ON LOCALLY CONFORMALLY KÄHLER THREEFOLDS WITH ALGEBRAIC DIMENSION TWO

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Abstract. The paper is part of an attempt of understanding non-Kähler threefolds. We start by looking at compact complex threefolds with algebraic dimension two and admitting locally conformally Kähler metrics. Under mild assumptions, we prove that they are blown-up quasi-bundles over a projective surface.

Motivation and outline of the paper

The paper is part of an attempt of understanding non-Moishezon non-Kähler threefolds. We start by looking at the simplest case: threefolds X with algebraic dimension $a(X) = 2$. In this case, it is classically known that they are bimeromorphic to elliptic fibrations over projective surfaces, that is, there exists a smooth bimeromorphic threefold X^* and a surjective holomorphic map $f\colon X^* \to B$ whose general fibres are smooth elliptic curves. In fact, X^* is an *algebraic reduction* of X. The main goal of the paper is to give a description of X^*, and then retrieve informations about X.

By generalizing what happens on non-Kähler surfaces S of algebraic dimension $a(S) = 1$ [Brî96, Proposition 3.17], we prove that, under mild assumptions that we now describe, the algebraic reduction is a *quasi-bundle*, namely, the fibres with the reduced structure are smooth elliptic isomorphic curves. The main idea of our strategy is inspired by the Lefschetz hyperplane theorems in algebraic geometry. More exactly, we consider divisors H on B, and look at their preimages $S_H := f^{-1}(H) \subset X$. Then we would like to take advantage of the known results for compact complex non-Kähler surfaces.

Unfortunately, the information about non-Kählerianity passes badly from X to S_H: we will exhibit in Example 1.1 a non-Kähler X having a smooth Kähler S_H. Still, if X^* is lcK and H is general enough, we are able to show that S_H is non-Kähler as well. Here, by *locally conformally Kähler (lcK)*,

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we mean that there is a Hermitian metric that locally admits a conformal change to a Kähler metric.

From here, and the description of non-Kähler elliptic surfaces, we are able to show that:

Theorem 2.7. *Let X be an lcK threefold, which is an elliptic fibration $f\colon X \to B$ over a projective surface B. Then the general fibres of f are isomorphic elliptic curves.*

In fact, the threefold X fibres over the projective surface B only up to bimeromorphisms,

$$\begin{array}{ccc} X & \overset{}{\underset{\psi}{\dashleftarrow\dashrightarrow}} & X^* \\ & & \Big\downarrow{\scriptstyle f} \\ & & B, \end{array}$$

that can be factorized as a sequence of blow-ups and blow-downs with smooth centres. Unfortunately Kählerianity in dimensions at least 3 behave badly with respect to bimeromorphisms: the blow-up of a Kähler manifold is Kähler, but the blow-down of a Kähler manifold may well be non-Kähler. With lcK metrics, the situation is even worse: it may happen that even a blow-up of an lcK manifold carries no lcK metric. So, we introduce the *ad hoc* notion of *weak locally conformally Kähler* (wlcK, for short) structure, see Section 2.1. The main point is that the wlcK condition is behaving well under blow-up: is X is wlcK and $\hat{X}$ is an arbitrary blow-up of it, then $\hat{X}$ has a wlcK structure too. With this notion, we are able to show that Theorem 2.7 holds if one relaxes the lcK hypothesis to just wlcK.

Returning to bimeromorphisms, we already noticed that one can always reach the algebraic reduction X^* starting from X by means of a chain of blow-ups and blow-downs with smooth centres, the so-called Weak Factorization Theorem. Since wlcK structures behave badly with respect to blow-downs, we will work under the assumption that the Strong Factorization Conjecture holds true: that is, there are a (smooth) threefold $\hat{X}$ and maps σ and c compositions of blow-ups with smooth centres such that

$$\begin{array}{ccccc} & & \hat{X} & & \\ & {\scriptstyle\sigma}\swarrow & & \searrow{\scriptstyle c} & \\ X & & \underset{\psi}{\dashleftarrow\dashrightarrow} & & X^*. \end{array}$$

Thus, if X has an lcK metric, then $\hat{X}$ still has a wlcK structure, so by Theorem 2.7, we know the structure of the general fibres of $F := f \circ \sigma$.

