Supplementary material of

High frequency multi-field continualization scheme for layered magneto-electro-elastic materials

Rosaria Del Toro[1], Maria Laura De Bellis[1], Andrea Bacigalupo[2]

[1] University of Chieti-Pescara, Department INGEO, Viale Pindaro 42, Pescara, Italy
[2] University of Genova, Department DICCA, via Montallegro 1, Genova, Italy

Section A Field equations of periodic MEE layered materials

In Section A, the manipulation to derive the system (7a)-(7e) is described. Expliciting the components of the governing equations (7*b*)-(7*c*)

$$(\breve{e}_{1s1}^{(m,\varepsilon)} u_{s,1})_{,1} + (\eta_{1s}^{(m,\varepsilon)} E_s)_{,1} + (\lambda_{1s}^{(m,\varepsilon)} H_s)_{,1} = 0, \tag{1a}$$
$$(\breve{q}_{1s1}^{(m,\varepsilon)} u_{s,1})_{,1} + (\lambda_{1s}^{(m,\varepsilon)} E_s)_{,1} + (\mu_{1s}^{(m,\varepsilon)} H_s)_{,1} = 0, \tag{1b}$$

provides

$$\begin{aligned}&(\breve{e}_{111}^{(m,\varepsilon)} u_{1,1} + \breve{e}_{121}^{(m,\varepsilon)} u_{2,1} + \breve{e}_{131}^{(m,\varepsilon)} u_{3,1} + \eta_{11}^{(m,\varepsilon)} E_1 + \eta_{12}^{(m,\varepsilon)} E_2 + \\ &+ \eta_{13}^{(m,\varepsilon)} E_3 + \lambda_{11}^{(m,\varepsilon)} H_1 + \lambda_{12}^{(m,\varepsilon)} H_2 + \lambda_{13}^{(m,\varepsilon)} H_3)_{,1} = 0,\end{aligned} \tag{2a}$$
$$\begin{aligned}&(\breve{q}_{111}^{(m,\varepsilon)} u_{1,1} + \breve{q}_{121}^{(m,\varepsilon)} u_{2,1} + \breve{q}_{131}^{(m,\varepsilon)} u_{3,1} + \lambda_{11}^{(m,\varepsilon)} E_1 + \lambda_{12}^{(m,\varepsilon)} E_2 + \\ &+ \lambda_{13}^{(m,\varepsilon)} E_3 + \mu_{11}^{(m,\varepsilon)} H_1 + \mu_{12}^{(m,\varepsilon)} H_2 + \mu_{13}^{(m,\varepsilon)} H_3)_{,1} = 0.\end{aligned} \tag{2b}$$

Moreover, developping the equations (7*d*)-(7*e*) for $i = 1$ derives

$$\begin{aligned}&\breve{e}_{111}^{(m,\varepsilon)} \dot{u}_{1,1} + \breve{e}_{121}^{(m,\varepsilon)} \dot{u}_{2,1} + \breve{e}_{131}^{(m,\varepsilon)} \dot{u}_{3,1} + \eta_{11}^{(m,\varepsilon)} \dot{E}_1 + \eta_{12}^{(m,\varepsilon)} \dot{E}_2 + \\ &+ \eta_{13}^{(m,\varepsilon)} \dot{E}_3 + \lambda_{11}^{(m,\varepsilon)} \dot{H}_1 + \lambda_{12}^{(m,\varepsilon)} \dot{H}_2 + \lambda_{13}^{(m,\varepsilon)} \dot{H}_3 = 0,\end{aligned} \tag{3a}$$
$$\begin{aligned}&\breve{q}_{111}^{(m,\varepsilon)} \dot{u}_{1,1} + \breve{q}_{121}^{(m,\varepsilon)} \dot{u}_{2,1} + \breve{q}_{131}^{(m,\varepsilon)} \dot{u}_{3,1} + \lambda_{11}^{(m,\varepsilon)} \dot{E}_1 + \lambda_{12}^{(m,\varepsilon)} \dot{E}_2 + \\ &+ \lambda_{13}^{(m,\varepsilon)} \dot{E}_3 + \mu_{11}^{(m,\varepsilon)} \dot{H}_1 + \mu_{12}^{(m,\varepsilon)} \dot{H}_2 + \mu_{13}^{(m,\varepsilon)} \dot{H}_3 = 0,\end{aligned} \tag{3b}$$

and, therefore, manipulating the equations (2a)–(2b) and (3a)–(3b) gives

$$\begin{aligned}&(\breve{e}_{111}^{(m,\varepsilon)} \dot{u}_{1,1} + \breve{e}_{121}^{(m,\varepsilon)} \dot{u}_{2,1} + \breve{e}_{131}^{(m,\varepsilon)} \dot{u}_{3,1} + \eta_{11}^{(m,\varepsilon)} \dot{E}_1 + \\ &\eta_{12}^{(m,\varepsilon)} \dot{E}_2 + \eta_{13}^{(m,\varepsilon)} \dot{E}_3 + \lambda_{11}^{(m,\varepsilon)} \dot{H}_1 + \lambda_{12}^{(m,\varepsilon)} \dot{H}_2 + \lambda_{13}^{(m,\varepsilon)} \dot{H}_3)_{,1} = 0,\end{aligned} \tag{4a}$$
$$\begin{aligned}&(\breve{q}_{111}^{(m,\varepsilon)} \dot{u}_{1,1} + \breve{q}_{121}^{(m,\varepsilon)} \dot{u}_{2,1} + \breve{q}_{131}^{(m,\varepsilon)} \dot{u}_{3,1} + \lambda_{11}^{(m,\varepsilon)} \dot{E}_1 + \\ &\lambda_{12}^{(m,\varepsilon)} \dot{E}_2 + \lambda_{13}^{(m,\varepsilon)} \dot{E}_3 + \mu_{11}^{(m,\varepsilon)} \dot{H}_1 + \mu_{12}^{(m,\varepsilon)} \dot{H}_2 + \mu_{13}^{(m,\varepsilon)} \dot{H}_3)_{,1} = 0.\end{aligned} \tag{4b}$$

Section B Components of matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$, $\boldsymbol{\Omega}$, $\boldsymbol{\Gamma}$

Section B collects the components of the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ in equation (18), the matrix $\boldsymbol{D}$ in equation (27) and the matrices $\boldsymbol{\Omega}$ and $\boldsymbol{\Gamma}$ in equation (23). The non-null components of the 3−by−3 matrix $\boldsymbol{A}^{(LL)}$ are

$$A_{11}^{(LL)} = C_{1111}^{(m,\varepsilon)},\ A_{21}^{(LL)} = \breve{e}_{111}^{(m,\varepsilon)},\ A_{31}^{(LL)} = \breve{q}_{111}^{(m,\varepsilon)}, \tag{5a}$$

the non-null components of the 3−by−6 matrix $\boldsymbol{A}^{(LT)}$ are

$$\begin{aligned}A_{11}^{(LT)} =&C_{1121}^{(m,\varepsilon)},\ A_{12}^{(LT)} = C_{1131}^{(m,\varepsilon)},\\ A_{21}^{(LT)} =&\breve{e}_{121}^{(m,\varepsilon)},\ A_{22}^{(LT)} = \breve{e}_{131}^{(m,\varepsilon)},\\ A_{31}^{(LT)} =&\breve{q}_{121}^{(m,\varepsilon)},\ A_{32}^{(LT)} = \breve{q}_{131}^{(m,\varepsilon)},\end{aligned} \tag{5b}$$

the non-null components of the 6−by−3 matrix $\boldsymbol{A}^{(TL)}$ are

$$A_{11}^{(TL)} =C_{2111}^{(m,\varepsilon)},\ A_{21}^{(TL)} = C_{3111}^{(m,\varepsilon)}, \tag{5c}$$

the non-null components of the 6−by−6 symmetric matrix $\boldsymbol{A}^{(TT)}$ are

$$A_{11}^{(TT)} =C_{2121}^{(m,\varepsilon)},\ A_{12}^{(TT)} = C_{2131}^{(m,\varepsilon)},\ A_{22}^{(TT)} = C_{3131}^{(m,\varepsilon)}. \tag{5d}$$

The non-null components of the 3−by−3 matrix $\boldsymbol{B}^{(LL)}$ are

$$\begin{aligned}B_{11}^{(LL)} =&2\iota k_1 C_{1111}^{(m,\varepsilon)}, B_{12}^{(LL)} = -e_{111}^{(m,\varepsilon)}, B_{13}^{(LL)} = -q_{111}^{(m,\varepsilon)},\\ B_{21}^{(LL)} =&\iota k_1 \breve{e}_{111}^{(m,\varepsilon)}, B_{22}^{(LL)} = \eta_{11}^{(m,\varepsilon)}, B_{23}^{(LL)} = B_{32}^{(LL)} = \lambda_{11}^{(m,\varepsilon)},\\ B_{31}^{(LL)} =&\iota k_1 \breve{q}_{111}^{(m,\varepsilon)}, B_{33}^{(LL)} = \mu_{11}^{(m,\varepsilon)},\end{aligned} \tag{6a}$$

the non-null components of the 3−by−6 matrix $\boldsymbol{B}^{(LT)}$ are

$$\begin{aligned}&B_{11}^{(LT)} = 2\iota k_1 C_{1121}^{(m,\varepsilon)}, B_{12}^{(LT)} = 2\iota k_1 C_{1131}^{(m,\varepsilon)}, B_{13}^{(LT)} = -e_{112}^{(m,\varepsilon)},\\ &B_{14}^{(LT)} = -e_{113}^{(m,\varepsilon)}, B_{15}^{(LT)} = -q_{112}^{(m,\varepsilon)}, B_{16}^{(LT)} = -q_{113}^{(m,\varepsilon)},\\ &B_{21}^{(LT)} = -\omega k_1 \breve{e}_{121}^{(m,\varepsilon)}, B_{22}^{(LT)} = -\omega k_1 \breve{e}_{131}^{(m,\varepsilon)}, B_{23}^{(LT)} = \iota\omega\eta_{12}^{(m,\varepsilon)},\\ &B_{24}^{(LT)} = \iota\omega\eta_{13}^{(m,\varepsilon)}, B_{25}^{(LT)} = \iota\omega\lambda_{12}^{(m,\varepsilon)}, B_{26}^{(LT)} = \iota\omega\lambda_{13}^{(m,\varepsilon)}\\ &B_{31}^{(LT)} = -\omega k_1 \breve{q}_{121}^{(m,\varepsilon)}, B_{32}^{(LT)} = -\omega k_1 \breve{q}_{131}^{(m,\varepsilon)}, B_{33}^{(LT)} = \iota\omega\lambda_{12}^{(m,\varepsilon)},\\ &B_{34}^{(LT)} = \iota\omega\lambda_{13}^{(m,\varepsilon)}, B_{35}^{(LT)} = \iota\omega\mu_{12}^{(m,\varepsilon)}, B_{36}^{(LT)} = \iota\omega\mu_{13}^{(m,\varepsilon)},\end{aligned} \tag{6b}$$

