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Obliquely reflecting Brownian motion in nonpolyhedral, piecewise smooth cones, with an example of application to diffusion approximation of bandwidth-sharing queues

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Abstract
This work gives sufficient conditions for uniqueness in distribution of semimartingale, obliquely reflecting Brownian motion in a nonpolyhedral, piecewise $\mathcal{C}^2$ cone, with radially constant, Lipschitz continuous direction of reflection on each face. The conditions are shown to be verified by the conjectured diffusion approximation to the workload in a class of bandwidth-sharing networks, thus ensuring that the conjectured limit is uniquely characterized. This is an essential step in proving the diffusion approximation. This uniqueness result is made possible by replacing the Krein–Rutman theorem used by Kwon and Williams (1993) in a smooth cone with the recent reverse ergodic theorem for inhomogeneous, killed Markov chains of Costantini and Kurtz (Ann Appl Probab, 2024. https://doi.org/10.1214/23-AAP2047; Stoch Process Appl 170:104295, 2024. https://doi.org/10.1016/j.spa.2024.104295).

Keywords Semimartingale obliquely reflecting Brownian motion · Piecewise smooth cone · Stochastic networks · Bandwidth sharing · Input-queued switches

Mathematics Subject Classification 60J60 · 60J55 · 60K3 · 90B15

1 Introduction

Bandwidth-sharing networks operating under a weighted α-fair bandwidth-sharing policy are used to model Internet congestion control and were studied in depth in Kelly and Williams [15] and Kang et al. [12]. Under this policy, the proportion of the capacity of each network resource allocated to each route is determined by maximizing a reward function: The parameter α is essentially the exponent in the reward function (see Sect. 2 for details). The diffusion approximation of the workload process, in the heavy traffic regime, is conjectured to be a semimartingale obliquely reflecting Brownian motion

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(ORBM) in a piecewise smooth cone. If there are only two resources, the cone is a wedge in $\mathbb{R}^2$. If there are more than two resources, the parameter α determines the shape of the cone: For $\alpha = 1$, the cone is polyhedral; for $\alpha \neq 1$, the cone has curved faces that intersect nonsmoothly and can even meet in cusp-like singularities (see the example of Sect. 2.2). In the $\alpha = 1$ case, Kang et al. [12] prove the diffusion approximation by applying the invariance principle of Kang and Williams [11] and by the uniqueness result of Dai and Williams [6]. Roughly speaking, the invariance principle of Kang and Williams [11] ensures that the sequence of the scaled workload processes is relatively compact and that every limit point (in distribution) is a solution of a stochastic differential equation with reflection in the cone. (Therefore, as a by-product, it ensures also existence of a solution.) Convergence then follows if the solution to the stochastic differential equation with reflection is unique in distribution, which, for a polyhedral cone, is proved in Dai and Williams [6]. For $\alpha \neq 1$, the authors of Kang et al. [12] say that their argument for the $\alpha = 1$ case does not carry over, in particular due to the lack of a uniqueness theory for semimartingale ORBM in nonpolyhedral cones (see the discussion of the literature below) and the conjecture is still open.

Another notable class of networks in the study of which piecewise smooth cones show up is input-queued switches under a maximum-weight-α policy. Input-queued switches are widely used in Internet routing and were studied under a maximum-weight-α policy in Shah and Wischik [21] and Kang and Williams [14]. In this case, the parameter α is the exponent in the criterion for determining an optimal matching. As for bandwidth-sharing networks, the diffusion approximation to the workload process is conjectured to be a semimartingale obliquely reflecting Brownian motion (ORBM) in a piecewise smooth cone, which is polyhedral for $\alpha = 1$, and has curved faces for $\alpha \neq 1$. Kang and Williams [14] prove the diffusion approximation for the $\alpha = 1$ case, by Kang and Williams [11] and Dai and Williams [6], but for $\alpha \neq 1$ the conjecture is still open for the same reasons as for bandwidth-sharing networks.