In the above notations, it may still happen that the map F is non-flat morphism, *i.e.* some of its fibres may have dimension 2. Applying the Hironaka flattening [Hir75] to F may result into putting heavy singularities on the flatenning, so we will make the further assumption that F can be factored as above with f *flat morphism* (compare Examples 1.1 and 1.2). Moreover, we will also assume that there is such an X^* which is *minimal*, – in the sense of Mori program, that is K_{X^*} is nef. Finally, we will also assume that the singular locus $S(f)$ of the fibration f is a simple normal crossing divisor. In our case, we wonder whether the assumptions above should hold in general: but we were so far unable to give a rigourous proof.

Under these assumptions, we are able to prove:

Theorem 2.10. *All the singular fibres of f are in fact just multiple of the general fibre, that is, X^* is a quasi-bundle.*

Moreover, using the properties of non-Kähler quasi-bundles, we can prove:

Theorem 2.11. *One can take $X = \hat{X}$; moreover, the chain of blow-ups σ is special, in the sense that any blow-up in it is either a blow-up of a point, or a blow-up along a curve contained in a (previously produced) exceptional divisor.*

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1. Prerequisites and examples

Let X be a compact threefold of algebraic dimension $a(X) = 2$. We known that X fibres over a smooth projective surface up to bimeromorphisms [Uen75, page 25]: more precisely, there is an *algebraic reduction*

$$\begin{array}{ccc} X & \overset{\psi}{\dashleftarrow\dashrightarrow} & X^* \\ & & \downarrow{\scriptstyle f} \\ & & B \end{array} \tag{1}$$

where X^* is a compact complex threefold bimeromorphic to X through the bimeromorphism ψ, and f is a surjective, proper, holomorphic map onto the compact projective surface B, with connected fibres [Uen75, Proposition 3.4]. We know that the general fibre of f is smooth [Uen75, Corollary 1.8] and elliptic [Kaw65, Theorem 2], [Uen75, Theorem 12.4].

In the following, we describe the setting for the main results.

Flatness. The following two examples show that f being non-flat might result in unexpected behaviour for the degeration of fibres.

Example 1.1. *Let $H^3 = \mathbb{C}^3 \setminus \{0\}/\mathbb{Z}$ be the Hopf threefold, where $\mathbb{Z}$ acts as $z \mapsto 2z$, and $H^3 \ni (z_1, z_2, z_3) \mapsto [z_1 : z_2 : z_3] \in \mathbb{P}^2$ be the Hopf fibration. Consider the blow-up of $\mathbb{P}^2$ at a point $p \in \mathbb{P}^2$, and consider the following diagram:*

$$\begin{array}{ccc} H^3 & \longleftarrow & H^3 \times_{\mathbb{P}^1} \mathrm{Bl}_p\mathbb{P}^2 \\ {\scriptstyle \pi}\downarrow & \overset{f}{\swarrow} & \downarrow{\scriptstyle \pi'} \\ \mathbb{P}^2 & \underset{\sigma}{\longleftarrow} & \mathrm{Bl}_p\mathbb{P}^2 \end{array}$$

where in fact $H^3 \times_{\mathbb{P}^1} \mathrm{Bl}_p\mathbb{P}^2 \simeq \mathrm{Bl}_{\pi^{-1}(p)}H^3$. We consider the fibration

$$f := \pi' \circ \sigma \colon H^3 \times_{\mathbb{P}^1} \mathrm{Bl}_p\mathbb{P}^2 \to \mathbb{P}^2.$$

Over any point $q \neq p$, the fibre is the same as the fibre of π at p, namely, the smooth elliptic curve $\mathcal{E} := \mathbb{C} \setminus \{0\}/\mathbb{Z}$. Over p, the fibre is $f^{-1}(p) = \mathbb{P}^1 \times \mathcal{E}$. Therefore the family of elliptic curves degenerates to a Kähler surface.

In this same example, we also notice that $H^3 \times_{\mathbb{P}^1} \mathrm{Bl}_p\mathbb{P}^2$ is a non-Kähler manifold fibring over $\mathrm{Bl}_p\mathbb{P}^2$ via π', with a Kähler fibre. In fact, $H^3 \times_{\mathbb{P}^1} \mathrm{Bl}_p\mathbb{P}^2$ does not admit any lcK metric. Indeed, let (ω, θ) be an lcK metric on $H^3 \times_{\mathbb{P}^1} \mathrm{Bl}_p\mathbb{P}^2$. Then $\omega|_{\mathbb{P}^1 \times \mathcal{E}}$ is lcK on a Kähler surface, whence $[\theta]|_{\mathbb{P}^1 \times \mathcal{E}} = 0$. In particular, $[\theta]|_{\mathcal{E}} = 0$. Therefore $[\theta] = (\pi')^{-1}([\alpha])$ for $[\alpha] \in H^1(\mathrm{Bl}_p\mathbb{P}^2; \mathbb{R})$. By Lemma 2.4 in the next section, then $[\theta] = 0$.

