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Editorial

Algebraic, Analytic, and Computational Number Theory and Its Applications

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Analytic number theory is a branch of number theory which inherits methods from mathematical analysis in order to solve difficult problems about the integers. Analytic number theory can be split into two major areas: multiplicative number theory and additive number theory. Bernhard Riemann made some very important contributions to the field of analytic number theory: among others, he investigated the Riemann zeta function and he established its importance for understanding the distribution of prime numbers. Some of the most useful tools in multiplicative number theory are the Dirichlet series and the technique of partial summation, which can be used to characterize the coefficients of the Dirichlet series [1]. A typical problem of analytic number theory is the enumeration of number-theoretic objects like primes, solutions of Diophantine equations, etc.

Algebraic number theory, on the other hand, studies the arithmetic of algebraic number fields, i.e., the ring of integers of arbitrary number fields. It embraces, among others, the study of the ideals and of the group of units in the ring of integers, the extent to which unique factorization holds, and so on [2–4]. Algebraic number theory has become an important branch of pure mathematics, on a par with algebraic geometry. There are two standard ways to approach algebraic number theory, one by means of ideals and the other by means of valuations [5]. Factorization in a field makes sense only with respect to a subring, and so we must resurrect the ring of integers in a number field in order to define it. Since the unique factorization property will fail in general, we need a way to measure how much it fails. The class number of a number field is, by definition, the order of the ideal class group of its ring of integers—it measures how far our ring of integers is from being a unique factorization domain. The divisibility properties of class numbers are very important for the investigation of the structure of ideal class groups of number fields. There has been considerable investigation in the past on the divisibility of the class numbers of quadratic number fields [6–14].

Lastly, since any factorization of an algebraic integer is defined up to multiplicative units, we need to understand the structure of the group of units in the ring of integers in order to fully understand the arithmetic of the number field.

Some important properties of associative algebra can be studied by employing the standard tools of algebraic number theory, elementary number theory, computational number theory, and combinatorics (see [15–20]). Three papers of this Special Issue deal with sequences of special numbers and special quaternions [21–23].

The purpose and scope of this Special Issue is to collect new results in algebraic number theory and analytic number theory (namely, in the areas of ramification theory in algebraic number fields, class field theory, arithmetic functions, L-functions, modular forms, and elliptic curves) and in some close research areas (namely, associative algebra, logical algebra, elementary number theory, combinatorics, difference equations, group rings, and algebraic hyperstructures).

In the following part of this Editorial, we discuss the manuscripts which have been selected for publication in this Special Issue. These papers were written by scientists

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working in leading universities or leading research centers in China, Czech Republic, Korea, Lithuania, Romania, Taiwan, Thailand, Turkey, United Kingdom, India, Pakistan, Saudi Arabia, Indonesia, and Morocco. The contributions are listed in the List of Contributions.

Contribution 1 is from D. Andrica and O. Bagdasar. In the paper titled "On Generalized Lucas Pseudoprimality of Level k", they investigate the Fibonacci pseudoprimes of level k. They disprove a statement concerning the relationship between sets of different levels and prove a counterpart of this result for the Lucas pseudoprimes of level k. They use some recently found properties of the generalized Lucas pseudoprimes and generalized Pell–Lucas sequences to define new kinds of pseudoprimes of levels k^+ and k^- and parameter a. For these novel pseudoprime sequences, they investigated some basic properties and computed several associated integer sequences, which were added to the Online Encyclopedia of Integer Sequences.

Contribution 2 is from E. Trojovská and P. Trojovský. In the paper "On Fibonacci Numbers of Order r Which Are Expressible as Sum of Consecutive Factorial Numbers", they investigate the sequence of the generalized Fibonacci number of order r. Let $(t_n^{(r)})_{n\geq 0}$ be the sequence of the generalized Fibonacci number of order r, which is defined by the recurrence $t_n^{(r)} = t_{n-1}^{(r)} + \ldots + t_{n-r}^{(r)}$ for $n \geq r$, with initial values $t_0^{(r)} = 0$ and $t_i^{(r)} = 1$, for all $1 \leq i \leq r$. In 2002, Grossman and Luca searched for terms of the sequence $(t_n^{(2)})_n$ which are expressible as a sum of factorials. In this paper, the authors continue this program by proving that, for any $l \geq 1$, there exists an effectively computable constant $C = C(l) > 0$ (only depending on l), such that, if (m, n, r) is a solution of $t_m^{(r)} = n! + (n+1)! + \ldots + (n+l)!$, with r even, then $\max\{m, n, r\} < C$. As an application, they solve the previous equation for all $1 \leq l \leq 5$.

