 5 \pmod 8$ and either $(\frac{\Delta_K}{p}) = 1$ or $d \equiv 1 \pmod 8$;*

3. *p and q are odd, with $p \equiv q \equiv 3 \pmod 4$, and*

   - *$(\frac{q}{p}) \neq 1$ and either $(\frac{\Delta_K}{p}) = 1$ or $d \equiv 1 \pmod 8$;*
     or
   - *$(\frac{p}{q}) \neq 1$ and either $(\frac{\Delta_K}{q}) = 1$ or $d \equiv 1 \pmod 8$.*

Proof. The proof of the first assertion is a simple deduction from Theorem 4.2. It suffices to take $\alpha, m \in \{p, q\}$ such that $\alpha \neq m$ and $m \equiv 1 \pmod 4$. Assertions 2 and 3 follow easily applying other results which we obtained in this article, namely Propositions 4.5, 4.6 and 4.7 below, after taking into account Lemma 4.2 and Lemma 4.4 below. □

To complete the proof of this second main result, we need some preliminary results. Let K be a number field and $\mathcal{O}_K$ its ring of integers. If $\mathcal{O}_K$ is a principal ideal domain, then we may identify the ideals of $\mathcal{O}_K$ with their generators, up to units. Thus, in a quaternion algebra H over $\mathbb{Q}$, the element D_H (the discriminant of H) turns out to be an integer, and H is split if and only if $D_H = 1$. On the other hand, a quaternion algebra $H_{\mathbb{Q}}(\alpha, \beta)$ is a division algebra if and only if there is a prime p such that $p | D_{H_{\mathbb{Q}}(\alpha,\beta)}$. We continue this note with two statements following from the classical Albert-Brauer-Hasse-Noether theorem. Proofs of specific formulations of this theorem can be found in [13, 4].

Theorem 4.7. *Let H_F be a quaternion algebra over a number field F and let K be a quadratic extension of F. Then there is an embedding of K into H_F if and only if no prime of F which ramifies in H_F splits in K.*

Proposition 4.1. *Let F be a number field and let K be a quadratic extension of F. Let H_F be a quaternion algebra over F. Then K splits H_F if and only if there exists an F-embedding $K \hookrightarrow H_F$.*

In [15] the second author obtained the following result about quaternion algebras over the field $\mathbb{Q}(i)$ which is also a simple deduction from Corollary 4.1.

Proposition 4.2. *Let $p \equiv 1 \pmod 4$ be a prime integer and let m be an integer which is not a quadratic residue modulo p. Then the quaternion algebra $H_{\mathbb{Q}(i)}(m,p)$ is a division algebra.*

In [16] the second author obtained some sufficient conditions for a quaternion algebra $H_{\mathbb{Q}(i)}(p,q)$ to split, where p and q are two distinct primes:

Proposition 4.3. *Let $d \neq 0,1$ be a squarefree integer such that $d \not\equiv 1 \pmod 8$, and let p and q be two primes, with $q \geq 3$ and $p \neq q$. Let $\mathcal{O}_K$ be the ring of integers of the quadratic field $K = \mathbb{Q}(\sqrt{d})$ and let Δ_K be its discriminant.*

1. *If $p \geq 3$ and both $(\frac{\Delta_K}{p})$ and $(\frac{\Delta_K}{q})$ are not equal to 1, then the quaternion algebra $H_{\mathbb{Q}(\sqrt{d})}(p,q)$ splits;*

2. *If $p = 2$ and $(\frac{\Delta_K}{q}) \neq 1$, then the quaternion algebra $H_{\mathbb{Q}(\sqrt{d})}(2,q)$ splits.*

From the aforementioned results we deduce easily a necessary and sufficient condition for a quaternion algebra $H_{\mathbb{Q}(i)}(p,q)$ to be a division algebra:

Proposition 4.4. *Let p and q be two distinct odd primes, such that $(\frac{q}{p}) \neq 1$. Then the quaternion algebra $H_{\mathbb{Q}(i)}(p,q)$ is a division algebra if and only if $p \equiv 1 \pmod 4$ or $q \equiv 1 \pmod 4$.*

Proof. To prove the necessity, note that if $H_{\mathbb{Q}(i)}(p,q)$ is a division algebra, then Proposition 4.3 and Theorem 2.1 tell us that $(\frac{\Delta_K}{p}) = 1$ or $(\frac{\Delta_K}{q}) = 1$. This is equivalent to $p \equiv 1 \pmod 4$ or $q \equiv 1 \pmod 4$.

To prove the sufficiency, we must distinguish amongst two cases:

- $p \equiv 1 \pmod 4$:
  Since $(\frac{q}{p}) \neq 1$, Proposition 4.2 tells us that $H_{\mathbb{Q}(i)}(p,q)$ is a division algebra.