the non-null components of the 6−by−3 matrix $\boldsymbol{B}^{(TL)}$ are

$$\begin{aligned}&B_{11}^{(TL)} = 2\iota k_1 C_{2111}^{(m,\varepsilon)}, B_{12}^{(TL)} = -e_{211}^{(m,\varepsilon)}, B_{13}^{(TL)} = -q_{211}^{(m,\varepsilon)},\\ &B_{21}^{(TL)} = 2\iota k_1 C_{3111}^{(m,\varepsilon)}, B_{22}^{(TL)} = -e_{311}^{(m,\varepsilon)}, B_{23}^{(TL)} = -q_{311}^{(m,\varepsilon)},\\ &B_{31}^{(TL)} = \iota\omega\breve{e}_{211}^{(m,\varepsilon)}, B_{41}^{(TL)} = \iota\omega\breve{e}_{311}^{(m,\varepsilon)}, B_{51}^{(TL)} = \iota\omega\breve{q}_{211}^{(m,\varepsilon)}, B_{61}^{(TL)} = \iota\omega\breve{q}_{311}^{(m,\varepsilon)},\end{aligned} \tag{6c}$$

the non-null components of the 6−by−6 symmetric matrix $\boldsymbol{B}^{(TT)}$ are

$$\begin{aligned}&B_{11}^{(TT)} = 2\iota k_1 C_{2121}^{(m,\varepsilon)}\iota\omega,\ B_{12}^{(TT)} = 2\iota k_1 C_{2131}^{(m,\varepsilon)}\iota\omega,\ B_{13}^{(TT)} = -\iota\omega e_{212}^{(m,\varepsilon)},\\ &B_{14}^{(TT)} = -\iota\omega e_{213}^{(m,\varepsilon)},\ B_{15}^{(TT)} = -\iota\omega q_{212}^{(m,\varepsilon)},\ B_{16}^{(TT)} = -\iota\omega q_{213}^{(m,\varepsilon)},\\ &B_{22}^{(TT)} = 2\iota k_1 C_{3131}^{(m,\varepsilon)}\iota\omega,\ B_{23}^{(TT)} = -\iota\omega e_{312}^{(m,\varepsilon)},\ B_{24}^{(TT)} = -\iota\omega e_{313}^{(m,\varepsilon)},\\ &B_{25}^{(TT)} = -\iota\omega q_{312}^{(m,\varepsilon)},\ B_{26}^{(TT)} = -\iota\omega q_{313}^{(m,\varepsilon)},\ B_{36}^{(TT)} = 1,\ B_{45}^{(TT)} = -1.\end{aligned} \tag{6d}$$

The non-null components of the 3−by−3 matrix $\boldsymbol{C}^{(LL)}$ are

$$C_{11}^{(LL)} = \rho^{(m,\varepsilon)}\omega^2 - C_{1111}^{(m,\varepsilon)}k_1^2,\ C_{12}^{(LL)} = -\iota k_1 e_{111}^{(m,\varepsilon)},\ C_{13}^{(LL)} = -\iota k_1 q_{111}^{(m,\varepsilon)}, \tag{7a}$$

the non-null components of the 3−by−6 matrix $\boldsymbol{C}^{(LT)}$ are

$$C_{11}^{(LT)} = -C_{1121}^{(m,\varepsilon)}k_1^2,\ C_{12}^{(LT)} = -C_{1131}^{(m,\varepsilon)}k_1^2,\ C_{13}^{(LT)} = -e_{112}^{(m,\varepsilon)}\iota k_1,$$

$$C_{14}^{(LT)} = -e_{113}^{(m,\varepsilon)} \iota k_1,\ C_{15}^{(LT)} = -q_{112}^{(m,\varepsilon)} \iota k_1,\ C_{16}^{(LT)} = -q_{113}^{(m,\varepsilon)} \iota k_1, \tag{7b}$$

the non-null components of the 6−by−3 matrix $\boldsymbol{C}^{(TL)}$ are

$$\begin{aligned}
&C_{11}^{(TL)} = -C_{2111}^{(m,\varepsilon)} k_1^2, C_{12}^{(TL)} = -\iota k_1 e_{211}^{(m,\varepsilon)}, C_{13}^{(TL)} = -\iota k_1 q_{211}^{(m,\varepsilon)},\\
&C_{21}^{(TL)} = -C_{3111}^{(m,\varepsilon)} k_1^2,\ C_{22}^{(TL)} = -\iota k_1 e_{311}^{(m,\varepsilon)},\ C_{23}^{(TL)} = -\iota k_1 q_{311}^{(m,\varepsilon)},\\
&C_{31}^{(TL)} = -\omega k_1 \breve{e}_{211}^{(m,\varepsilon)},\ C_{32}^{(TL)} = \iota\omega\eta_{21}^{(m,\varepsilon)},\ C_{33}^{(TL)} = \iota\omega\lambda_{21}^{(m,\varepsilon)},\\
&C_{41}^{(TL)} = -\omega k_1 \breve{e}_{311}^{(m,\varepsilon)},\ C_{42}^{(TL)} = \iota\omega\eta_{31}^{(m,\varepsilon)},\ C_{43}^{(TL)} = \iota\omega\lambda_{31}^{(m,\varepsilon)},\\
&C_{51}^{(TL)} = -\omega k_1 \breve{q}_{211}^{(m,\varepsilon)},\ C_{52}^{(TL)} = \iota\omega\lambda_{21}^{(m,\varepsilon)},\ C_{53}^{(TL)} = \iota\omega\mu_{21}^{(m,\varepsilon)},\\
&C_{61}^{(TL)} = -\omega k_1 \breve{q}_{311}^{(m,\varepsilon)},\ C_{62}^{(TL)} = \iota\omega\lambda_{31}^{(m,\varepsilon)},\ C_{63}^{(TL)} = \iota\omega\mu_{31}^{(m,\varepsilon)},
\end{aligned} \tag{7c}$$

the non-null components of the 6−by−6 symmetric matrix $\boldsymbol{C}^{(TT)}$ are

$$\begin{aligned}
&C_{11}^{(TT)} = \iota\omega(C_{2121}^{(m,\varepsilon)} k_1^2 - \rho^{(m,\varepsilon)}\omega^2),\ C_{12}^{(TT)} = \iota\omega C_{2131}^{(m,\varepsilon)} k_1^2,\ C_{13}^{(TT)} = -\omega k_1 e_{212}^{(m,\varepsilon)},\\
&C_{14}^{(TT)} = -\omega k_1 e_{213}^{(m,\varepsilon)},\ C_{15}^{(TT)} = -\omega k_1 q_{212}^{(m,\varepsilon)},\ C_{16}^{(TT)} = -\omega k_1 q_{213}^{(m,\varepsilon)},\\
&C_{22}^{(TT)} = \iota\omega(C_{3131}^{(m,\varepsilon)} k_1^2 - \rho^{(m,\varepsilon)}\omega^2),\ C_{23}^{(TT)} = -\omega k_1 e_{312}^{(m,\varepsilon)},\ C_{24}^{(TT)} = -\omega k_1 e_{313}^{(m,\varepsilon)},\ C_{25}^{(TT)} = -\omega k_1 q_{312}^{(m,\varepsilon)},\\
&C_{26}^{(TT)} = -\omega k_1 q_{313}^{(m,\varepsilon)},\ C_{33}^{(TT)} = \iota\omega\eta_{22}^{(m,\varepsilon)},\ C_{34}^{(TT)} = \iota\omega\eta_{23}^{(m,\varepsilon)},\ C_{35}^{(TT)} = \iota\omega\lambda_{22}^{(m,\varepsilon)},\\
&C_{36}^{(TT)} = \iota\omega\lambda_{23}^{(m,\varepsilon)} - \iota k_1,\ C_{44}^{(TT)} = \iota\omega\eta_{33}^{(m,\varepsilon)},\ C_{45}^{(TT)} = \iota\omega\lambda_{32}^{(m,\varepsilon)} + \iota k_1,\ C_{46}^{(TT)} = \iota\omega\lambda_{33}^{(m,\varepsilon)},\\
&C_{55}^{(TT)} = \iota\omega\mu_{22}^{(m,\varepsilon)},\ C_{56}^{(TT)} = \iota\omega\mu_{23}^{(m,\varepsilon)},\ C_{66}^{(TT)} = \iota\omega\mu_{33}^{(m,\varepsilon)}.
\end{aligned} \tag{7d}$$

The components of the matrices $\boldsymbol{D}^{(LL)}$, $\boldsymbol{D}^{(LT)}$, $\boldsymbol{D}^{(TL)}$, $\boldsymbol{D}^{(TT)}$ related to equation (27) are shown. In particular, the non-null components of the 4−by−4 matrix $\boldsymbol{D}^{(LL)}$ are

$$D_{21}^{(LL)} = C_{1111}^{(m,\varepsilon)}, D_{22}^{(LL)} = \iota k_1 C_{1111}^{(m,\varepsilon)}, D_{23}^{(LL)} = -e_{111}^{(m,\varepsilon)}, D_{24}^{(LL)} = -q_{111}^{(m,\varepsilon)}, D_{12}^{(LL)} = D_{33}^{(LL)} = D_{34}^{(LL)} = 1, \tag{8a}$$

the non-null components of the 4−by−8 matrix $\boldsymbol{D}^{(LT)}$ are

$$\begin{aligned}
&D_{21}^{(LT)} = C_{1121}^{(m,\varepsilon)}, D_{22}^{(LT)} = C_{1131}^{(m,\varepsilon)}, D_{23}^{(LT)} = \iota k_1 C_{1121}^{(m,\varepsilon)}, D_{24}^{(LT)} = \iota k_1 C_{1131}^{(m,\varepsilon)},\\
&D_{25}^{(LT)} = -e_{112}^{(m,\varepsilon)}, D_{26}^{(LT)} = -e_{113}^{(m,\varepsilon)}, D_{27}^{(LT)} = -q_{112}^{(m,\varepsilon)}, D_{28}^{(LT)} = -q_{113}^{(m,\varepsilon)},
\end{aligned} \tag{8b}$$

the non-null components of the 8−by−4 matrix $\boldsymbol{D}^{(TL)}$ are

$$\begin{aligned}
&D_{31}^{(TL)} = C_{2111}^{(m,\varepsilon)}, D_{32}^{(TL)} = \iota k_1 C_{2111}^{(m,\varepsilon)}, D_{33}^{(TL)} = -e_{211}^{(m,\varepsilon)}, D_{34}^{(TL)} = -q_{211}^{(m,\varepsilon)},\\
&D_{41}^{(TL)} = C_{3111}^{(m,\varepsilon)}, D_{42}^{(TL)} = \iota k_1 C_{3111}^{(m,\varepsilon)}, D_{44}^{(TL)} = -e_{311}^{(m,\varepsilon)}, D_{41}^{(TL)} = -q_{311}^{(m,\varepsilon)}.
\end{aligned} \tag{8c}$$