Contribution 3 is from B. Aiewcharoen et al. In the paper "Global and Local Behavior of the System of Piecewise Linear Difference Equations $x_{n+1} = |x_n| - y_n - b$ and $y_{n+1} = x_n - |y_n| + 1$, where $b \geq 4$", they study the system of piecewise linear difference equations $x_{n+1} = |x_n| - y_n - b$ and $y_{n+1} = x_n - |y_n| + 1$ where $n \geq 0$. The global behavior at $b = 4$ shows that all solutions become the equilibrium point. For a large value of $|x_0|$ and $|y_0|$, they prove that (i) if $b = 5$, then the solution becomes the equilibrium point, and (ii) if $b \geq 6$, then the solution becomes the periodic solution of prime period 5.

Contribution 4 is from N. Minculete and D. Savin. In the paper "Some Properties of Euler's Function and of the Function τ and Their Generalizations in Algebraic Number Fields", they prove some inequalities which involve Euler's function, the extended Euler's function, the function τ, and the generalized Euler's function τ in algebraic number fields, thus extending the results they obtained in [24,25].

Contribution 5 is from J. Daengsaen and S. Leeratanavalee. In the paper "Regularities in Ordered n–Ary Semihypergroups", they approach a class of hyperstructures called ordered n–ary semihypergroups and study them using j-hyperideals for all positive integers $1 \leq j \leq n$ and $n \geq 3$. They first introduce the notion of (softly) left regularity, (softly) right regularity, (softly) intra-regularity, complete regularity, and generalized regularity of ordered n–ary semihypergroups and investigate their related properties. They present several of their characterizations in terms of j-hyperideals. Finally, they also establish the relationships between various classes of regularities in ordered n–ary semihypergroups.

Contribution 6 is from W. Ding et al. In the paper "New Zero-Density Results for Automorphic L–Functions of $GL(n)$", they study the automorphic L-function of $GL(n)$. Let $L(s, \pi)$ be an automorphic L-function of $GL(n)$, where π is an automorphic representation of group $GL(n)$ over the rational number field $\mathbb{Q}$. They study the zero-density estimates for $L(s, \pi)$. If $N_\pi(\sigma, T_1, T_2)$=$\sharp\{\rho = \beta + i\gamma : L(\rho, \pi) = 0, \sigma < \beta < 1, T_1 \leq \gamma \leq T_2\}$, where $0 \leq \sigma < 1$ and $T_1 < T_2$; then, they establish an upper bound for $N_\pi(\sigma, T, 2T)$ when σ is close to 1. They restrict the imaginary part γ into a narrow strip $[T, T + T^\alpha]$ with $0 < \alpha \leq 1$ and prove some new zero-density results on $N_\pi(\sigma, T, T + T^\alpha)$ (under specific conditions), thus improving the previous results when σ is near 34 and 1, respectively. Their proofs rely on the zero-detecting method and the Halász-Montgomery method.

Contribution 7 is from N. Terzioğlu et al. In the paper "New Properties and Identities for Fibonacci Finite Operator Quaternions", they define a new family of quaternions whose components are the Fibonacci finite operator numbers. They also prove some properties of this new family of quaternions. Moreover, by using their matrix representation, they present many identities related to Fibonacci finite operator quaternions.