- $q \equiv 1 \pmod 4$:
  Since $(\frac{q}{p}) = -1$, the quadratic reciprocity law implies $(\frac{p}{q}) = -1$. Proposition 4.2 then tells us that $H_{\mathbb{Q}(i)}(p,q)$ is a division algebra. □

We ask ourselves whether we can obtain a necessary and sufficient explicit condition for $H_{\mathbb{Q}(\sqrt{d})}(p,q)$ to be a division algebra when d is arbitrary. From Proposition 4.3 we obtain a necessary explicit condition for $H_{\mathbb{Q}(\sqrt{d})}(p,q)$ to be a division algebra, namely: if $H_{\mathbb{Q}(\sqrt{d})}(p,q)$ is a division algebra, then $(\frac{\Delta_K}{p}) = 1$ or $(\frac{\Delta_K}{q}) = 1$. However this condition is not sufficient: for example, if we let $K = \mathbb{Q}(\sqrt{3})$, $p = 7$, $q = 47$, then $(\frac{\Delta_K}{p}) \neq 1$ and $(\frac{\Delta_K}{q}) = 1$, the quaternion algebra $H_{\mathbb{Q}}(7,47)$ is a division algebra, but the quaternion algebra $H_{\mathbb{Q}(\sqrt{3})}(7,47)$ is not a division algebra.

It is known [11] that if a prime integer p divides $D_{H(\alpha,\beta)}$, then it must divide $2\alpha\beta$, hence we may restrict our attention to these primes. In other words, in order to obtain a sufficient condition for a quaternion algebra $H_{\mathbb{Q}(\sqrt{d})}(p,q)$ to be a division algebra, it is important to study the ramification of the primes $2, p, q$ in the algebra $H_{\mathbb{Q}}(p,q)$. The following lemma [1, Lemma 1.21] gives us a hint:

Lemma 4.2. *Let p and q be two primes, and let $H_{\mathbb{Q}}(p,q)$ be a quaternion algebra of discriminant D_H.*

1. *If $p \equiv q \equiv 3 \pmod 4$ and $(\frac{q}{p}) \neq 1$, then $D_H = 2p$;*
2. *If $q = 2$ and $p \equiv 3 \pmod 8$, then $D_H = pq = 2p$;*
3. *If p or $q \equiv 1 \pmod 4$, with $p \neq q$ and $(\frac{p}{q}) = -1$, then $D_H = pq$.*

In addition, the following lemma [1, Lemma 1.20] tells us precisely when a quaternion algebra $H_{\mathbb{Q}}(p,q)$ splits.

Lemma 4.3. *Let p and q be two prime integers. Then, $H_{\mathbb{Q}}(p,q)$ is a matrix algebra if and only if one of the following conditions is satisfied:*

1. *$p = q = 2$;*
2. *$p = q \equiv 1 \pmod 4$;*
3. *$q = 2$ and $p \equiv \pm 1 \pmod 8$;*
4. *$p \neq q$, $p \neq 2$, $q \neq 2$, $(\frac{q}{p}) = 1$ and either p or q is congruent to $1 \pmod 4$.*

The next theorem [1, Theorem 1.22] describes the discriminant of $H_{\mathbb{Q}}(p,q)$, where p and q are primes:

Theorem 4.8. *Let $H = (\frac{a,b}{\mathbb{Q}})$ be a quaternion algebra.*

1. *If $D_H = 1$, then H splits.*
2. *If $D_H = 2p$, p prime and $p \equiv 3 \pmod 4$, then $H \cong (\frac{p,-1}{\mathbb{Q}})$.*
3. *If $D_H = pq$, p, q primes, $q \equiv 1 \pmod 4$ and $(\frac{p}{q}) = -1$, then $H \cong (\frac{p,\,q}{\mathbb{Q}})$.*

If a and b are prime numbers, then the algebra H satisfies one and only one of the above statements.

Note that when $q = 2$ and p is a prime such that $p \equiv 3 \pmod 8$, then, according to Lemma 4.2 the discriminant $D_{H_{\mathbb{Q}(p,q)}}$ is equal to $2p$, so $H_{\mathbb{Q}}(p,q)$ is a division algebra. The next proposition shows what happens when we extend the field of scalars from $\mathbb{Q}$ to $\mathbb{Q}(\sqrt{d})$.

Proposition 4.5. *Let p be an odd prime, with $p \equiv 3 \pmod 8$. Let $K = \mathbb{Q}(\sqrt{d})$ and let Δ_K be the discriminant of K. Then $H_{\mathbb{Q}(\sqrt{d})}(p,2)$ is a division algebra if and only if $(\frac{\Delta_K}{p}) = 1$ or $d \equiv 1 \pmod 8$.*

Proof. If $H_{\mathbb{Q}(\sqrt{d})}(p,2)$ is a division algebra then, from Proposition 4.3, Theorem 2.1, Theorem 4.7 and Proposition 4.1, we conclude that $(\frac{\Delta_K}{p}) = 1$ or $d \equiv 1 \pmod 8$.