Finally, the non-null components of the 8−by−8 matrix $\boldsymbol{D}^{(TT)}$ are

$$\begin{aligned}
&D_{31}^{(TT)} = C_{2121}^{(m,\varepsilon)}, D_{32}^{(TT)} = C_{2131}^{(m,\varepsilon)}, D_{33}^{(TT)} = \iota k_1 C_{2121}^{(m,\varepsilon)}, D_{34}^{(TT)} = \iota k_1 C_{2131}^{(m,\varepsilon)},\\
&D_{35}^{(TT)} = -e_{212}^{(m,\varepsilon)}, D_{36}^{(TT)} = -e_{213}^{(m,\varepsilon)}, D_{37}^{(TT)} = -q_{212}^{(m,\varepsilon)}, D_{38}^{(TT)} = -q_{213}^{(m,\varepsilon)},\\
&D_{41}^{(TT)} = C_{3121}^{(m,\varepsilon)}, D_{42}^{(TT)} = C_{3131}^{(m,\varepsilon)}, D_{43}^{(TT)} = \iota k_1 C_{3121}^{(m,\varepsilon)}, D_{44}^{(TT)} = \iota k_1 C_{3131}^{(m,\varepsilon)},\\
&D_{45}^{(TT)} = -e_{312}^{(m,\varepsilon)}, D_{46}^{(TT)} = -e_{313}^{(m,\varepsilon)}, D_{47}^{(TT)} = -q_{312}^{(m,\varepsilon)}, D_{48}^{(TT)} = -q_{313}^{(m,\varepsilon)},\\
&D_{13}^{(TT)} = D_{24}^{(TT)} = D_{55}^{(TT)} = D_{66}^{(TT)} = D_{77}^{(TT)} = D_{88}^{(TT)} = 1.
\end{aligned} \tag{8d}$$

Then, the components of the matrices $\boldsymbol{\Omega}$ and $\boldsymbol{\Gamma}$ in equation (23) are displayed. The non-null components of the 4−by−4 matrix $\boldsymbol{\Gamma}^{(LL)}$ are

$$\begin{aligned}
&\Gamma_{11}^{(LL)} = C_{1111}^{(m,\varepsilon)},\ \Gamma_{12}^{(LL)} = 2\iota k_1 C_{1111}^{(m,\varepsilon)},\ \Gamma_{13}^{(LL)} = -e_{111}^{(m,\varepsilon)},\ \Gamma_{14}^{(LL)} = -q_{111}^{(m,\varepsilon)},\\
&\Gamma_{21}^{(LL)} = \breve{e}_{111}^{(m,\varepsilon)},\ \Gamma_{22}^{(LL)} = \iota k_1 \breve{e}_{111}^{(m,\varepsilon)}, \Gamma_{23}^{(LL)} = \eta_{11}^{(m,\varepsilon)},\ \Gamma_{24}^{(LL)} = \Gamma_{33}^{(LL)} = \lambda_{11}^{(m,\varepsilon)},
\end{aligned} \tag{9a}$$

$$\Gamma_{31}^{(LL)} = \breve{q}_{111}^{(m,\varepsilon)}, \Gamma_{32}^{(LL)} = \iota k_1 \breve{q}_{111}^{(m,\varepsilon)}, \Gamma_{34}^{(LL)} = \mu_{11}^{(m,\varepsilon)}, \Gamma_{42}^{(LL)} = 1,$$

the non-null components of the 4−by−8 matrix $\mathbf{\Gamma}^{(LT)}$ are

$$\begin{aligned}
\Gamma_{11}^{(LT)} =& C_{1121}^{(m,\varepsilon)}, \Gamma_{12}^{(LT)} = C_{1131}^{(m,\varepsilon)}, \Gamma_{13}^{(LT)} = 2\iota k_1 C_{1121}^{(m,\varepsilon)}, \Gamma_{14}^{(LT)} = 2\iota k_1 C_{1131}^{(m,\varepsilon)}, \\
\Gamma_{15}^{(LT)} =& -e_{112}^{(m,\varepsilon)}, \Gamma_{16}^{(LT)} = -e_{113}^{(m,\varepsilon)}, \Gamma_{17}^{(LT)} = -q_{112}^{(m,\varepsilon)}, \Gamma_{18}^{(LT)} = -q_{113}^{(m,\varepsilon)}, \\
\Gamma_{21}^{(LT)} =& \breve{e}_{121}^{(m,\varepsilon)}, \Gamma_{22}^{(LT)} = \breve{e}_{131}^{(m,\varepsilon)}, \Gamma_{23}^{(LT)} = -\omega k_1 \breve{e}_{121}^{(m,\varepsilon)}, \Gamma_{24}^{(LT)} = -\omega k_1 \breve{e}_{131}^{(m,\varepsilon)}, \\
\Gamma_{25}^{(LT)} =& \iota\omega\eta_{12}^{(m,\varepsilon)}, \Gamma_{26}^{(LT)} = \iota\omega\eta_{13}^{(m,\varepsilon)}, \Gamma_{27}^{(LT)} = \iota\omega\lambda_{12}^{(m,\varepsilon)}, \Gamma_{28}^{(LT)} = \iota\omega\lambda_{13}^{(m,\varepsilon)}, \\
\Gamma_{31}^{(LT)} =& \breve{q}_{121}^{(m,\varepsilon)}, \Gamma_{32}^{(LT)} = \breve{q}_{131}^{(m,\varepsilon)}, \Gamma_{33}^{(LT)} = -\omega k_1 \breve{q}_{121}^{(m,\varepsilon)}, \Gamma_{34}^{(LT)} = -\omega k_1 \breve{q}_{131}^{(m,\varepsilon)}, \\
\Gamma_{35}^{(LT)} =& \iota\omega\lambda_{12}^{(m,\varepsilon)}, \Gamma_{36}^{(LT)} = \iota\omega\lambda_{13}^{(m,\varepsilon)}, \Gamma_{37}^{(LT)} = \iota\omega\mu_{12}^{(m,\varepsilon)}, \Gamma_{38}^{(LT)} = \iota\omega\mu_{13}^{(m,\varepsilon)},
\end{aligned} \tag{9b}$$

the non-null components of the 8−by−4 matrix $\mathbf{\Gamma}^{(TL)}$ are

$$\begin{aligned}
\Gamma_{11}^{(TL)} =& C_{2111}^{(m,\varepsilon)}, \Gamma_{12}^{(TL)} = 2\iota k_1 C_{2111}^{(m,\varepsilon)}, \Gamma_{13}^{(TL)} = -e_{211}^{(m,\varepsilon)}, \Gamma_{14}^{(TL)} = -q_{211}^{(m,\varepsilon)}, \\
\Gamma_{21}^{(TL)} =& C_{3111}^{(m,\varepsilon)}, \Gamma_{22}^{(TL)} = 2\iota k_1 C_{3111}^{(m,\varepsilon)}, \Gamma_{23}^{(TL)} = -e_{311}^{(m,\varepsilon)}, \Gamma_{24}^{(TL)} = -q_{311}^{(m,\varepsilon)}, \\
\Gamma_{32}^{(TL)} =& \iota\omega\breve{e}_{211}^{(m,\varepsilon)}, \Gamma_{42}^{(TL)} = \iota\omega\breve{e}_{311}^{(m,\varepsilon)}, \Gamma_{52}^{(TL)} = \iota\omega\breve{q}_{211}^{(m,\varepsilon)}, \Gamma_{62}^{(TL)} = \iota\omega\breve{q}_{311}^{(m,\varepsilon)},
\end{aligned} \tag{9c}$$

the non-null components of the 8−by−8 matrix $\mathbf{\Gamma}^{(TT)}$ are

$$\begin{aligned}
\Gamma_{11}^{(TT)} =& C_{2121}^{(m,\varepsilon)}, \Gamma_{12}^{(TT)} = C_{2131}^{(m,\varepsilon)}, \Gamma_{13}^{(TT)} = 2\iota k_1 C_{2121}^{(m,\varepsilon)}, \Gamma_{14}^{(TT)} = 2\iota k_1 C_{2131}^{(m,\varepsilon)}, \\
\Gamma_{15}^{(TT)} =& -e_{212}^{(m,\varepsilon)}, \Gamma_{16}^{(TT)} = -e_{213}^{(m,\varepsilon)}, \Gamma_{17}^{(TT)} = -q_{212}^{(m,\varepsilon)}, \Gamma_{18}^{(TT)} = -q_{213}^{(m,\varepsilon)}, \\
\Gamma_{21}^{(TT)} =& C_{3121}^{(m,\varepsilon)}, \Gamma_{22}^{(TT)} = C_{3131}^{(m,\varepsilon)}, \Gamma_{23}^{(TT)} = 2\iota k_1 C_{3121}^{(m,\varepsilon)}, \Gamma_{24}^{(TT)} = 2\iota k_1 C_{3131}^{(m,\varepsilon)}, \\
\Gamma_{25}^{(TT)} =& -e_{312}^{(m,\varepsilon)}, \Gamma_{26}^{(TT)} = -e_{313}^{(m,\varepsilon)}, \Gamma_{27}^{(TT)} = -q_{312}^{(m,\varepsilon)}, \Gamma_{28}^{(TT)} = -q_{313}^{(m,\varepsilon)}, \\
\Gamma_{33}^{(TT)} =& \iota\omega\breve{e}_{221}^{(m,\varepsilon)}, \Gamma_{34}^{(TT)} = \iota\omega\breve{e}_{231}^{(m,\varepsilon)}, \Gamma_{43}^{(TT)} = \iota\omega\breve{e}_{321}^{(m,\varepsilon)}, \Gamma_{44}^{(TT)} = \iota\omega\breve{e}_{331}^{(m,\varepsilon)}, \\
\Gamma_{53}^{(TT)} =& \iota\omega\breve{q}_{221}^{(m,\varepsilon)}, \Gamma_{54}^{(TT)} = \iota\omega\breve{q}_{231}^{(m,\varepsilon)}, \Gamma_{63}^{(TT)} = \iota\omega\breve{q}_{321}^{(m,\varepsilon)}, \Gamma_{64}^{(TT)} = \iota\omega\breve{q}_{331}^{(m,\varepsilon)}, \\
\Gamma_{47}^{(TT)} =& \Gamma_{56}^{(TT)} = \Gamma_{73}^{(TT)} = \Gamma_{84}^{(TT)} = 1, \Gamma_{38}^{(TT)} = \Gamma_{65}^{(TT)} = -1.
\end{aligned} \tag{9d}$$