Contribution 8 is from D. Piciu and D. Savin. In the paper "Residuated Lattices with Noetherian Spectrum", they characterize the residuated lattices for which the topological space of prime ideals is a Noetherian space. They introduce the notion of *i*-Noetherian residuated lattice and investigate its properties. They prove that a residuated lattice is *i*-Noetherian if and only if every ideal is principal. Moreover, they show that a residuated lattice has the spectrum of a Noetherian space if and only if it is *i*-Noetherian. In the last section of the paper, the authors compare ideals in residuated lattices with ideals in unitary commutative rings. They prove that if $(R, +.)$ is a Boolean ring, then any ideal of R is idempotent. Additionally, the authors prove that any Boolean ring is a Bezout ring with zero divisors.

Contribution 9 is from A. Laurinčikas and R. Macaitienė. In the paper "A Generalized Bohr-Jessen Type Theorem for the Epstein Zeta–Function", they study some properties of the the Epstein zeta function. Let Q be a positive defined $n \times n$ matrix and $Q[\underline{x}] = \underline{x}^T Q \underline{x}$. The Epstein zeta function $\zeta(s; Q)$, $s = \sigma + it$, is defined for $\sigma > \frac{n}{2}$ by the meromorphic continuation of the series $\zeta(s; Q) = \sum_{\underline{x} \in \mathbb{Z}^n - \{0\}} (Q[\underline{x}])^{-s}$ to the whole complex plane. Suppose that $n \geq 4$ is even and $\phi(t)$ is a differentiable function with a monotonic derivative. They prove that $\frac{1}{T} meas\{t \in [0, T] : \zeta(\sigma + i\phi(t); Q) \in A\}$; $A \in B(\mathbb{C})$, converges weakly to an explicitly given probability measure on $(\mathbb{C}, B(\mathbb{C}))$ as $T \to \infty$.

Contribution 10 is from K.-S. Kim, S. In the paper "Some Remarks on the Divisibility of the Class Numbers of Imaginary Quadratic Fields" [6], he investigates, for a given integer n, some families of imaginary quadratic number fields of the form $\mathbb{Q}(\sqrt{4q^2 - p^n})$ whose ideal class group has a subgroup isomorphic to $\mathbb{Z}/n\mathbb{Z}$, thus continuing the work of K. Chakraborty, A. Hoque, Y. Kishi, and P.P. Pandey, who studied the more restricted family $\mathbb{Q}(\sqrt{q^2 - p^n})$ with p and q as distinct odd prime numbers and $n \geq 3$ as an odd integer (see Theorem 1.2 of [8]).

Contribution 11 is from A. Vijayarangan, V. Narayanan, V. Natarajan, and S. Raghavendran. In the paper "Novel Authentication Protocols Based on Quadratic Diophantine Equations", they determine some geometric properties of positive integral solutions of the quadratic Diophantine equation $x_1^2 + x_2^2 = y_1^2 + y_2^2$. Moreover, in the same paper, the authors develop a new authentication protocol based on the geometric properties of the solutions of this quadratic Diophantine equation. The paper successfully depicts the role of number theory—especially Diophantine equations—in cryptography.

Contribution 12 is from Y. Wang, M.A. Binyamin, I. Amin, A. Aslam, and Y. Rao. In the paper "On the Classification of Telescopic Numerical Semigroups of Some Fixed Multiplicity", they expand on the results of Suer and Ilhan [26–28] for telescopic numerical semigroups of multiplicities 8 and 12 with embedding dimension four. Moreover, the authors compute the Frobenius number and genus for these classes in terms of the minimal system of generators.

Contribution 13 is from A.Z. Azak. In the paper "Pauli Gaussian Fibonacci and Pauli Gaussian Lucas Quaternions", she investigates the Pauli–Fibonacci quaternions (resp. Pauli–Lucas quaternions) whose coefficients consist of Gaussian–Fibonacci numbers (resp. Gaussian–Lucas numbers). Gaussian–Fibonacci numbers and Gaussian–Lucas numbers were introduced by Jordan in [29]. In [30], S. Halici introduced complex Fibonacci quaternions. Recently, the investigation of Gaussian–Lucas numbers has become an active research topic, and many of their properties have been exploited [31,32]. In the paper published in this Special Issue, the authors prove the Binet formulas for Pauli–Gaussian–Fibonacci and Pauli–Gaussian–Lucas quaternions and they also prove the Honsberger's, Catalan's and Cassini's identities for Pauli–Gaussian–Fibonacci quaternions.