Conversely, since $p \equiv 3 \pmod 8$ then, according to Lemma 4.2(ii) we must have $D_H = 2p$. It follows that the primes which ramify in $H_{\mathbb{Q}}(p,2)$ are precisely p and 2. Since either $(\frac{\Delta_K}{p}) = 1$ or $d \equiv 1 \pmod 8$ then, by the decomposition of primes in quadratic fields, we obtain that either p or 2 splits in the ring or integers of K. From Theorem 4.7, Proposition 4.1 and Theorem 2.1, we conclude that $H_{\mathbb{Q}(\sqrt{d})}(p,2)$ is a division algebra. □

We next study the case when p and q are primes, both congruent to 3 modulo 4. If $(\frac{q}{p}) \neq 1$, then, according to Lemma 4.2(i), the discriminant $D_{H_{\mathbb{Q}}(p,q)}$ is equal to $2p$, so $H_{\mathbb{Q}}(p,q)$ is a division algebra. The next proposition tells us when the quaternion algebra $H_{\mathbb{Q}(\sqrt{d})}(p,q)$ is still a division algebra.

Proposition 4.6. *Let p and q be two odd prime integers, with $p \equiv q \equiv 3$ (mod 4) and $(\frac{q}{p}) \neq 1$. Let $K = \mathbb{Q}(\sqrt{d})$ and let Δ_K be the discriminant of K. Then the quaternion algebra $H_{\mathbb{Q}(\sqrt{d})}(p,q)$ is a division algebra if and only if $(\frac{\Delta_K}{p}) = 1$ or $d \equiv 1 \pmod 8$.*

Proof. If $H_{\mathbb{Q}(\sqrt{d})}(p,q)$ is a division algebra then from Proposition 4.3(i) it follows that either $(\frac{\Delta_K}{p}) = 1$ or $(\frac{\Delta_K}{q}) = 1$. But, according to Lemma 4.2(i) we must have $D_{H_{\mathbb{Q}}(p,q)} = 2p$. So the integral primes which ramify in $H_{\mathbb{Q}}(p,q)$ and could split in K are precisely p and 2. Finally, after applying Theorem 2.1, Theorem 4.7, Proposition 4.1 and the decomposition of primes in quadratic fields, we obtain that either $(\frac{\Delta_K}{p}) = 1$ or $d \equiv 1 \pmod 8$. The proof of the converse is similar to the proof of sufficiency of Proposition 4.5. □

The only case left out is $q = 2$, $p \equiv 5 \pmod 8$. We consider first the quaternion algebra $H_{\mathbb{Q}}(p,q)$, and we get the following result:

Lemma 4.4. *Let $p \equiv 5 \pmod 8$ be a prime integer. Then the discriminant of the quaternion algebra $H_{\mathbb{Q}}(p,2)$ is equal to $2p$, and hence $H_{\mathbb{Q}}(p,2)$ is a division algebra.*

Proof. We give here a simple proof which is independent of the theorems stated above. We know that if a prime divides the discriminant of $H_{\mathbb{Q}}(a,b)$ then it must divide $2ab$. Since $p \equiv 5 \pmod 8$, from the properties of the Hilbert symbol and of the Legendre symbol we obtain: $(2,p)_p = (\frac{2}{p}) = (-1)^{\frac{p^2-1}{8}} = -1$ and $(2,p)_2 = (-1)^{\frac{p-1}{2}\cdot\frac{1-1}{2}+\frac{p^2-1}{8}} = -1$. Hence the primes which ramify in $H_{\mathbb{Q}}(p,2)$ are exactly 2 and p. Therefore, the reduced discriminant of $H_{\mathbb{Q}}(p,2)$ must be equal to $2p$. □

We turn now our attention to the quaternion algebra $H_{\mathbb{Q}(\sqrt{d})}(p,q)$, where $q = 2$ and $p \equiv 5 \pmod 8$.

Proposition 4.7. *Let p be an odd prime, with $p \equiv 5 \pmod 8$. Let $K = \mathbb{Q}(\sqrt{d})$ and let Δ_K be the discriminant of K. Then $H_{\mathbb{Q}(\sqrt{d})}(p, 2)$ is a division algebra if and only if $(\frac{\Delta_K}{p}) = 1$ or $d \equiv 1 \pmod 8$.*

Proof. The proof is similar to the proof of Proposition 4.5, after replacing Lemma 4.2 with Lemma 4.4. □

Taking into account these results and Theorems 2.1 and 4.6, we are able to understand when a quaternion algebra $H_{\mathbb{Q}(\sqrt{d})}(p, q)$ splits. It is clear also that in each one of the cases covered by Lemma 4.3, a quaternion algebra $H_{\mathbb{Q}(\sqrt{d})}(p, q)$ splits.