The contribution of this work is precisely to prove, in the nonpolyhedral case, a parallel result to Dai and Williams [6], that is to give sufficient conditions under which uniqueness in distribution holds for semimartingale ORBM in a nonpolyhedral, piecewise smooth cone. The parallelism is incomplete because the conditions of Dai and Williams [6] are also necessary, at least for simple polyhedrons, while the conditions given here are only sufficient. As an example, it will be shown that the conditions given here are verified by the conjectured diffusion approximation to the workload in a class of bandwidth-sharing networks. In order to complete the proof of the diffusion approximation, one needs to show that the invariance principle of Kang and Williams [11] can be applied, or prove a suitable result analogous to the invariance principle of Kang and Williams [11], but this is left for future work. These results would also guarantee the existence of a semimartingale ORBM, so the issue of existence is also not treated here. (However, see Remark 2.5 for the existence of a semimartingale reflecting ORBM in the example of Sect. 2.2.)

Only partial uniqueness results are available for obliquely reflecting diffusion in nonsmooth domains.

Most results concern piecewise smooth domains. There are of course the seminal works of Harrison and Reiman [10] in the orthant and Varadhan and Williams [23] and

Williams [24] in the wedge. The above-mentioned paper Dai and Williams [6] proves the existence and uniqueness in distribution for semimartingale ORBM in a convex polyhedron, with constant direction of reflection on each face, under a condition on the directions of reflection that generalizes the so-called completely-$\mathcal{S}$ condition used by Taylor and Williams [22] in the orthant. This condition is also necessary for simple polyhedrons. Strong existence and uniqueness for a class of ORBMs in polyhedral domains that do not satisfy the condition of Dai and Williams [6] are obtained by Ramanan [20]. In this case, the ORBM is not a semimartingale. For a piecewise smooth domain with curved faces and no cusp-like points, the best available result is Dupuis and Ishii [9], which proves strong existence and uniqueness of the solution to a stochastic differential equation with reflection, under a restriction on the directions of reflection. In dimension two, recently Costantini and Kurtz [5] have obtained existence and uniqueness in distribution, in a piecewise smooth domain allowing for cusps, under a condition that is optimal in the sense that it reduces to that of Dai and Williams [6] in the case of a convex polygon with constant direction of reflection on each face. This condition is also the same under which, in a piecewise smooth domain in arbitrary dimension, Kang and Ramanan [13] prove the equivalence between existence and uniqueness in distribution of the solution to a stochastic differential equation with reflection, well-posedness of the corresponding submartingale problem and well-posedness of the corresponding constrained martingale problem (see Sect. 3.2 for constrained martingale problems). However, Kang and Ramanan [13] do not show that the above condition is sufficient for existence and uniqueness in distribution. In dimension two, the cusp case has been studied by DeBlassie and Toby [7], when, on each side of the cusp, the direction of reflection forms a constant angle with the inward normal, and by Costantini and Kurtz [2] under the same condition as in Costantini and Kurtz [5].

However, a piecewise smooth cone is the intersection of smooth cones, which are not smooth domains; hence, it is not a piecewise smooth domain. Therefore, none of the above results applies to piecewise smooth cones. Moreover, the arguments used for piecewise smooth domains do not extend to piecewise smooth cones.

The work that is most closely related to the present one is Kwon and Williams [19], which proves the existence and uniqueness of the solution to the submartingale problem for an ORBM in a smooth cone, with zero drift and radially constant, smooth direction of reflection. Kwon and Williams [19] use smoothness of the cone essentially in two points: to solve a boundary value problem (3.1)), the solution of which determines whether the vertex is reached; and to prove that the transition operator of a certain killed Markov chain is a strongly positive operator, and hence be able to apply the Krein–Rutman theorem (Kreĭn and Rutman [16]) to obtain uniqueness. The Krein–Rutman theorem assumes also compactness of the transition operator, and proving this requires some delicate oscillation estimates.

The main difference between this work and Kwon and Wil [19] is that uniqueness is obtained not by the Krein–Rutman theorem, but by a new reverse, ergodic theorem for inhomogeneous, killed Markov chains proved in Costantini and Kurtz [4] (recalled as Theorem 3.20). The assumptions of Theorem 3.20 have a clear probabilistic meaning and may be verified in a wider range of situations: piecewise smooth cones—the case of this paper; directions of reflection that are not radially constant; variable coefficients of the reflecting process; nonsmooth domains that are not cones but can be locally approximated by cones (see Costantini and Kurtz [4] in arbitrary dimension and Costantini and Kurtz [5] in dimension two), etc. In addition, the assumptions of Theorem 3.20 do not involve compactness of the transition operators and hence do not require oscillation estimates.