Example 1.2. *Consider again the blow-up of $\mathbb{P}^2$ at a point, $\mathrm{Bl}_p\mathbb{P}^2$. Since $\mathrm{Bl}_p\mathbb{P}^2$ is a projective manifold, let us embed it into $\mathbb{P}^N$ for N large enough. Consider the Hopf manifold of dimension $N+1$ and the diagram*

$$\begin{array}{ccccc} H^{N+1} & \longleftarrow & \pi^{-1}(\mathrm{Bl}_p\mathbb{P}^2) & & \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & \overset{f}{\searrow} & \\ \mathbb{P}^N & \hookleftarrow & \mathrm{Bl}_p\mathbb{P}^2 & \longrightarrow & \mathbb{P}^2 \end{array}$$

Here the restriction of π to any curve $C \subset \mathrm{Bl}_p\mathbb{P}^2$ is a non-trivial elliptic principal bundle, therefore $f^{-1}(p)$ is a Hopf surface, in particular, non-Kähler.

Multiple fibres. The next example is meant to illustrate that usually one also has multiple fibres, but they can be "resolved" by means of finite Galois coverings of the base.

Example 1.3. *Let a, b, c be positive real numbers for which there exists $n_1, n_2, n_3 \in \mathbb{N} \setminus \{0\}$ such that $a^{n_1} = b^{n_2} = c^{n_3} > 1$. Define $H_{a,b,c} := \mathbb{C}^3 \setminus \{0\}/\mathbb{Z}$ where the generator g of $\mathbb{Z}$ acts by*

$$g \cdot (z_1, z_2, z_3) := (az_1, bz_2, cz_3).$$

Then one has a map $f : H_{a,b,c} \to \mathbb{P}^2$ defined by

$$f(z_1, z_2, z_3) := [z_1^{n_1} : z_2^{n_2} : z_3^{n_3}].$$

Then f has multiple fibres: namely, for $h \neq 0$ and $k \neq 0$, the fibre $f^{-1}([0 : h : k])$ has multiplicity n_1, the fibre $f^{-1}([h : 0 : k]$ has multiplicity n_2, the fibre $f^{-1}([h : k : 0])$ has multiplicity n_3; the fibre $f^{-1}([0 : 0 : 1])$ has multiplicity $n_1 \cdot n_2$, the fibre $f^{-1}([0 : 1 : 0])$ has multiplicity $n_1 \cdot n_3$, the fibre $f^{-1}([1 : 0 : 0])$ has multiplicity $n_2 \cdot n_3$.

Let μ to be the common value $\mu := a^{n_1} = b^{n_2} = c^{n_3}$, and let λ be any solution of $\lambda^{n_1 n_2 n_3} = \mu$ such that $\lambda^{n_2 n_3} = a$, $\lambda^{n_1 n_3} = b$, $\lambda^{n_1 n_2} = c$. Then we have a natural map $\pi : H_{\lambda,\lambda,\lambda} \to H_{a,b,c}$

$$\pi(z_1, z_2, z_3) = (z_1^{n_2 n_3}, z_2^{n_1 n_3}, z_3^{n_1 n_2})$$

whch is a (finite) Galois covering. In fact, the natural map

$$f' \colon H_{\lambda,\lambda,\lambda} \to \mathbb{P}^2, \qquad f'(z_1, z_2, z_3) := [z_1, z_2, z_3]$$

turns $H_{\lambda,\lambda,\lambda}$ into an elliptic principal bundle, and $H_{\lambda,\lambda,\lambda}$ is the fibred product obtained from the diagram

$$\begin{array}{ccc} H_{a,b,c} & \longleftarrow & H_{a,b,c} \times_{\mathbb{P}^2} \mathbb{P}^2 = H_{\lambda,\lambda,\lambda} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ \mathbb{P}^2 & \underset{\pi'}{\longleftarrow} & \mathbb{P}^2 \end{array}$$

where the map π' is given by

$$\pi'([z_1 : z_2 : z_3]) = [z_1^{n_1 n_2 n_3} : z_2^{n_1 n_2 n_3} : z_3^{n_1 n_2 n_3}].$$

Put in other words, $H_{\lambda,\lambda,\lambda}$ is finite Galois cover of $H_{a,b,c}$ (with deck group $G = \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z} \times \mathbb{Z}/n_3\mathbb{Z}$). That is, one can pass form an elliptic fibration with multiple fibres to one with no multiple fibres by means of a suitable finite cover.