The non-null components of the 4−by−4 matrix $\mathbf{\Omega}^{(LL)}$ are

$$\begin{aligned}
\Omega_{12}^{(LL)} =& \rho^{(m,\varepsilon)}\omega^2 - C_{1111}^{(m,\varepsilon)} k_1^2, \Omega_{13}^{(LL)} = -\iota k_1 e_{111}^{(m,\varepsilon)}, \Omega_{14}^{(LL)} = -\iota k_1 q_{111}^{(m,\varepsilon)}, \\
\Omega_{41}^{(LL)} =& -1,
\end{aligned} \tag{10a}$$

the non-null components of the 4−by−8 matrix $\mathbf{\Omega}^{(LT)}$ are

$$\begin{aligned}
\Omega_{13}^{(LT)} =& -C_{1121}^{(m,\varepsilon)} k_1^2, \Omega_{14}^{(LT)} = -C_{1131}^{(m,\varepsilon)} k_1^2, \Omega_{15}^{(LT)} = -e_{112}^{(m,\varepsilon)} \iota k_1, \Omega_{16}^{(LT)} = -e_{113}^{(m,\varepsilon)} \iota k_1, \\
\Omega_{17}^{(LT)} =& -q_{112}^{(m,\varepsilon)} \iota k_1, \Omega_{18}^{(LT)} = -q_{113}^{(m,\varepsilon)} \iota k_1,
\end{aligned}$$

the non-null components of the 8−by−4 matrix $\mathbf{\Omega}^{(TL)}$ are

$$\begin{aligned}
\Omega_{12}^{(TL)} =& -C_{2111}^{(m,\varepsilon)} k_1^2, \Omega_{13}^{(TL)} = -\iota k_1 e_{211}^{(m,\varepsilon)}, \Omega_{14}^{(TL)} = -\iota k_1 q_{211}^{(m,\varepsilon)}, \\
\Omega_{22}^{(TL)} =& -C_{3111}^{(m,\varepsilon)} k_1^2, \Omega_{23}^{(TL)} = -\iota k_1 e_{311}^{(m,\varepsilon)}, \Omega_{24}^{(TL)} = -\iota k_1 q_{311}^{(m,\varepsilon)}, \\
\Omega_{32}^{(TL)} =& -\omega k_1 \breve{e}_{211}^{(m,\varepsilon)}, \Omega_{33}^{(TL)} = \iota\omega\eta_{21}^{(m,\varepsilon)}, \Omega_{34}^{(TL)} = \iota\omega\lambda_{21}^{(m,\varepsilon)}, \\
\Omega_{42}^{(TL)} =& -\omega k_1 \breve{e}_{311}^{(m,\varepsilon)}, \Omega_{43}^{(TL)} = \iota\omega\eta_{31}^{(m,\varepsilon)}, \Omega_{44}^{(TL)} = \iota\omega\lambda_{31}^{(m,\varepsilon)}, \\
\Omega_{52}^{(TL)} =& -\omega k_1 \breve{q}_{211}^{(m,\varepsilon)}, \Omega_{53}^{(TL)} = \iota\omega\lambda_{21}^{(m,\varepsilon)}, C_{53}^{(TL)} = \iota\omega\mu_{21}^{(m,\varepsilon)}, \\
\Omega_{62}^{(TL)} =& -\omega k_1 \breve{q}_{311}^{(m,\varepsilon)}, \Omega_{63}^{(TL)} = \iota\omega\lambda_{31}^{(m,\varepsilon)}, \Omega_{64}^{(TL)} = \iota\omega\mu_{31}^{(m,\varepsilon)},
\end{aligned} \tag{10b}$$

the non-null components of the 8−by−8 matrix $\mathbf{\Omega}^{(TT)}$ are

$$\begin{aligned}
&\Omega_{13}^{(TT)}=\rho^{(m,\varepsilon)}\omega^2-C_{2121}^{(m,\varepsilon)}k_1^2,\ \Omega_{14}^{(TT)}=-C_{2131}^{(m,\varepsilon)}k_1^2,\ \Omega_{15}^{(TT)}=-\iota k_1e_{212}^{(m,\varepsilon)},\ \Omega_{16}^{(TT)}=-\iota k_1e_{213}^{(m,\varepsilon)},\\
&\Omega_{17}^{(TT)}=-\iota k_1q_{212}^{(m,\varepsilon)},\ \Omega_{18}^{(TT)}=-\iota k_1q_{213}^{(m,\varepsilon)},\Omega_{23}^{(TT)}=-C_{3121}^{(m,\varepsilon)}k_1^2,\ \Omega_{24}^{(TT)}=\rho^{(m,\varepsilon)}\omega^2-C_{3131}^{(m,\varepsilon)}k_1^2,\\
&\Omega_{25}^{(TT)}=-\iota k_1e_{312}^{(m,\varepsilon)},\ \Omega_{26}^{(TT)}=-\iota k_1e_{313}^{(m,\varepsilon)},\ \Omega_{27}^{(TT)}=-\iota k_1q_{312}^{(m,\varepsilon)},\ \Omega_{28}^{(TT)}=-\iota k_1q_{313}^{(m,\varepsilon)},\\
&\Omega_{33}^{(TT)}=-\omega k_1\breve{e}_{221}^{(m,\varepsilon)},\ \Omega_{34}^{(TT)}=-\omega k_1\breve{e}_{231}^{(m,\varepsilon)},\ \Omega_{35}^{(TT)}=\iota\omega\eta_{22}^{(m,\varepsilon)},\ \Omega_{36}^{(TT)}=\iota\omega\eta_{23}^{(m,\varepsilon)},\\
&\Omega_{37}^{(TT)}=\iota\omega\lambda_{22}^{(m,\varepsilon)},\ \Omega_{38}^{(TT)}=\iota\omega\lambda_{23}^{(m,\varepsilon)}-\iota k_1,\ \Omega_{43}^{(TT)}=-\omega k_1\breve{e}_{321}^{(m,\varepsilon)},\ \Omega_{44}^{(TT)}=-\omega k_1\breve{e}_{331}^{(m,\varepsilon)},\\
&\Omega_{45}^{(TT)}=\iota\omega\eta_{32}^{(m,\varepsilon)},\ \Omega_{46}^{(TT)}=\iota\omega\eta_{33}^{(m,\varepsilon)},\ \Omega_{47}^{(TT)}=\iota\omega\lambda_{32}^{(m,\varepsilon)}+\iota k_1,\ \Omega_{48}^{(TT)}=\iota\omega\lambda_{33}^{(m,\varepsilon)},\ \Omega_{73}^{(TT)}=-1\\
&\Omega_{53}^{(TT)}=-\omega k_1\breve{q}_{221}^{(m,\varepsilon)},\ \Omega_{54}^{(TT)}=-\omega k_1\breve{q}_{231}^{(m,\varepsilon)},\ \Omega_{55}^{(TT)}=\iota\omega\lambda_{22}^{(m,\varepsilon)},\ \Omega_{56}^{(TT)}=\iota\omega\lambda_{23}^{(m,\varepsilon)}+\iota k_1,\\
&\Omega_{57}^{(TT)}=\iota\omega\mu_{22}^{(m,\varepsilon)},\ \Omega_{58}^{(TT)}=\iota\omega\mu_{23}^{(m,\varepsilon)},\ \Omega_{63}^{(TT)}=-\omega k_1\breve{q}_{321}^{(m,\varepsilon)},\ \Omega_{64}^{(TT)}=-\omega k_1\breve{q}_{331}^{(m,\varepsilon)},\ \Omega_{84}^{(TT)}=-1,\\
&\Omega_{65}^{(TT)}=\iota\omega\lambda_{32}^{(m,\varepsilon)}-\iota k_1,\ \Omega_{66}^{(TT)}=\iota\omega\lambda_{33}^{(m,\varepsilon)},\ \Omega_{67}^{(TT)}=\iota\omega\mu_{32}^{(m,\varepsilon)},\ \Omega_{68}^{(TT)}=\iota\omega\mu_{63}^{(m,\varepsilon)}.
\end{aligned}\tag{10c}$$

The components of the matrices $\tilde{\mathbf{\Gamma}}^{(TT)}$ and $\tilde{\mathbf{\Omega}}^{(TT)}$ that appear in equation (49) are expressed. Hence, the non-null components of the 4−by−4 matrix $\tilde{\mathbf{\Gamma}}^{(TT)}$ are

$$\tilde{\Gamma}_{11}^{(TT)}=G^{(m,\varepsilon)},\tilde{\Gamma}_{13}^{(TT)}=-e^{(m,\varepsilon)},\tilde{\Gamma}_{14}^{(TT)}=-q^{(m,\varepsilon)},\tilde{\Gamma}_{22}^{(TT)}=1,\tilde{\Gamma}_{34}^{(TT)}=\tilde{\Gamma}_{33}^{(TT)}=\iota.\tag{11a}$$

Moreover, the non-null components of the 4−by−4 matrix $\tilde{\mathbf{\Omega}}^{(TT)}$ are

$$\begin{aligned}
&\tilde{\Omega}_{11}^{(TT)}=2\iota k_1G^{(m,\varepsilon)},\tilde{\Omega}_{12}^{(TT)}=\rho^{(m,\varepsilon)}\omega^2-k_1^2G^{(m,\varepsilon)},\tilde{\Omega}_{13}^{(TT)}=-\iota k_1e^{(m,\varepsilon)},\tilde{\Omega}_{14}^{(TT)}=-\iota k_1q^{(m,\varepsilon)},\tilde{\Omega}_{21}^{(TT)}=-1,\\
&\tilde{\Omega}_{31}^{(TT)}=\iota\omega e^{(m,\varepsilon)},\tilde{\Omega}_{32}^{(TT)}=\iota^2\omega k_1,\tilde{\Omega}_{33}^{(TT)}=\iota\omega\eta^{(m,\varepsilon)},\tilde{\Omega}_{34}^{(TT)}=\iota^2k_1+\iota\omega\lambda^{(m,\varepsilon)},\\
&\tilde{\Omega}_{41}^{(TT)}=-\iota\omega q^{(m,\varepsilon)},\tilde{\Omega}_{42}^{(TT)}=-\iota^2\omega k_1,\tilde{\Omega}_{43}^{(TT)}=-\iota\omega\lambda^{(m,\varepsilon)}+\iota^2k_1,\tilde{\Omega}_{44}^{(TT)}=-\iota\omega\mu^{(m,\varepsilon)},
\end{aligned}\tag{11b}$$

where, due to the cubic symmetry of the layer-composing materials, the tensors that appear in matrices (11a)–(11b) are introduced as

$$\begin{aligned}
&G^{(m,\varepsilon)}:=C_{3131}^{(m,\varepsilon)}=C_{2121}^{(m,\varepsilon)},\ e^{(m,\varepsilon)}:=e_{313}^{(m,\varepsilon)}=e_{212}^{(m,\varepsilon)},\ q^{(m,\varepsilon)}:=q_{313}^{(m,\varepsilon)}=q_{212}^{(m,\varepsilon)},\\
&\eta^{(m,\varepsilon)}:=\eta_{33}^{(m,\varepsilon)}=\eta_{22}^{(m,\varepsilon)},\ \lambda^{(m,\varepsilon)}:=\lambda_{33}^{(m,\varepsilon)}=\lambda_{22}^{(m,\varepsilon)},\ \mu^{(m,\varepsilon)}:=\mu_{33}^{(m,\varepsilon)}=\mu_{22}^{(m,\varepsilon)}.
\end{aligned}\tag{12}$$