In the present context, Theorem 3.20 allows to avoid assuming smoothness of the cone. As in Kwon and Williams [19], here too a function (or two functions) is employed to analyze the time till the vertex of the cone is reached, but, for the purpose of applying Theorem 3.20, the functions do not need to satisfy the equalities in (3.1), but only corresponding inequalities (Condition 3.5). Thus, these functions can be found without solving the boundary value problem. As an example, see the function in Sect. 2.5. Assumption (ii) of Theorem 3.20 is related to the strong positivity condition of the Krein–Rutman theorem. Here it is verified by a coupling result proved in Costantini and Kurtz [2] and an extension to piecewise smooth cones of the support theorem that Kwon and Williams [19] use to prove strong positivity in the smooth case (Lemmas 3.22 and 3.23). This extension is possible thanks to a result of Kang and Ramanan [13]. A more detailed discussion of how the assumptions of Theorem 3.20 are verified and a more detailed comparison with Kwon and Williams [19] are provided at the beginning of Sect. 3.

As in Kwon and Williams [19], the proof of the main result of this paper requires several steps. To better illustrate the arguments, first the application to bandwidth-sharing queues is discussed and a specific proof for this example is given: Sects. 2.1 and 2.2 present the model; Sect. 2.3 provides an outline of the proof; the various steps of the proof are then carried out each in one of Sects. 2.4 to 2.8. Next the general result is presented, discussed and proved in Sect. 3. The assumptions and the main result are stated precisely in Sect. 3.1: As in Kwon and Williams [19], it is assumed that the direction of reflection on each face is radially constant, but the result can be extended to variable directions of reflection (see Remark 3.7); $\mathcal{C}^2$ smoothness of each face is needed in order to be able to apply the above-mentioned result by Kang and Ramanan [13]. The proof is presented following the outline of the specific proof for the example: The differences between the proof of the general result and that for the example are highlighted at the end of Sect. 3.1. Sections 3.3 and 3.9 carry out one step of the proof each. Section 3.2 contains some preliminary material, and Appendix contains the proof of a more technical point.

1.1 Notation

For a set E, $\mathbf{1}_E$ is the indicator function. For a finite set E, $|E|$ denotes its cardinality.

$\mathbb{Z}_+$ denotes the set of nonnegative integers and $\mathbb{N}$ the set of strictly positive integers.

$\mathbb{R}^d_+$ denotes the nonnegative orthant in $\mathbb{R}^d$ and $\mathbb{R}^d_{++}$ denotes the strictly positive orthant.

For $e,\ g \in \mathbb{R}^d$, $e \cdot g$ denotes the scalar product. For $M \in \mathbb{R}^{d_1 \times d_2}$, M^T denotes its transpose $|M| := \sup_{x \neq 0} \frac{|Mx|}{|x|}$ its norm.

For $a,\ b \in \mathbb{R}$, $a \wedge b$ denotes the minimum between a and b, while $a \vee b$ denotes the maximum. A central dot $\cdot$ sometimes replaces the argument of a function. $f \circ g$

denotes the composition of two functions. For $i, j \in \mathbb{N}$, δ_{ij} denotes the Kronecker symbol, i.e., $\delta_{ii} = 1$, $\delta_{ij} = 0$ for $i \neq j$.

For $E \subseteq \mathbb{R}^d$, $\overset{\circ}{E}$ denotes the interior of E and $\overline{E}$ its closure. $d(x, C)$ denotes the distance of $x \in \mathbb{R}^d$ from a closed set C, i.e., $d(x, C) := \inf_{y \in C} |x - y|$. $\mathcal{C}^1(C)$ denotes the set of functions continuously differentiable on some open neighborhood of C and analogously $\mathcal{C}^2(C)$. $\mathcal{C}^1_b(C)$ denotes the set of functions in $\mathcal{C}^1(C)$ bounded, with bounded derivatives, and analogously $\mathcal{C}^2_b(C)$. S^{d-1} denotes the unit sphere in $\mathbb{R}^d$. $\mathcal{C}^2(\mathbb{R}^d, \mathbb{R}^d)$ and $\mathcal{C}^2(S^{d-1}, \mathbb{R}^d)$ denote the set of twice continuously differentiable vector fields defined on $\mathbb{R}^d$ and S^{d-1}, respectively. $B_\delta(x)$ denotes the ball of radius δ centered at x. B_δ denotes the ball centered at the origin.