Bimeromorphisms. We recall the basic facts on factorization of bimeromorphic map, according to [AKMWo02]. The *Weak Factorization Theorem* [AKMWo02, Theorem 0.3.1] assures that any bimeromorphism ψ can be factorized as a sequence of blow-ups and blow-downs with smooth centres. It is conjectured that the following stronger result holds true, and we will further on assume it throught the paper:

Conjecture 1.4 (Strong Factorization Conjecture [Hir64, AKMWo02])**.** *Any bimeromophism $\psi\colon X \dashrightarrow X^*$ between compact complex manifolds can be factorized as*

$$\begin{array}{ccc} & \hat{X} & \\ {\scriptstyle \sigma}\swarrow & & \searrow{\scriptstyle c} \\ X & \underset{\psi}{\dashrightarrow} & X^* \end{array} \tag{2}$$

where σ and c are (compositions of) blow-ups with smooth centres.

Non-Kähler surfaces. We recollect below the basic facts about non-Kähler sufaces with algebraic dimension one; the main references here are [Kod60, Kod63, Uen75, BHPVdV04, Brî96].

If X is a (smooth compact complex) surface X with $a(X) = 1$, then there exists a projective curve B and a surjective holomorphic map $f\colon X \to B$ whose general fibres are smooth elliptic curves [Uen75, Theorem 12.4].

Next, if X is non-Kähler, then all the smooth fibres of f are isomorphic. If X is also a *minimal model*, that is, K_X is nef, then the structure of the singular fibres of f is also easy to determine: any such fibre is actually a multiple fibre, whose reduction is isomorphic to the general fibre [Brî96, Proposition 3.17]. It follows that, for a general such X (*i.e.* not necessary minimal), the singular fibres are all of the form

$$mE + \text{tree of rational curves}$$

where E is the general fibre and $m \geq 1$.

Locally conformally Kähler structures. In higher dimension, we work in the non-Kähler setting, more precisely, in the realm of locally conformally

Kähler geometry. We recall that a *locally conformally Kähler structure* (lcK, for short) on a complex manifold is a Hermitian metric that locally admits a conformal change to a Kähler metric. If ω denotes the associated $(1,1)$-form of the metric, then the lcK condition corresponds to the equations

$$d\omega = \theta \wedge \omega, \qquad d\theta = 0,$$

where θ is a 1-form, called the *Lee form* of ω. More precisely, if $\theta \stackrel{\text{loc}}{=} df$ by the Poincaré Lemma, then $\exp(-f)\omega$ is a local Kähler metric.

On the connected component of two such open sets where ω is conformally Kähler, the corresponding Kähler metrics differ up to homothety: in this sense, locally conformally Kähler geometry can be intended as a sort of "equivariant homothetic Kähler geometry" [GOPP06].

We refer to [DO98, Orn05, Baz18] for an open-ended account of lcK geometry and its applications. We are interested in lcK geometry as a fruitful extension of the Kähler condition. In this direction, it is promising that almost any compact complex surface admits an lcK metric, with the exception of some Inoue surfaces [Bel00].

One can prove that the class $[\theta] \in H^1(X;\mathbb{R})$ being zero corresponds to ω being globally conformally Kähler. In fact, it has been proven by Vaisman [Vai80] that an lcK structure on a Kähler manifold (or, more generally, a compact complex manifold satisfying the $\partial\overline{\partial}$-Lemma [KK10, Proposition 5.1]) is necessarily gcK. Hereafter, when talking about lcK structures, we always assume the Lee class to be non-zero.

Coming back to the diagram (2), we notice that, if X is (non-Kähler) lcK, then also X^* is non-Kähler. Otherwise, $\hat{X}$ would be Käler too, see *e.g.* [Voi07, Proposition 3.24]. Then, by Lemma 2.6, we would get $[\theta] = 0$.

2. Results

2.1. **Weak locally conformally Kähler structures.**

It is known that the property of admitting a Kähler metric is not stable under bimeromorphisms: more precisely, it is preserved under blow-up, see *e.g.* [Voi07, Proposition 3.24], but in general it is not preserved under blow-down [Hir62], except when the centre is a point [Miy74]. On the other side, the lcK property is not preserved even under blow-up (see Example 1.1), except when the centre is a point [Tri82, Vul09]. For this reason, we introduce the following weaker notion:

Definition 2.1. *A* weak locally conformally Kähler *(shortly, wlcK) structure on a complex manifold X is given by a $(1,1)$-form ω and a real 1-form θ such that*

- $d\omega = \theta \wedge \omega$ *and* $d\theta = 0$, *and*
- $\omega > 0$ *outside a proper analytic subset, that we will call the* bad locus N_ω.