The non-null components of the non-singular 4−by−4 matrix $\tilde{\boldsymbol{D}}^{(TT)}$ in equation (51) are

$$\begin{aligned}
&\tilde{D}_{12}^{(TT)}=\tilde{D}_{33}^{(TT)}=\tilde{D}_{44}^{(TT)}=1,\\
&\tilde{D}_{21}^{(TT)}=G^{(m,\varepsilon)},\tilde{D}_{22}^{(TT)}=\iota k_1G^{(m,\varepsilon)},\tilde{D}_{23}^{(TT)}=-e^{(m,\varepsilon)},\tilde{D}_{24}^{(TT)}=-q^{(m,\varepsilon)}.
\end{aligned}\tag{13}$$

Section C Recursive algorithm to determine the invariants of a characteristic polynomial

In Section C the procedure to obtain the invariants of the characteristic polynomials (35), ($41a$) and ($41b$) is described. The twelfth degree palindromic characteristic polynomial $\mathcal{G}(\varphi,\omega)$ reported on equation (35) is defined by means of invariant coefficients as

$$\mathcal{G}(\varphi,\omega)=\det\big(\boldsymbol{T}(\omega)-\varphi\boldsymbol{I}\big)=\sum_{j=0}^{12}\varphi^jI_{12-j}(\omega),\tag{14}$$

where the relations $I_{12}(\omega)=I_0(\omega)=1,\ I_{11}(\omega)=I_1(\omega),\ I_{10}(\omega)=I_2(\omega),\ I_9(\omega)=I_3(\omega),\ I_8(\omega)=I_4(\omega)$ and $I_7(\omega)=I_5(\omega)$ are fulfilled. The roots of the characteristic polynomial (14) can be obtained by exploiting the Faddeev-LeVerrier recursive formula [1] as

$$I_1(\omega)=-\mathrm{tr}(\boldsymbol{T}(\omega)),\quad I_2(\omega)=-\tfrac{1}{2}\mathrm{tr}(\boldsymbol{T}(\omega)^2)+\tfrac{1}{2}(\mathrm{tr}(\boldsymbol{T}(\omega)))^2,\tag{15a}$$

$$
\begin{aligned}
I_3(\omega) &= -\tfrac{1}{3}\mathrm{tr}(\boldsymbol{T}(\omega)^3) + \tfrac{1}{2}\mathrm{tr}(\boldsymbol{T}(\omega))\mathrm{tr}(\boldsymbol{T}(\omega)^2) - \tfrac{1}{6}(\mathrm{tr}(\boldsymbol{T}(\omega)))^3, && \text{(15b)}\\
I_4(\omega) &= -\tfrac{1}{4}\mathrm{tr}(\boldsymbol{T}^4(\omega)) - \tfrac{1}{4}(\mathrm{tr}(\boldsymbol{T}(\omega)))^2\mathrm{tr}(\boldsymbol{T}(\omega)^2) + \tfrac{1}{3}\mathrm{tr}(\boldsymbol{T}(\omega))\mathrm{tr}(\boldsymbol{T}(\omega)^3) + \tfrac{1}{8}(\mathrm{tr}(\boldsymbol{T}(\omega)^2))^2 + \\
&\quad + \tfrac{1}{24}(\mathrm{tr}(\boldsymbol{T}(\omega)))^4, && \text{(15c)}\\
I_5(\omega) &= -\tfrac{1}{5}\mathrm{tr}(\boldsymbol{T}(\omega)^5) + \tfrac{1}{4}\mathrm{tr}(\boldsymbol{T}(\omega)^4)\mathrm{tr}(\boldsymbol{T}(\omega)) + \tfrac{1}{12}\mathrm{tr}(\boldsymbol{T}(\omega))^3\mathrm{tr}(\boldsymbol{T}(\omega)^2) + \tfrac{1}{5}\mathrm{tr}(\boldsymbol{T}(\omega)^3)\mathrm{tr}(\boldsymbol{T}(\omega)^2) + \\
&\quad - \tfrac{1}{8}(\mathrm{tr}(\boldsymbol{T}(\omega)^2))^2\mathrm{tr}(\boldsymbol{T}(\omega)) - \tfrac{1}{6}(\mathrm{tr}(\boldsymbol{T}(\omega)))^2\mathrm{tr}(\boldsymbol{T}(\omega)^3) - \tfrac{1}{120}(\mathrm{tr}(\boldsymbol{T}(\omega)))^5. && \text{(15d)}
\end{aligned}
$$

The fourth degree palindromic characteristic polynomial $\mathcal{G}^{(L)}(\varphi,\omega)$ expressed in $(41a)$ can be written as

$$\mathcal{G}^{(L)}(\varphi,\omega) = \det(\boldsymbol{T}^{(LL)}(\omega) - \varphi\boldsymbol{I}) = \sum_{j=0}^{4}\varphi^j I_{4-j}^{(L)}(\omega), \tag{16}$$

by taking into account of the symmetry relations of the invariants $I_4^{(L)}(\omega) = I_0^{(L)}(\omega) = 1$, $I_3^{(L)}(\omega) = I_1^{(L)}(\omega)$, which result to be

$$I_1^{(L)}(\omega) = -\mathrm{tr}(\boldsymbol{T}^{(LL)}(\omega)), \quad I_2^{(L)}(\omega) = -\tfrac{1}{2}\mathrm{tr}(\boldsymbol{T}^{(LL)}(\omega)^2) + \tfrac{1}{2}(\mathrm{tr}(\boldsymbol{T}^{(LL)}(\omega)))^2. \tag{17}$$

The eight degree palindromic characteristic polynomial $\mathcal{G}^{(T)}(\varphi,\omega)$ reported on $(41b)$ can be expressed as

$$\mathcal{G}^{(T)}(\varphi,\omega) = \det(\boldsymbol{T}^{(TT)}(\omega) - \varphi\boldsymbol{I}) = \sum_{j=0}^{8}\varphi^j I_{8-j}^{(T)}(\omega), \tag{18}$$

and considering the symmetry relations $I_8^{(T)}(\omega) = I_0^{(T)}(\omega) = 1$, $I_7^{(T)}(\omega) = I_1^{(T)}(\omega)$, $I_6^{(T)}(\omega) = I_2^{(T)}(\omega)$ and $I_5^{(T)}(\omega) = I_3^{(T)}(\omega)$, the Faddeev-LeVerrier recursive formula enables to derive the invariants as

$$
\begin{aligned}
I_1^{(T)}(\omega) &= -\mathrm{tr}(\boldsymbol{T}^{(TT)}(\omega)), \quad I_2^{(T)}(\omega) = -\tfrac{1}{2}\mathrm{tr}(\boldsymbol{T}^{(TT)}(\omega)^2) + \tfrac{1}{2}(\mathrm{tr}(\boldsymbol{T}^{(TT)}(\omega)))^2, && \text{(19a)}\\
I_3^{(T)}(\omega) &= -\tfrac{1}{3}\mathrm{tr}((\boldsymbol{T}^{(TT)}(\omega))^3) + \tfrac{1}{2}\mathrm{tr}(\boldsymbol{T}^{(TT)}(\omega))\mathrm{tr}((\boldsymbol{T}^{(TT)}(\omega))^2) - \tfrac{1}{6}(\mathrm{tr}(\boldsymbol{T}^{(TT)}(\omega)))^3, && \text{(19b)}\\
I_4^{(T)}(\omega) &= -\tfrac{1}{4}\mathrm{tr}((\boldsymbol{T}^{(TT)})^4(\omega)) - \tfrac{1}{4}(\mathrm{tr}(\boldsymbol{T}^{(TT)}(\omega)))^2\mathrm{tr}((\boldsymbol{T}^{(TT)}(\omega))^2) + \tfrac{1}{3}\mathrm{tr}(\boldsymbol{T}^{(TT)}(\omega))\mathrm{tr}((\boldsymbol{T}^{(TT)}(\omega))^3) + \\
&\quad + \tfrac{1}{8}(\mathrm{tr}((\boldsymbol{T}^{(TT)}(\omega))^2))^2 + \tfrac{1}{24}(\mathrm{tr}(\boldsymbol{T}^{(TT)}(\omega)))^4. && \text{(19c)}
\end{aligned}
$$

The palindromic characteristic polynomial $\tilde{\mathcal{G}}^{(T)}(\varphi,\omega)$ arose in (53) is

$$\tilde{\mathcal{G}}^{(T)}(\varphi,\omega) = \det(\tilde{\boldsymbol{T}}^{(TT)}(\omega) - \varphi\boldsymbol{I}) = \sum_{j=0}^{4}\varphi^j \tilde{I}_{4-j}^{(T)}(\omega), \tag{20}$$

where the following symmetry relations $\tilde{I}_4^{(T)}(\omega) = \tilde{I}_0^{(T)}(\omega) = 1$, $\tilde{I}_3^{(T)}(\omega) = \tilde{I}_1^{(T)}(\omega)$ hold and the invariants are

$$\tilde{I}_1(\omega) = -\mathrm{tr}(\tilde{\boldsymbol{T}}^{(TT)}(\omega)), \quad \tilde{I}_2(\omega) = -\tfrac{1}{2}\mathrm{tr}((\tilde{\boldsymbol{T}}^{(TT)}(\omega))^2) + \tfrac{1}{2}(\mathrm{tr}(\tilde{\boldsymbol{T}}^{(TT)}(\omega)))^2. \tag{21}$$