For $E \subseteq \mathbb{R}^d$, $\mathcal{C}_{\overline{E}}[0, \infty)$ denotes the set of continuous functions from $[0, \infty)$ to $\overline{E}$ and $\mathcal{D}_{\overline{E}}[0, \infty)$ denotes the set of functions from $[0, \infty)$ to $\overline{E}$ that are right continuous with left-hand limits. For bounded, continuous functions f, $\|f\|$ denotes the supremum norm.

For $E \subseteq \mathbb{R}^d$, $\mathcal{P}(\overline{E})$ denotes the set of probability measures on $\overline{E}$. For a random variable Z, $\mathcal{L}(Z)$ denotes the law of Z. For a stochastic process Z, $\{\mathcal{F}^Z_t\}$ denotes the filtration generated by Z, i.e., $\mathcal{F}^Z_t := \sigma\{Z(s),\ s \leq t\}$. For a stochastic process Z and a finite random time τ, defined on the same probability space, $Z(\tau + \cdot)$ is the time shift of Z by τ, that is, $Z(\tau + \cdot)$ is defined pathwise as

$$Z(\tau + \cdot)(t) := Z(\tau + t), \quad t \geq 0.$$

2 An example from diffusion approximation of bandwidth-sharing queues

2.1 Diffusion approximation of bandwidth-sharing queues

Kelly and Williams [15] and Kang et al. [12] studied a model of Internet congestion control in which the flows present in the network share the bandwidth according to a weighted α-fair policy.

In their model, there is a set $\mathbb{J} = \{1, \ldots, d\}$ of resources and each flow corresponds to the continuous, simultaneous transmission of a document through a subset of resources that is called a route. The set of routes is $\mathbb{I} = \{1, \ldots, m\}$, and for each $i \in \mathbb{I}$ route, i is described by the ith column of an incidence matrix A, where $A_{ji} := 1$ if route i uses resource j and $A_{ji} := 0$ otherwise; it is assumed that for each $j \in \mathbb{J}$ there is at least one $i \in \mathbb{I}$ such that $A_{ji} = 1$, $A_{li} = 0$ for $l \neq j$ (Assumption 5.1 in Kang et al. [12]). In particular, A has rank d. There is no loss of generality in assuming, for each $j = 1, \ldots, d$, $A_{ji} = \delta_{ji}$ for $i = 1, \ldots, d$ (while there is no assumption on $A_{j,i}$ for $i = d+1, \ldots, m$, when $m > d$).

At each route $i \in \mathbb{I}$, the arrival times of documents are the jump times of a Poisson process of parameter ν^r_i, and each document has an exponentially distributed size of parameter μ^r_i; document sizes are independent of one another and independent of all arrival times; the initial number of documents is independent of the remaining sizes of the documents, which are independent exponentials of parameter μ^r_i. All the variables and processes for different routes are mutually independent. r is a positive, integer scaling parameter going to infinity.

Each resource $j \in \mathbb{J}$ has a capacity C_j. If $N_i^r(t)$ denotes the number of documents on route i at time t and $N^r(t)$ denotes the vector of components $N_i^r(t)$, $i \in \mathbb{I}$, for each route i and each resource j used by route i, the proportion of the capacity C_j allocated to route i at time t is $P_i(N^r(t))$, where the function P, which represents the allocation policy, is defined in the following way. Let k_i, $i \in \mathbb{I}$, be strictly positive parameters. For each $n \in \mathbb{R}_+^m - \{0\}$, let $\mathbb{I}_+(n) := \{l : n_l > 0\}$ and $P^+(n)$ be the unique maximizer, over all $p \in \mathbb{R}_+^{\mathbb{I}_+(n)}$ such that $\sum_{l \in \mathbb{I}_+(n)} A_{jl} p_l \leq C_j$ for all $j \in \mathbb{J}$, of the reward function

$$F_n(p) = \begin{cases} \sum_{l \in \mathbb{I}_+(n)} k_l n_l^\alpha \frac{p_l^{1-\alpha}}{1-\alpha}, & \text{if} \alpha \neq 1, \\ \sum_{l \in \mathbb{I}_+(n)} k_l n_l \log(p_l), & \text{if} \alpha = 1, \end{cases}$$