This notion is motivated by the following straightforward remark:

Proposition 2.2. *Let $\sigma \colon \hat{X} \to X$ be a proper modification (e.g. a composition of blow-ups). If X admits a wlcK metric, then $\hat{X}$ does, too.*

Remark 2.3. *The manifold X underlying $\mathbb{S}^3 \times \mathbb{S}^3$ with the Calabi-Eckmann complex structure does not admit any wlcK structure. Indeed, if (ω, θ) would be a wlcK structure, then since $b_1(X) = 0$, we get θ exact. That is, by a conformal rescaling, we can assume ω satisfying $d\omega = 0$. As $b_2(X) = 0$, we get ω is exact, hence $\int_X \omega^3 = 0$. On the other side, since $\omega > 0$ outside a set of Lebesgue measure zero, we get $\int_X \omega^3 > 0$: contradiction.*

In the next sections, we will use the following lemmata. The first one deals with wlcK structures on fibrations:

Lemma 2.4 ("Lemma on fibrations")**.** *Let X and B be complex manifolds with $\dim X > \dim B$. Let $f \colon X \to B$ be a surjective proper holomorphic map with connected fibres. Let (ω, θ) be a wlcK structure on X such that the bad locus N_ω is contained into some proper analytic subspace of X. If the Lee class $[\theta] = f^*[\alpha]$ in the image of the pull-back induced by f, where $[\alpha] \in H^1(B; \mathbb{R})$, then $[\theta] = 0$.*

Proof. Up to global conformal changes, let us assume $\theta = f^*\alpha$. We denote by S_f the union of the non-flat and singular loci, namely, the set of points $b \in B$ such that the corresponding fibre $f^{-1}(b)$ is either singular, or has dimension strictly greater than $k := \dim X - \dim B > 0$. That S_f is an analytic subspace. Moreover, the restriction of f to $X \setminus f^{-1}(S_f) \to B \setminus S_f$ is now a submersion. We also consider the open subset $B' \subset B \setminus S_f$ such that, for any $b \in B'$, we have

$$v(b) := (f_*\omega^k)(b) = \int_{f^{-1}(b)} \omega^k|_{f^{-1}(b)} = \mathrm{vol}_\omega(f^{-1}(b)) > 0$$

is a positive function. By the weak assumption on the metric, the complemente of B' in S is an analytic subset. By the wlcK condition $d\omega = f^*\alpha \wedge \omega$ and by the projection formula, we compute

$$d(f_*\omega^k) = f_*(d\omega^k) = k f_*(f^*\alpha \wedge \omega^k) = k\alpha \wedge f_*\omega^k,$$

that we rewrite as

$$dv = k\alpha \wedge v.$$

This says that

$$\alpha|_{B'} = \frac{1}{k}\frac{dv}{v} = \frac{1}{k} d \lg v,$$

namely, $[\alpha]|_{B'} = 0$. The map $H^1(B; \mathbb{R}) \to H^1(B'; \mathbb{R})$ induced by the inclusion is injective. This follows, for example, by taking the long exact sequence in cohomology for the pair (B, B') and by noticing that $H^1(B, B'; \mathbb{R}) = 0$. Therefore we get that $[\alpha] = 0$ in $H^1(B; \mathbb{R})$, whence the statement. □

Remark 2.5. *As a consequence we have that if $X \to B$ is an elliptic bundle and X is wlcK, then X is in fact an elliptic principal bundle. Indeed, if X is not a principal bundle, it follows that the natural map $H^1(B) \to H^1(X)$ is an isomorphism.*

The second lemma is a generalization of the Vaisman lemma [Vai80] to the weak lcK context:

Lemma 2.6. *Let X be a compact complex manifold endowed with a wlcK structure (ω, θ). If X admits Kähler metrics, then $[\theta] = 0$.*