Contribution 14 is from A. Altassan and M. Alan. In the paper "Almost Repdigit k-Fibonacci Numbers with an Application of k-Generalized Fibonacci Sequences", they introduce the notion of *almost repdigit*. In Theorem 1, which is the main result of this paper, the authors determine all the terms of the k-generalized Fibonacci sequence, which are almost repdigits. In order to prove this theorem, the authors make use of linear forms in the logarithms of algebraic numbers, Matveev's Theorem [33], some results from Baker and Davenport [34], the reduction algorithm from Dujella and Pethö [35], and various properties of k-Fibonacci numbers.

Contribution 15 is from A. Dubickas. In the paper "Density of Some Special Sequences Modulo 1", he explicitly describes all the elements of the sequence of fractional parts $\left\{a^{f(n)}/n\right\}$ where $f \in \mathbb{Z}[X]$ is a non constant polynomial with a positive leading coefficient and $a \geq 2$ is an integer. The author proves that this sequence is dense everywhere in $[0,1]$, thus expanding on the result of Cilleruelo, Kumchev, Luca, Rué, and Shparlinski [36], who required f to be the identity function and used a very different method. The author then proved that the result still holds true for the sequence of fractional parts $\left\{a^{f(n)}/n^d\right\}$ if we impose suitable condition on d, i.e., if $d \geq 1$ has no prime divisors other than those of a. In particular, this implies that for any pair of integers $a \geq 2$ and $b \geq 1$, the sequence of fractional parts $\left\{a^n/\sqrt[b]{n}\right\}$ is dense everywhere in $[0,1]$.

Contribution 16 is from T. Srichan. In the paper "A Bound for a Sum of Products of Two Characters and Its Application", he obtains a nice bound on the sum $\sum_{m^a n^b \leq x} \chi_1^a(m)\chi_2^b(n)$, where χ_i is the primitive Dirichlet character modulus q_i; the numbers a and b are fixed positive integers; and χ^a and χ^b are not principal characters. The main result of this paper is Theorem 1. In the last section of the article, the author presents an interesting application of this theorem, namely, a nice estimate of the error term in the problem of finding an asymptotic estimate of the number of full-square integers simultaneously belonging to two arithmetic progressions.

Contribution 17 is from M. Nur, M. Bahri, A. Islamiyati, and H. Batkunde. In the paper "A New Semi-Inner Product and p_n-Angle in the Space of p-Summable Sequences", they define a semi-inner product in the space of p-summable sequences equipped with an n-norm. The authors also introduce some concepts of functional analysis with connections to number theory, namely, the concept of p_n-orthogonality, a p_n-angle between two vectors in the space of p-summable sequences (resp. p_n-orthogonality, p_n-angle between one-dimensional subspaces and arbitrary-dimensional subspaces). They also obtain some interesting results involving these concepts.

Contribution 18 is from D. Andrica and O. Bagdasar. In the paper "Remarks on the Coefficients of Inverse Cyclotomic Polynomials", they study some properties of the n-th inverse cyclotomic polynomial, i.e., those polynomials whose roots are exactly all the non-primitive n-th roots of unity. These polynomials, which have been an object of investigation recently, are defined as the ratio of the polynomial $x^n - 1$ to the n-th cyclotomic polynomial $\Phi_n(x)$, and they satisfy some very nice properties (e.g., while a cyclotomic polynomial is palindromic, an inverse cyclotomic polynomial is anti-palindromic). After reviewing some known formulae for the calculation of the coefficients of cyclotomic polynomials, the authors derive two new recursive formulas for the coefficients of inverse cyclotomic polynomials (Theorems 4 and 6). These formulas are expressed in terms of Ramanujan sums and provide counterparts to similar formulae that were obtained by the same authors for cyclotomic polynomials. In the last section of the paper, the authors apply these recursive formulas to compute the coefficients of some ternary and quaternary inverse cyclotomic polynomials, i.e., when n is a product of three or four distinct primes.