5. Examples involving prime Fibonacci numbers

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence which is defined by the following recurrence relation:

$$F_0 = 0,\ F_1 = 1 \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.$$

Binet's formula, which has been discovered by the mathematician J. P. Marie Binet (1786- 1856), states that:

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right].$$

By means of this formula, one can show that (see [8]):

$$F_{2n+1} = F_{n+1}^2 + F_n^2 \text{ and } 4^n F_{2n+1} = \sum_{k=0}^{n} 5^k \mathrm{C}_{2n+1}^{2k+1}.$$

where C_m denotes the m-th Catalan number. From this formula we deduce that if p is an odd prime number > 5, then $F_p \equiv 1 \pmod 4$ and $F_p \equiv \left(\frac{p}{5}\right) \pmod p$. This implies the following lemma.

Lemma 5.1. *Let p be a prime > 5. Then the Fibonacci number F_p has the following properties:*

1. *if $p \equiv 1 \pmod 4$, then $(\frac{F_p}{p}) = 1$;*
2. *if $p \equiv 3 \pmod 4$, then $(\frac{F_p}{p}) = (\frac{p}{5})$.*

To prove the last theorem of this paper, we also need the following lemma.

Lemma 5.2 ([19])**.** *Let F_p be a Fibonacci number with prime index $p \equiv 1 \pmod 4$. Then*

$$\left(\frac{F_p}{p}\right)_4 \left(\frac{p}{F_p}\right)_4 = \begin{cases} 1, & \text{if } p \equiv 1 \pmod 3, \\ \left(\frac{2}{p}\right), & \text{if not.} \end{cases}$$

Theorem 5.1. *Let F_p be a Fibonacci prime number with $p > 5$ and let $K = \mathbb{Q}(\sqrt{p})$ be a quadratic number field. Then $H_K(\varepsilon_p, F_p)$ splits if and only if one of the following conditions holds:*

1. $p \equiv 1 \pmod{12}$;
2. $p \equiv 17 \pmod{24}$;
3. $p \equiv 3 \pmod 4$, $p^2 \equiv 1 \pmod 5$ *and* $(\frac{2}{F_p}) = 1$;
4. $p \equiv 3 \pmod 4$ *and* $p^2 \equiv 4 \pmod 5$.

Proof. First of all, recall that if F_n is prime, then n is prime; and n divides m if and only if F_n divides F_m, hence since F_p is a prime Fibonacci number such that $p > 5$, it follows that p is an odd prime.

• If $p \equiv 1 \pmod 4$, then $N(\varepsilon_p) = -1$ and $(\frac{p}{F_p}) = (\frac{F_p}{p}) = 1$ ((1) of Lemma 5.1). We can then apply Corollary 4.4 and find that $H_K(\varepsilon_p, F_p)$ splits if and only if $(\frac{F_p}{p})_4(\frac{p}{F_p})_4 = 1$. According to Lemma 5.2, this is equivalent to $p \equiv 1 \pmod 3$ or $p \equiv 1 \pmod 8$. By the Chinese remainder theorem, we deduce that $p \equiv 1 \pmod{12}$ or $p \equiv 17 \pmod{24}$.

• If $p \equiv 3 \pmod 4$, then $N(\varepsilon_p) = 1$. We must now study the value of the symbol $(\frac{F_p}{p})$. If $(\frac{F_p}{p}) = 1$, then the second assertion of Lemma 5.1 implies that $(\frac{5}{p}) = 1$, i.e., $p^2 \equiv 1 \pmod 5$. In this case $H_K(\varepsilon_p, F_p)$ splits if and only if $(\frac{2}{F_p}) = 1$ ([2, Lemma 6] and 2. of Theorem 4.5). Finally, if $p^2 \equiv 4 \pmod 5$, then Theorem 4.2 shows that $H_K(\varepsilon_p, F_p)$ splits. □

As an application of this theorem, we used the SageMath software [18] to find all prime or probably prime Fibonacci numbers F_p ($p > 5$) known until today (see sequence A001605 in the OEIS or see Wikipedia) such that $H_K(\varepsilon_p, F_p)$ and p satisfy conditions 1, 2, 3 and 4 above:

1. $p \in \{13, 433, 25561, 30757, 50833, 130021, 590041, 593689, 1968721\}$;
2. $p \in \{17, 137, 449, 569, 1049897, 2904353\}$;
3. $p \in \{11, 131, 359, 431, 9311, 35999, 37511, 81839, 148091, 397379, 1803059\}$;
4. $p \in \{7, 23, 43, 47, 83, 4723, 5387, 201107, 1285607, 1636007\}$.

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