Proof. Denote by n the complex dimension of X. Fix g_0 a Kähler metric on X. Let θ_0 be the harmonic representative of $[\theta]$ with respect to g_0. Up to global conformal change, we can take the wlcK structure ω_0 with Lee form θ_0. Let $\theta_0 = \alpha + \bar{\alpha}$ be the decomposition of θ_0 into pure-type components, where $\alpha \in \wedge^{1,0}X$. Multiplying the condition $d\omega_0^{n-1} = (n-1)\theta_0 \wedge \omega_0^{n-1}$ by α, we get

$$\alpha \wedge d\omega_0^{n-1} = (n-1)\alpha \wedge \bar{\alpha} \wedge \omega_0^{n-1}.$$

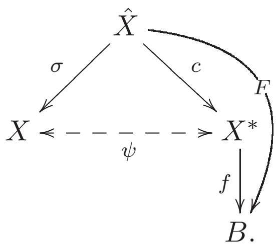
By the strong Hodge decomposition, α is itself harmonic: in particular, $d\alpha = 0$. It follows that

$$0 = \int_X \alpha \wedge d\omega_0^{n-1} = (n-1)\|\alpha\|^2_{\omega_0|_{X\setminus N_\omega}}.$$

Therefore $\alpha = 0$, yielding the statement. □

2.2. **General fibres of the algebraic reduction.** Under the above assumptions, we can prove the first result: as in the case of surfaces, the general fibre of the algebraic reduction has constant j-invariant. The reasonement is pretty much an extension of Example 1.1, using this time the "Lemma on fibrations" 2.4 in its more general setup of wlcK.

Theorem 2.7. *Let X be a compact complex threefold with $a(X) = 2$, endowed with an lcK structure (ω, θ). Assume the Strong Factorization Conjecture, and consider the algebraic reduction*



(3)

Then all the smooth one-dimensional fibres of f are isomorphic.

Proof. It suffices to prove that the general fibres of f are isomorphic, since by continuity of the j-invariant, the claim follows.

Denote by E the exceptional divisor of the blow-up σ. Consider $F = f \circ c \colon \hat{X} \to B$. Fix a very ample line bundle $\mathcal{H} \in \mathrm{Pic}(B)$ on the projective surface B, and set $\mathcal{L} := F^*(\mathcal{H})$. Note that $\mathcal{L}$ is globally generated. By applying twice the Bertini theorem for a general section $H \in |\mathcal{H}|$, both H and $S_H := F^{-1}(H) \in |\mathcal{L}|$ are smooth, and we can also assume that $S_H \cap E \subsetneq S_H$. The last fact assures that $\sigma^*(\omega)|_{S_H}$ is wlcK.

We claim that S_H is non-Kähler. On the contrary, if S_H was Kähler, then $[\theta]|_{S_H} = 0$ in $H^1(\hat{X};\mathbb{R})$ by Lemma 2.6. In particular, the restriction of the class $[\theta]$ to the general fibre of F vanishes, namely, there exists an analytic subset $\mathcal{Z} \subset B$ such that $F \colon \hat{X} \setminus F^{-1}(\mathcal{Z}) \to B \setminus \mathcal{Z}$ is a submersion with fibre $\mathcal{E}$ and such that $[\theta]|_{\mathcal{E}} = 0$. By the long exact sequence of the fibration, we get $[\theta]|_{\hat{X}\setminus F^{-1}(\mathcal{Z})} = F^*([\alpha])$ for some $[\alpha] \in H^1(B \setminus \mathcal{Z};\mathbb{R})$. By Lemma 2.4, we get $[\theta]|_{\hat{X}\setminus F^{-1}(\mathcal{Z})} = 0$. Since the map $H^1(\hat{X};\mathbb{R}) \to H^1(\hat{X} \setminus F^{-1}(\mathcal{Z});\mathbb{R})$ is injective and the map $H^1(\hat{X};\mathbb{R}) \to H^1(X;\mathbb{R})$ is an isomorphism, we get that $[\theta] = 0$ in $H^1(X;\mathbb{R})$.

Therefore, S_H is non-Kähler. By a theorem of [Brî96, Proposition 3.17], we get that the general fibres of $S_H \to H$ are isomorphic. Therefore, the general fibres of f are isomorphic. □

Remark 2.8. *By induction, the statement of Theorem 2.7 holds true for lcK manifolds X of arbitrary dimension n having $a(X) = n - 1$.*

2.3. **Singular fibres of the algebraic reduction.** First, we state a lemma which will be used later. To emphasize the usefulness of the lemma, notice that for elliptic (Kähler) surfaces there may exist isolated fibres that are singular when given the reduced structure [Kod63]. The next lemma simply says that such a situation cannot occur for our 3-folds.