Section D Multi-field integral-type non-local first order continua

In Section D, the procedure to achieve multi-field integral-type non-local continua at the first order is detailed. Performing the time Fourier transform $\mathcal{F}_t$ to the system $(10a)$-$(10e)$ for the single layer of the MEE material, after some manipulations, it is possible to derive the first order system

$$\boldsymbol{\Psi}\check{\boldsymbol{b}}_{,1} + \boldsymbol{\Phi}\check{\boldsymbol{b}} = \boldsymbol{0}, \tag{22}$$

where the auxiliary vector $\check{\boldsymbol{b}}(x_1,\omega) = (\check{\boldsymbol{b}}^{(L)}(x_1,\omega)\,\check{\boldsymbol{b}}^{(T)}(x_1,\omega))^\top$ contains the subvector $\check{\boldsymbol{b}}^{(L)}(x_1,\omega)$ such that

$$\check{\boldsymbol{b}}^{(L)}(x_1,\omega) = (\mathcal{F}_t(u_{1,1})\,\mathcal{F}_t(u_1)\,\mathcal{F}_t(E_1)\,\mathcal{F}_t(H_1))^\top, \tag{23}$$

and the subvector $\check{\boldsymbol{b}}^{(T)}(x_1,\omega)$ such that

$$\check{\boldsymbol{b}}^{(T)}(x_1,\omega) = (\mathcal{F}_t(u_{2,1})\,\mathcal{F}_t(u_{3,1})\,\mathcal{F}_t(u_2)\,\mathcal{F}_t(u_3)\,\mathcal{F}_t(E_2)\,\mathcal{F}_t(E_3)\,\mathcal{F}_t(H_2)\,\mathcal{F}_t(H_3))^\top \tag{24}$$

and the non-singular matrices $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ can be decomposed as

$$\boldsymbol{\Psi} = \begin{pmatrix} \boldsymbol{\Psi}^{(LL)} & \boldsymbol{\Psi}^{(LT)} \\ \boldsymbol{\Psi}^{(TL)} & \boldsymbol{\Psi}^{(TT)} \end{pmatrix}, \quad \boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\Phi}^{(LL)} & \boldsymbol{\Phi}^{(LT)} \\ \boldsymbol{\Phi}^{(TL)} & \boldsymbol{\Phi}^{(TT)} \end{pmatrix}, \tag{25}$$

whose non-null components are reported on Appendix D.1. Let us consider the non-singular matrix $\boldsymbol{E}$ as

$$\boldsymbol{E} = \begin{pmatrix} \boldsymbol{E}^{(LL)} & \boldsymbol{E}^{(LT)} \\ \boldsymbol{E}^{(TL)} & \boldsymbol{E}^{(TT)} \end{pmatrix}, \tag{26}$$

such that $\check{\boldsymbol{P}} = \boldsymbol{E}\check{\boldsymbol{b}}$, where the continuous field vector $\check{\boldsymbol{P}}$ has been defined in (59). The non-null components of matrix $\boldsymbol{E}$ are reported on Appendix D.2. Since the matrix $\boldsymbol{E}$ is invertible, replacing the relation $\check{\boldsymbol{b}} = \boldsymbol{E}^{-1}\check{\boldsymbol{P}}$ into the matricial system (22) leads to

$$\frac{\partial \check{\boldsymbol{P}}}{\partial x_1} + (\boldsymbol{\Psi}\boldsymbol{E}^{-1})^{-1}\boldsymbol{\Phi}\boldsymbol{E}^{-1}\check{\boldsymbol{P}} = \boldsymbol{0}, \tag{27}$$

which has the same formal mathematical structure as the equation (77) at order $M=1$. Finally, invoking the spatial Fourier transform with complex argument $\mathcal{F}$ to equation (27) yields

$$((\boldsymbol{\Psi}\boldsymbol{E}^{-1})^{-1}\boldsymbol{\Phi}\boldsymbol{E}^{-1} + \iota k_1 \boldsymbol{I})\hat{\boldsymbol{P}}(k_1,\omega) = \boldsymbol{0}, \tag{28}$$

which displays the same formal mathematical structure as the eigenproblem (79) at the first order.

Section D.1 Components of the matrices $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$

Section D.1 gathers the components of the matrices $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ expressed in (25). The non-null components of the 4−by−4 matrix $\Psi^{(LL)}$ are

$$\begin{aligned} &\Psi^{(LL)}_{11} = C^{(m,\varepsilon)}_{1111},\ \Psi^{(LL)}_{13} = -\breve{e}^{(m,\varepsilon)}_{111},\ \Psi^{(LL)}_{14} = -\breve{q}^{(m,\varepsilon)}_{111}, \\ &\Psi^{(LL)}_{21} = \breve{e}^{(m,\varepsilon)}_{111},\ \Psi^{(LL)}_{23} = \eta^{(m,\varepsilon)}_{11},\ \Psi^{(LL)}_{24} = \lambda^{(m,\varepsilon)}_{11}, \\ &\Psi^{(LL)}_{31} = \breve{q}^{(m,\varepsilon)}_{111},\ \Psi^{(LL)}_{33} = \lambda^{(m,\varepsilon)}_{11},\ \Psi^{(LL)}_{34} = \mu^{(m,\varepsilon)}_{11},\ \Psi^{(LL)}_{41} = 1. \end{aligned} \tag{29a}$$

The non-null components of the 4−by−8 matrix $\Psi^{(LT)}$ are

$$\begin{aligned} &\Psi^{(LT)}_{11} = C^{(m,\varepsilon)}_{1121},\ \Psi^{(LT)}_{12} = C^{(m,\varepsilon)}_{1131},\ \Psi^{(LT)}_{15} = -e^{(m,\varepsilon)}_{112},\ \Psi^{(LT)}_{16} = -e^{(m,\varepsilon)}_{113},\ \Psi^{(LT)}_{17} = -q^{(m,\varepsilon)}_{112},\ \Psi^{(LT)}_{18} = -q^{(m,\varepsilon)}_{113}, \\ &\Psi^{(LT)}_{11} = \breve{e}^{(m,\varepsilon)}_{121},\ \Psi^{(LT)}_{12} = \breve{e}^{(m,\varepsilon)}_{131},\ \Psi^{(LT)}_{15} = \eta^{(m,\varepsilon)}_{12},\ \Psi^{(LT)}_{16} = \eta^{(m,\varepsilon)}_{13},\ \Psi^{(LT)}_{17} = \lambda^{(m,\varepsilon)}_{12},\ \Psi^{(LT)}_{18} = \lambda^{(m,\varepsilon)}_{13}, \\ &\Psi^{(LT)}_{31} = \breve{q}^{(m,\varepsilon)}_{121},\ \Psi^{(LT)}_{32} = \breve{q}^{(m,\varepsilon)}_{131},\ \Psi^{(LT)}_{35} = \lambda^{(m,\varepsilon)}_{12},\ \Psi^{(LT)}_{36} = \lambda^{(m,\varepsilon)}_{13},\ \Psi^{(LT)}_{37} = \mu^{(m,\varepsilon)}_{12},\ \Psi^{(LT)}_{38} = \mu^{(m,\varepsilon)}_{13}. \end{aligned} \tag{29b}$$

The non-null components of the 8−by−4 matrix $\Psi^{(TL)}$ are

$$\begin{aligned} &\Psi^{(TL)}_{11} = C^{(m,\varepsilon)}_{2111},\ \Psi^{(TL)}_{13} = -e^{(m,\varepsilon)}_{211},\ \Psi^{(TL)}_{14} = -q^{(m,\varepsilon)}_{211}, \\ &\Psi^{(LL)}_{21} = C^{(m,\varepsilon)}_{3111},\ \Psi^{(LL)}_{23} = -e^{(m,\varepsilon)}_{311},\ \Psi^{(LL)}_{24} = -q^{(m,\varepsilon)}_{311}, \\ &\Psi^{(LL)}_{32} = \iota\omega\breve{e}^{(m,\varepsilon)}_{211},\ \Psi^{(LL)}_{43} = \iota\omega\breve{e}^{(m,\varepsilon)}_{311},\ \Psi^{(LL)}_{54} = \iota\omega\breve{q}^{(m,\varepsilon)}_{211},\ \Psi^{(LL)}_{62} = \iota\omega\breve{q}^{(m,\varepsilon)}_{311}. \end{aligned} \tag{29c}$$

The non-null components of the 8−by−8 matrix $\Psi^{(TT)}$ are

$$\begin{aligned} &\Psi^{(TT)}_{11} = C^{(m,\varepsilon)}_{2121},\ \Psi^{(TT)}_{12} = C^{(m,\varepsilon)}_{2131},\ \Psi^{(TT)}_{15} = -e^{(m,\varepsilon)}_{212},\ \Psi^{(TT)}_{16} = -e^{(m,\varepsilon)}_{213},\ \Psi^{(TT)}_{17} = -q^{(m,\varepsilon)}_{212},\ \Psi^{(TT)}_{18} = -q^{(m,\varepsilon)}_{213}, \\ &\Psi^{(TT)}_{21} = C^{(m,\varepsilon)}_{3121},\ \Psi^{(TT)}_{22} = C^{(m,\varepsilon)}_{3131},\ \Psi^{(TT)}_{25} = -e^{(m,\varepsilon)}_{312},\ \Psi^{(TT)}_{26} = -e^{(m,\varepsilon)}_{313},\ \Psi^{(TT)}_{27} = -q^{(m,\varepsilon)}_{312},\ \Psi^{(TT)}_{28} = -q^{(m,\varepsilon)}_{313}, \end{aligned}$$

$$\Psi_{33}^{(TT)} = \iota\omega\breve{\breve{e}}_{221}^{(m,\varepsilon)},\ \Psi_{34}^{(TT)} = \iota\omega\breve{\breve{e}}_{231}^{(m,\varepsilon)},\ \Psi_{43}^{(TT)} = \iota\omega\breve{\breve{e}}_{321}^{(m,\varepsilon)},\ \Psi_{44}^{(TT)} = \iota\omega\breve{\breve{e}}_{331}^{(m,\varepsilon)},\ \Psi_{38}^{(TT)} = -1,\ \Psi_{47}^{(TT)} = \Psi_{84}^{(TT)} = 1,$$
$$\Psi_{53}^{(TT)} = \iota\omega\breve{\breve{q}}_{221}^{(m,\varepsilon)},\ \Psi_{54}^{(TT)} = \iota\omega\breve{\breve{q}}_{231}^{(m,\varepsilon)},\ \Psi_{63}^{(TT)} = \iota\omega\breve{\breve{q}}_{321}^{(m,\varepsilon)},\ \Psi_{64}^{(TT)} = \iota\omega\breve{\breve{q}}_{331}^{(m,\varepsilon)},\ \Psi_{56}^{(TT)} = \Psi_{73}^{(TT)} = 1,\ \Psi_{65}^{(TT)} = -1. \tag{29d}$$

The non-null components of the 4−by−4 matrix $\Phi^{(LL)}$ are

$$\Phi_{11}^{(LL)} = \omega^2\rho^{(m,\varepsilon)},\ \Phi_{41}^{(LL)} = -1. \tag{30a}$$

The non-null components of the 8−by−4 matrix $\Phi^{(TL)}$ are

$$\Phi_{33}^{(TL)} = \iota\omega\eta_{21}^{(m,\varepsilon)},\ \Phi_{34}^{(TL)} = \iota\omega\lambda_{21}^{(m,\varepsilon)},\ \Phi_{43}^{(TL)} = \iota\omega\eta_{31}^{(m,\varepsilon)},\ \Phi_{44}^{(TL)} = \iota\omega\lambda_{31}^{(m,\varepsilon)},$$
$$\Phi_{53}^{(TL)} = \iota\omega\lambda_{21}^{(m,\varepsilon)},\ \Phi_{54}^{(TL)} = \iota\omega\mu_{21}^{(m,\varepsilon)},\ \Phi_{63}^{(TL)} = \iota\omega\lambda_{31}^{(m,\varepsilon)},\ \Phi_{64}^{(TL)} = \iota\omega\mu_{31}^{(m,\varepsilon)}. \tag{30b}$$