Contribution 19 is from L. El Fadil. In the paper "On Indices of Septic Number Fields Defined by Trinomials $x^7 + ax + b$", he computes, for every prime integer p, the highest power $\nu_p(i(K))$ of p by dividing the index $i(K)$ of the number field K generated by the root of an irreducible trinomial $x^7 + ax + b$. This allowed him to compute the index $i(K)$ of the number field K, which is defined as the greatest common divisor of the indices of all the integral primitive elements of K. In particular, when the index of K is not trivial, then we can assert that K is not monogenic, i.e., its ring of integers cannot be generated by a single integral element. The monogenity of number fields is a classical problem of algebraic number fields, going back to Dedekind, Hasse, and Hensel. Recently, the monogenity of a number field and the construction of all possible generators of power integral bases have been intensively studied among others, such as Gaál, Nakahara, Pohst, and their collaborators. In the classical algebraic number theory book [37], Narkiewicz asked for an explicit formula to compute $\nu_p(i(K))$ for a given p. The method employed by the author of the paper in this Special Issue, who solves the problem posed by Narkiewicz in a very special case, is based on the study of the factorization of the prime ideals in the ring of integers of the septic field K.

Contribution 20 is from Z. Cheddour, A. Chillali, and A. Mouhib. In the paper "Generalized Fibonacci Sequences for Elliptic Curve Cryptography", they propose a generalization of the Fibonacci sequence based on elliptic curves. The study of repeating sequences in algebraic structures began with the early work of Wall, who studied regular Fibonacci sequences in cyclic groups. In this paper, the authors continue along this path and explore the concept of Fibonacci sequences for groups generated by points on an elliptic curve. They propose an encryption system which makes use of this sequence, and it is based on the discrete logarithm problem on elliptic curves. The authors compare the memory consumption of the proposed elliptic-curve-based cryptosystem with the Cramer–Shoup signature scheme relying on a strong RSA, and they show that in the proposed cryptosystem, it is significantly lower. Moreover, a notable advantage of the proposed scheme lies in its ability to generate a larger number of points with the same prime order p of the finite field.

Contribution 21 is from E. Tan, D. Savin, and S. Yılmaz. In the paper "A New Class of Leonardo Hybrid Numbers and Some Remarks on Leonardo Quaternions over Finite Fields", they introduce the generalized Leonardo hybrid numbers. Moreover, the authors define a new class of Leonardo hybrid numbers called q-generalized Leonardo hybrid numbers. Some properties of Leonardo numbers were studied by Catarino and Borges [38,39], by Alp and Kocer [40], and by Tan and Leung [41]. In this Special Issue paper, the authors obtain many important properties of generalized Leonardo hybrid numbers and q-generalized Leonardo hybrid numbers, including recurrence relations, the Binet formula, the exponential generating function, Vajda's identity, and the summation formulas. In [42], Savin determined which of the Fibonacci quaternions are invertible (resp. divisors of zero) in quaternion algebra over finite fields, namely, in quaternion algebra $Q_{\mathbb{Z}_p}(-1,-1)$. In [43] Mangueira, Alves, and Catarino introduced the notion of Leonardo Quaternions. In this Special Issue paper, the authors determine the Leonardo quaternions which are divisors of zero in the quaternion algebras $Q_{\mathbb{Z}_3}(-1,-1)$ and $Q_{\mathbb{Z}_5}(-1,-1)$. Moreover, the authors identify certain Leonardo quaternions that are invertible in the quaternion algebra $Q_{\mathbb{Z}_p}(-1,-1)$ for a prime integer $p \geq 7$.

In conclusion, we think that this Special Issue will be of great interest to all mathematicians specializing in algebraic, analytic, and computational number theory and their applications. The techniques employed in the papers of this Special Issue can prove to be very effective in obtaining new results in these areas of mathematics.

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List of Contributions

1. Andrica, D.; Bagdasar, O. On Generalized Lucas Pseudoprimality of Level k. *Mathematics* **2021**, *9*, 838. https://doi.org/10.3390/math9080838.
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