Lemma 2.9. *Denote by B° be the $2-$dimensional ball with origin removed, $B^\circ := B^2 \setminus \{0\}$, and let U be a 3-manifold such that there is a flat map $f\colon U \to B^2$. Assume that all the fibres of f are smooth isomorphic elliptic curves, call them $\mathcal{E}$, except maybe for the central one, $F_0 := f^{-1}(0)$. Then the central fibre F_0 is smooth as well, and isomorphic to the general fibre.*

Proof. Let $V := U \setminus F_0$: then the map $f_0 := f|_V\colon V \to B^\circ$ is a submersion. Since all fibres of f_0 are smooth isomorphic, by the Fischer-Grauert theorem and by [Brî96, Corollary 2.12], we see that f_0 is actually an elliptic bundle (*a priori* not necessary principal). Using the same notations as in [BHPVdV04, Section V.5], we denote by $\mathcal{A}_\mathcal{E}$ group of automorphisms of $\mathcal{E}$: then we have a short exact sequences of (non-Abelian) groups

$$0 \to \mathcal{E} \to \mathcal{A}_\mathcal{E} \to \mathbb{Z}_m \to 0$$

for some $m \in \{2, 4, 6\}$. We denote by the same letters $\mathcal{E}$, respectively $\mathcal{A}_\mathcal{E}$, the sheaves of germs of holomorphic functions on B with values in $\mathcal{E}$, respectively $\mathcal{A}_\mathcal{E}$. Then the set of elliptic bundles with typical fibre $\mathcal{E}$ over B° is in natural bijection with $H^1(B^\circ; \mathcal{A}_\mathcal{E})$, while the set of principal elliptic bundles is parametrised by $H^1(B^\circ; \mathcal{E})$. The above exact sequence and the fact that $H^1(B^\circ; \mathbb{Z}_m)$ is trivial tells us that in fact the map f_0 is a principal bundle.

Next, we fix a point P in the regular part of F_0; one can find local coordinates (z_1, z_2, z_3) on X in some neighbourhood of P such that $P = (0, 0, 0)$ and F_0 is given by $\{z_2 = z_3 = 0\}$. Then the set $\Sigma := \{(0, z_2, z_3) \ : \ |z_2|, |z_3| \text{ small enough}\}$ defines a local section of the principal bundle above. Up to take a smaller ball, it follows that f_0 is actually trivial, in particular, there exists an isomorphism $\varphi^\circ\colon U \setminus F_0 \to B^\circ \times \mathcal{E}$. We claim that φ° extends to an isomorphism $\varphi\colon U \to B \times \mathcal{E}$.

Indeed, the image $\Sigma' := \varphi^\circ(\Sigma)$ is a section of $B^\circ \times \mathcal{E}$; it follows that $B^\circ \times \mathcal{E} \setminus \Sigma'$ is Stein, hence the restriction of φ° to $(U \setminus \Sigma) \setminus F_0 \subset U \setminus \Sigma \subset \mathbb{C}^7$ extends over $F_0 \cap (U \setminus \Sigma)$ to a holomorphic map. Hence we extended φ° to all the points of U except P, in particular along $F_0 \setminus P$: but then Hartogs shows that eventually φ° extends also over P.

Now clearly the extension φ remains an isomorphism, since it is an isomorphism is codimension two. □

We now understand the singular fibres of the algebraic reduction. To this aim, we also assume that X^* in the algebraic reduction (1) is a *minimal* *model* (in the sense of Mori), in other words, that K_{X^*} is *nef*, namely, $(K_{X^*}.C) \geq 0$ for any curve C on X^*.

For the next theorem, compare Example 1.3.

Theorem 2.10. *Let X be a compact complex threefold with $a(X) = 2$, endowed with an lcK structure (ω, θ). Assume the Strong Factorization Conjecture, and consider the algebraic reduction as in (3). Assume that f is flat and $S(f)$ is a simple normal crossing divisor. Assume moreover that K_{X^*} is nef. Then all fibres of f with the reduced structure are elliptic and isomorphic, that is X^* is a quasi-bundle.*

Proof. First notice that, by the Bertini theorem, through the fibre over the general point of $S(f)$ there is a smooth S_H. By [Brî96, Corollary 3.17], it follows that the general singular fibre of f is of the form

$$mE + \text{ tree of rational curves},$$

for $m \in \mathbb{N}$.