The non-null components of the 8−by−8 matrix $\Phi^{(TT)}$ are

$$\Phi_{13}^{(TT)} = \Phi_{24}^{(TT)} = \omega^2\rho^{(m,\varepsilon)},\ \Phi_{35}^{(TT)} = \iota\omega\eta_{22}^{(m,\varepsilon)},\ \Phi_{36}^{(TT)} = \iota\omega\eta_{23}^{(m,\varepsilon)},\ \Phi_{37}^{(TT)} = \iota\omega\lambda_{22}^{(m,\varepsilon)},\ \Phi_{38}^{(TT)} = \iota\omega\lambda_{23}^{(m,\varepsilon)},$$
$$\Phi_{45}^{(TT)} = \iota\omega\eta_{32}^{(m,\varepsilon)},\ \Phi_{46}^{(TT)} = \iota\omega\eta_{33}^{(m,\varepsilon)},\ \Phi_{47}^{(TT)} = \iota\omega\lambda_{32}^{(m,\varepsilon)},\ \Phi_{48}^{(TT)} = \iota\omega\lambda_{33}^{(m,\varepsilon)},$$
$$\Phi_{55}^{(TT)} = \iota\omega\lambda_{22}^{(m,\varepsilon)},\ \Phi_{56}^{(TT)} = \iota\omega\lambda_{23}^{(m,\varepsilon)},\ \Phi_{57}^{(TT)} = \iota\omega\mu_{22}^{(m,\varepsilon)},\ \Phi_{58}^{(TT)} = \iota\omega\mu_{23}^{(m,\varepsilon)},$$
$$\Phi_{65}^{(TT)} = \iota\omega\lambda_{32}^{(m,\varepsilon)},\ \Phi_{66}^{(TT)} = \iota\omega\lambda_{33}^{(m,\varepsilon)},\ \Phi_{67}^{(TT)} = \iota\omega\mu_{32}^{(m,\varepsilon)},\ \Phi_{68}^{(TT)} = \iota\omega\mu_{33}^{(m,\varepsilon)},$$
$$\Phi_{73}^{(TT)} = \Phi_{84}^{(TT)} = -1. \tag{30c}$$

Section D.2 Components of the matrix $\boldsymbol{E}$

Section D.2 reports on the components of the matrix $\boldsymbol{E}$ expressed in (26). The non-null components of the 4−by−4 matrix $E^{(LL)}$ are

$$E_{21}^{(LL)} = C_{1111}^{(m,\varepsilon)},\ E_{23}^{(LL)} = -e_{111}^{(m,\varepsilon)},\ E_{24}^{(LL)} = -q_{111}^{(m,\varepsilon)}, E_{12}^{(LL)} = D_{33}^{(LL)} = E_{44}^{(LL)} = 1. \tag{31a}$$

The non-null components of the 4−by−8 matrix $E^{(LT)}$ are

$$E_{21}^{(LT)} = C_{1121}^{(m,\varepsilon)},\ E_{22}^{(LT)} = C_{1131}^{(m,\varepsilon)}, E_{25}^{(LT)} = -e_{112}^{(m,\varepsilon)},\ E_{26}^{(LT)} = -e_{113}^{(m,\varepsilon)}, E_{27}^{(LT)} = -q_{112}^{(m,\varepsilon)},\ E_{28}^{(LT)} = -q_{113}^{(m,\varepsilon)}. \tag{31b}$$

The non-null components of the 8−by−4 matrix $E^{(TL)}$ are

$$E_{31}^{(TL)} = C_{2111}^{(m,\varepsilon)},\ E_{33}^{(TL)} = -e_{211}^{(m,\varepsilon)},\ E_{34}^{(TL)} = -q_{211}^{(m,\varepsilon)},\ E_{41}^{(TL)} = C_{3111}^{(m,\varepsilon)},\ \breve{\breve{e}}_{43}^{(TL)} = -e_{311}^{(m,\varepsilon)},\ E_{44}^{(TL)} = -q_{311}^{(m,\varepsilon)}. \tag{31c}$$

The non-null components of the 8−by−8 matrix $E^{(TT)}$ are

$$E_{31}^{(TT)} = C_{2121}^{(m,\varepsilon)},\ E_{32}^{(TT)} = C_{2131}^{(m,\varepsilon)}, E_{35}^{(TT)} = -e_{212}^{(m,\varepsilon)},\ E_{36}^{(TT)} = -e_{213}^{(m,\varepsilon)}, E_{37}^{(TT)} = -q_{212}^{(m,\varepsilon)},\ E_{38}^{(TT)} = -q_{213}^{(m,\varepsilon)},$$
$$E_{41}^{(TT)} = C_{3121}^{(m,\varepsilon)},\ E_{42}^{(TT)} = C_{3131}^{(m,\varepsilon)}, E_{45}^{(TT)} = -e_{312}^{(m,\varepsilon)},\ E_{46}^{(TT)} = -e_{313}^{(m,\varepsilon)}, E_{47}^{(TT)} = -q_{312}^{(m,\varepsilon)},\ E_{48}^{(TT)} = -q_{313}^{(m,\varepsilon)},$$
$$E_{13}^{(TT)} = E_{24}^{(TT)} = E_{55}^{(TT)} = E_{66}^{(TT)} = E_{77}^{(TT)} = E_{88}^{(TT)} = 1. \tag{31d}$$

Section E Eigenproblem linearization

Section E highlights the linearization of eigenproblems (79) and (89). An alternative procedure to deal with the equation (80) is to rewrite the eigenproblem (79) at a generic order M as

$$(\bar{\boldsymbol{T}}(\omega) + \sum_{n=1}^{M} k_1^{2n-1}\boldsymbol{D}_{2n-1})\hat{\boldsymbol{P}}(k_1,\omega) = \boldsymbol{0}, \tag{32}$$

where $\boldsymbol{D}_{2n-1} = -\frac{(\iota S)^{2n-1}}{(2n-1)!}\boldsymbol{I}$, with $n \in \mathbb{N}$. The system (32) can be linearized as

$$(k_1\boldsymbol{G} + \boldsymbol{H})\hat{\boldsymbol{V}}(k_1, \omega) = \boldsymbol{0}, \tag{33}$$

where

$$\boldsymbol{G} = \begin{pmatrix} \boldsymbol{D}_{2n-1} & \boldsymbol{0} & \dots & \dots & \dots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \dots & \dots & \dots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \boldsymbol{0} \\ \boldsymbol{0} & \dots & \dots & \dots & \boldsymbol{0} & \boldsymbol{I} \end{pmatrix}, \quad \boldsymbol{H} = \begin{pmatrix} \boldsymbol{D}_{2n-3} & \boldsymbol{D}_{2n-5} & \dots & \dots & \boldsymbol{D}_1 & \bar{\boldsymbol{T}} \\ -\boldsymbol{I} & \boldsymbol{0} & \dots & \dots & \dots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \boldsymbol{0} \\ \boldsymbol{0} & \dots & \dots & \dots & -\boldsymbol{I} & \boldsymbol{0} \end{pmatrix}, \quad \hat{\boldsymbol{V}} = \begin{pmatrix} k_1^{2n-2}\hat{\boldsymbol{P}} \\ \vdots \\ \vdots \\ \hat{\boldsymbol{P}} \end{pmatrix}. \tag{34}$$

The eigenvalues of the problem (33) provides the complex spectrum of the gradient-type continuum at order $M = 2n-1$ and the polarization vectors.
The equation (89) can be recast into

$$(\bar{\boldsymbol{T}}(\omega) + \sum_{n=1}^{M}(k_1-\pi)^{2n-1}\boldsymbol{F}_{2n-1})\hat{\boldsymbol{P}}(k_1, \omega) = \boldsymbol{0}, \tag{35}$$

where $\boldsymbol{F}_{2n-1} = -e^{\iota\pi S}\frac{(\iota S)^{2n-1}}{(2n-1)!}\boldsymbol{I}$, with $n \in \mathbb{N}$. The system (35) can be linearized as

$$((k_1-\pi)\boldsymbol{L} + \boldsymbol{M})\hat{\boldsymbol{W}}(k_1, \omega) = \boldsymbol{0}, \tag{36}$$

with

$$\boldsymbol{L} = \begin{pmatrix} \boldsymbol{F}_{2n-1} & \boldsymbol{0} & \dots & \dots & \dots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \dots & \dots & \dots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \boldsymbol{0} \\ \boldsymbol{0} & \dots & \dots & \dots & \boldsymbol{0} & \boldsymbol{I} \end{pmatrix}, \quad \boldsymbol{M} = \begin{pmatrix} \boldsymbol{F}_{2n-3} & \boldsymbol{F}_{2n-5} & \dots & \dots & \boldsymbol{F}_1 & \bar{\boldsymbol{T}} \\ -\boldsymbol{I} & \boldsymbol{0} & \dots & \dots & \dots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \boldsymbol{0} \\ \boldsymbol{0} & \dots & \dots & \dots & -\boldsymbol{I} & \boldsymbol{0} \end{pmatrix}, \tag{37}$$

$$\hat{\boldsymbol{W}} = \begin{pmatrix} (k_1-\pi)^{2n-2}\hat{\boldsymbol{P}} \\ \vdots \\ \vdots \\ \hat{\boldsymbol{P}} \end{pmatrix}.$$

The eigenvalues of the eigenproblem (36) supplies the complex spectrum of the envelope-type continuum at order $M = 2n-1$ and the polarization vectors.