The tree of rational curves is easily excluded from the assumption that K_{X^*} is nef. Indeed, let $C \subset X^*$ be a rational curve in the fibre. By the adjunction formula, $K_C = K_{X^*}|_C \otimes \det N_{C|X^*}$, where $N_{C|X^*} = \mathcal{O}_C$ since C is in the fibre. Therefore $\deg K_C = \deg K_{X^*}|_C = (K_{X^*}.C)$. On the one side, $\deg K_C = 0$, the curve C being rational; on the other side $(K_{X^*}.C) \geq 0$ by the nef assumption. This is absurd.

Therefore the general singular fibre is of the form mE; notice that E is isomorphic to the general fibre by the continuity of j-invariant. It is easy to show that the multiplicity m is locally constant, so any singular fibre possibly not of this form lives either over an smooth point of the singular locus and all the neighbouring fibres are of a given multiplicity m, or lives above a node of $S(f)$ and the fibres over its neighbouring points have multiplicities m, respectively n on each branch of $s \circ f$. Taking a Galois cover of order m along $S(f)$ in the first case, or of order m along the first branch and respectively of order n along the second, we may now assume that f has finitely many singular fibres. But using the Lemma 2.9 above, we conclude that all the singular fibres of f are just multiple structures on the smooth elliptic general fibre, □

2.4. **Structure of LCK 3-folds.** Finally, we can further describe the structure of the algebraic reduction:

Theorem 2.11. *Let X be a compact complex threefold with $a(X) = 2$, endowed with an lcK structure (ω, θ). Assume the Strong Factorization Conjecture, and consider the algebraic reduction as in (3). Assume that f is flat and $S(f)$ is a simple normal crossing divisor. Assume moreover that K_{X^*} is nef. Then:*

- *the map σ in (3) can be taken to be the identity;*
- *and the blow-ups in c are "special", in the sense that any blow-up in it is either a blow-up of a point, or a blow-up along a curve contained in a (previously produced) exceptional divisor.*

Proof. Let us proceed with the proof.

- The map σ is a finite composition $\sigma_m \circ \cdots \circ \sigma_1$ of blow-ups. Take the last blow-up σ_m: denote by Z its centre (either a point or a curve), and by E its exceptional divisor (respectively, either $\mathbb{P}^2$ or a ruled surface). We claim that E is contracted under c. On the contrary, assume that E is not exceptional for c, namely, $S := c(E) \subset X^*$ is a surface. We look at the image $f(S) \subseteq B$.

  Three cases may occur: the image of S under f is either the whole B, or a curve in B, or just a point. If $f(S) = B$, then S intersects the general fibre in at least one and finitely-many points. On the other side, by Nakayama [Nak02, Theorem 7.4.4], the Poincaré dual $PD(F)$ of the general fibre F is 0, whence

$$S.F = \int_{X^*} PD(S) \wedge PD(F) = 0.$$

  We consider the case when $f(S)$ is a curve and Z is a point. Then $E = \mathbb{P}^2$. In this case, $f \circ c$ would be a non-constant map from $\mathbb{P}^2$ to a curve, which is not possible, because any two curves in $\mathbb{P}^2$ intersect.

  Let us consider now the case $f(S)$ is a curve and Z is a curve. In this case, E is a ruled surface: let $\ell \simeq \mathbb{P}^1$ be a general member of its ruling, and $\ell' \subset X^*$ its image under c. Notice that

$$(K_{\hat{X}}.\ell) = -1. \tag{4}$$

  If E is not contracted by c, we get that ℓ' is a curve and $c|_\ell : \ell \to \ell'$ is its normalization. As K_{X^*} is nef, we have $(K_{X^*}.\ell') \geq 0$. But since c is a blow-up, we get that $K_{\hat{X}} = c^*(K_{X^*}) + \Delta$ where Δ is some effective divisor on $\hat{X}$; notice that ℓ is not included in Δ. But then $(K_{\hat{X}}.\ell) = (c^*K_{X^*}.\ell) + (\Delta.\ell) \geq (c^*K_{X^*}.\ell) = (K_{X^*}.\ell') \geq 0$, contradicting (4).

  The last case when $f(S)$ is a point is excluded by the assumption that f is flat.
- The last thing to prove is that, when one blows-up curves, these must be contained in some exceptional divisor. In fact, all curves on a blow-up of X^* are either pullbacks of curves from X^* or contained into some exceptional divisor. But on X^* the only curves are the fibres of the fibration (see [Brî96, Proposition 2.14]), and blowing them up rules out the possibility for the blow-up manifold to have an lcK metric, or even a wlcK structure, by the same argument as in Example 1.1. □

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