Section F Procedure to determine the physically feasible solution

Section F proposes a procedure to select the physically feasible solution of the characteristic equations (80) and (90). Let $S_M^{(0)} = \left\{(k_{1,M}^{[j]}(\omega_*), \omega_*) : \mathcal{H}^{(0)}(k_1, \omega_*) = 0 \quad \forall\omega_* \in \mathbb{R}, \quad j \in \mathbb{N}_{\geq 1}^{\leq \mathcal{C}_M^{(0)}}\right\}$ be a numerable finite set with cardinality $\mathcal{C}_M^{(0)} = 12(M-1)$, which gathers the roots $k_{1,M}^{[j]}(\omega_*)$ of the equation (80), with $j \in \mathbb{N}_{\geq 1}^{\leq \mathcal{C}_{S,M}^{(0)}}$ and $\omega_* \in \mathbb{R}$. The $(k_{1,M}^{[j]}(\omega_*), \omega_*)$-pair determines a point of the complex spectrum in the (k_1, ω)-domain related to magneto-electro-elastic waves propagating in the model that is obtained via the continualization technique employed in Section 3.2. The infinite numerable set $\mathcal{S}_\infty^{(0)} = \Big\{(k_{1,\infty}^{[j]}(\omega_*), \omega_*) : \lim_{M\to\infty} \mathcal{H}^{(0)}(k_1, \omega_*) = 0 \quad \forall\omega_* \in$

$\mathbb{R}, \quad j \in \mathbb{N}_{\geq 1}\Big\}$, which is retrieved by means of a limit operation, contains the roots of the equation (65). It can be pointed out that the set $\mathcal{M} = \Big\{(k_1^{[j]}(\omega_*), \omega_*) : \mathcal{G}(k_1, \omega_*) = 0 \quad \forall \omega_* \in \mathbb{R}, \quad j \in \mathbb{N}_{\geq 1}^{\leq \mathcal{C}}\Big\}$ that contains the roots $k_1^{[j]}(\omega_*)$ of the equation (33) having cardinality $\mathcal{C} = 12$ is such that $\mathcal{M} \subset \mathcal{S}_\infty^{(0)}$. In addition, such roots $k_1^{[j]}(\omega_*) \in \mathcal{M} \subset \mathcal{S}_\infty^{(0)}$, $j \in \mathbb{N}_{\geq 1}^{\leq \mathcal{C}}$, represent the physically feasible solution $k_{1,\infty f}^{[j]}$ of equation (65), whereas the discardable roots are the ones that lie outside of the convergence region of the series defined in (76), namely at infinite. Infact, there is worth noticing that, for physically feasible solution, the relation $\lim_{M\to\infty} k_{1,Mf}^{[j]} = k_{1,\infty f}^{[j]} = k_1^{[j]} \in \mathcal{M} \subset \mathcal{S}_\infty^{(0)}$ holds. On the other hand, the same procedure can be applied to select the roots of the equation (90) by defining the numerable finite set $S_M^{(\pi)} = \Big\{(k_{1,M}^{[j,\pi]}(\omega_*), \omega_*) : \mathcal{H}^{(\pi)}(k_1, \omega_*) = 0 \quad \forall \omega_* \in \mathbb{R}, \quad j \in \mathbb{N}_{\geq 1}^{\leq \mathcal{C}_M^{(\pi)}}\Big\}$ having cardinality $\mathcal{C}_M^{(\pi)} = 12(M-1)$ and collecting the roots $k_{1,M}^{[j,\pi]}(\omega_*)$ of the equation (90), with $j \in \mathbb{N}_{\geq 1}^{\leq \mathcal{C}_M^{(\pi)}}$ and $\omega_* \in \mathbb{R}$. A limit operation provides the infinite numerable set $\mathcal{S}_\infty^{(\pi)} = \Big\{(k_{1,\infty}^{[j,\pi]}(\omega_*), \omega_*) : \lim_{M\to\infty} \mathcal{H}^{(\pi)}(k_1, \omega_*) = 0 \quad \forall \omega_* \in \mathbb{R}, \quad j \in \mathbb{N}_{\geq 1}\Big\}$ such that $k_1^{[j]}(\omega_*) \in \mathcal{M} \subset \mathcal{S}_\infty^{(\pi)}$, which represents the feasible solution $k_{1,\infty f}^{[j,\pi]}(\omega_*)$ of the infinite order equation (73), whereas the discardable roots can be found outside the convergence region of the Taylor series in equation (95).

Section G Representation of complex spectra via toroidal and cylindrical coordinates

Section G.1 Toroidal representation of complex spectra

Section G.1 proposes a representation of complex spectra and band structure related to equation (80) via toroidal coordinates. Indeed, according to [2], the relations between the cartesian and the toroidal coordinates related to the Figure 5, in the main text, are

$$\bar{\gamma}_1 = \frac{\sinh\left(\left|e^{\iota \bar{k}_1}\right|\right) \cos(\arg(e^{\iota \bar{k}_1}))}{\cosh\left(\left|e^{\iota \bar{k}_1}\right|\right) - \cos(\arg(e^{\iota \bar{\omega} \bar{t}}))}, \tag{38a}$$

$$\bar{\gamma}_2 = \frac{\sinh\left(\left|e^{\iota \bar{k}_1}\right|\right) \sin\left(\arg(e^{\iota \bar{k}_1})\right)}{\cosh\left(\left|e^{\iota \bar{k}_1}\right|\right) - \cos\left(\arg(e^{\iota \bar{\omega} \bar{t}})\right)}, \tag{38b}$$

$$\bar{\gamma}_3 = \frac{\sin\left(\arg(e^{\iota \bar{\omega} \bar{t}})\right)}{\cosh\left(\left|e^{\iota \bar{k}_1}\right|\right) - \cos\left(\arg(e^{\iota \bar{\omega} \bar{t}})\right)}. \tag{38c}$$

Section G.2 Cylindrical representation of complex spectra

Section G.2 exposes a representation of complex spectra and band structure related to equation (80) via cylindrical coordinates. Figure 1 portrays the behaviour of dimensionless homogenized complex spectra obtained by solving the characteristic equation (80), with M= 11 (white), and the exact one (red) derived by searching for the roots of the characteristic equation (53). The curves are plotted on a cylinder by varying the magnitude of $e^{\iota \bar{k}_1}$, the argument of $e^{\iota \bar{\omega} \bar{t}}$ and the real part of the dimensionless wavenumber $\bar{k}_1$. Referring to Figure 3-(c), (d) of the main text, the attenuation curves due to material damping occurring in $\mathcal{R}e(\bar{k}_1) = -\pi, \pi$ appear at the upper and lower border of the cylinder, whereas the acoustic branches related to wave propagation develop along the surface. In Figures 1, the spectrum derived from the proposed continualization scheme is in good agreement with the one related to the heterogeneous material obtained with the theory of Floquet-Bloch by selecting the dimensionless angular frequency range $\bar{\omega} \in [0, 10]$. According to [2], the relations between the cylindrical and the cartesian coordinates are

$$\bar{v}_1 = \left|e^{\iota \bar{k}_1}\right| \cos\left(\arg(e^{\iota \bar{\omega} \bar{t}})\right), \tag{39a}$$

$$\bar{v}_2 = \left|e^{\iota \bar{k}_1}\right| \sin\left(\arg(e^{\iota \bar{\omega} \bar{t}})\right), \tag{39b}$$

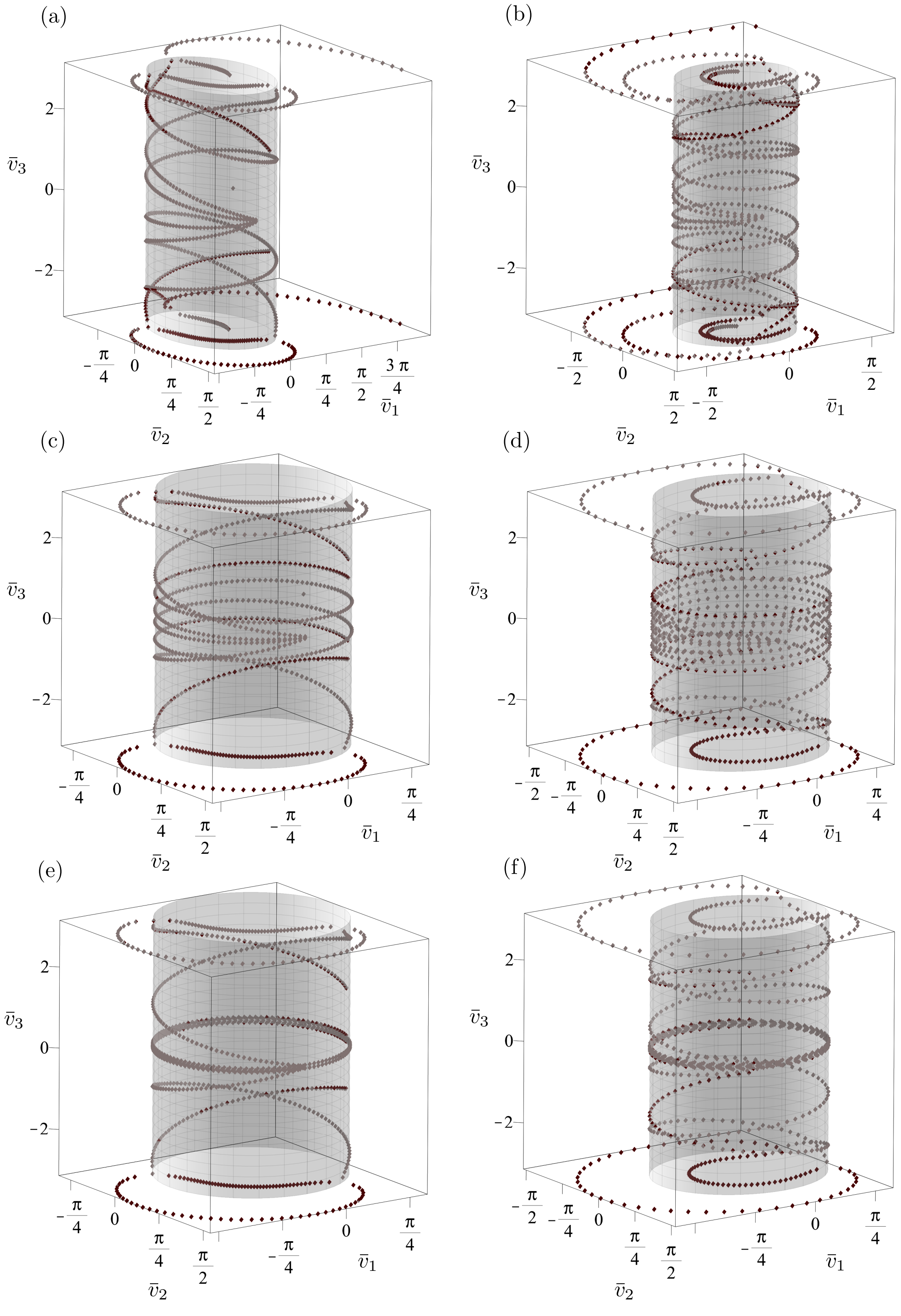

Figure 1: Exact (red) and homogenized complex spectra derived for $M=11$ (white) via the continualization scheme with the kernel developped as a Taylor series centered at $\bar{k}_1=0$ in the range $\bar{\omega}\in[0,10]$ and represented with cylindrical coordinates $\bar{v}_j$ for fixed non-null constitutive parameters $r_G=r_\rho=2$, $r_e=\frac{1}{100}$, $r_\mu=100$, $r_\eta=\frac{1}{100}$, $\upsilon=\frac{1}{3}$ by increasing the dimensionless time $\bar{t}$ as: $\bar{t}=4000$ in (a), (c), (e) and $\bar{t}=8000$ in (b), (d), (f) and by decreasing ψ as: $\psi=10^{-2}$ (a), (b), $\psi=10^{-3}$ (c), (d) and $\psi=10^{-5}$ (e), (f).

$$\bar{v}_3 = \mathcal{R}e(\bar{k}_1). \tag{39c}$$

References

[1] Helmberg, G., Wagner, P., Veltkamp, G. On Faddeev-Leverrier's method for the computation of the characteristic polynomial of a matrix and of eigenvectors. Linear algebra and its applications, 185, pp. 219–233, (1993).

[2] Aratyn, H., Rasinariu, C. *A short course in mathematical methods with Maple.* World Scientific Publishing Company, (2005).