where the value of the right member is taken to be $-\infty$ if $\alpha \in [1, \infty)$ and $p_l = 0$ for some $l \in \mathbb{I}_+(n)$. Then, for each $n \in \mathbb{R}_+^m - \{0\}$, P is defined as: $P_i(n) := P_i^+(n)$, for $i \in \mathbb{I}_+(n)$, $P_i(n) := 0$ for $i \in \mathbb{I} - \mathbb{I}_+(n)$; $P(0) := 0$. An explicit parametric form of the function P is given in Proposition 2.1 of Kang et al. [12]. (Note that P is denoted as Λ in Kang et al. [12].)

The network defined by the allocation policy P is studied in heavy traffic conditions, that is, denoting by $\rho_i^r := \nu_i^r/\mu_i^r$ and by ν^r, μ^r and ρ^r the vectors of components ν_i^r, μ_i^r and ρ_i^r, $i \in \mathbb{I}$, and by C the vector of components C_j, $j \in \mathbb{J}$, it is assumed that

$$\nu^r \to \nu \in \mathbb{R}_{++}^m, \quad \mu^r \to \mu \in \mathbb{R}_{++}^m, \quad r\big(A\rho^r - C\big) \to b \in \mathbb{R}^m, \qquad \text{as} r \to \infty. \tag{2.1}$$

Let $M^r := \mathrm{diag}(\mu^r)$, $M := \mathrm{diag}(\mu)$, $\rho_i := \nu_i/\mu_i$, $i \in \mathbb{I}$, and

$$\sigma := 2AM^{-1}\mathrm{diag}(\nu)M^{-1}A^T. \tag{2.2}$$

σ is always nonsingular because A has rank d. (2.1) implies

$$A\rho = C. \tag{2.3}$$

Building on the fluid model studied in Kelly and Williams [15], Kang et al. [12] conjecture the following.

Conjecture 2.1 *(Conjecture 5.1 of* Kang et al. [12]*) Consider the rescaled workload process X^r,*

$$X^r(t) := r^{-1}A(M^r)^{-1}N^r(r^2 t), \qquad t \geq 0,$$

(denoted as $\widehat{W}^r$ in Kang et al. [12]*). Under suitable assumptions on $N^r(0)$ and $X^r(0)$, X^r converges in distribution, as $r \to \infty$, to a process X that is a semimartingale obliquely reflecting Brownian motion (ORBM), namely a solution of the following*

stochastic differential equation with reflection:

$$
\begin{aligned}
& X(t) = X(0) + b\,t + \sigma\, W(t) + \int_0^t \gamma(s)\,\mathrm{d}\lambda(s), \quad t \geq 0, \\
& X(t) \in \overline{\mathcal{W}}, \quad \lambda(t) = \int_0^t \mathbf{1}_{\partial\mathcal{W}}(X(s))\,\mathrm{d}\lambda(s), \quad t \geq 0, \\
& \gamma(t) \in G(X(t)), \quad |\gamma(t)| = 1, \quad \mathrm{d}\lambda - a.e., \quad t \geq 0,
\end{aligned} \tag{2.4}
$$

where $\mathcal{W}$ is the cone

$$
\mathcal{W} := \left\{ x \in \mathbb{R}^d : x_j = \sum_{i \in \mathbb{I}} A_{ji} \frac{\rho_i}{\mu_i(k_i)^{1/\alpha}} \big((q^T A)_i\big)^{1/\alpha},\ q \in \mathbb{R}^d_{++} \right\}, \tag{2.5}
$$

$$
\partial_h \mathcal{W} := \{x \in \overline{\mathcal{W}} : q_h = 0,\ q_j > 0,\ j \neq h\}, \quad h \in \mathbb{J}, \tag{2.6}
$$

are the faces of $\mathcal{W}$, the direction of reflection on $\partial_h \mathcal{W}$ is g^h:

$$
g_j^h := \delta_{jh},\ j \in \mathbb{J}, \tag{2.7}
$$

$G(x)$ is the cone of directions of reflection at x, that is:

$$
G(x) := \left\{ g : g = \sum_{h:\, x \in \overline{\partial_h \mathcal{W}}} u_h g^h,\ u_h \geq 0 \right\}, \quad \text{for} x \in \partial\mathcal{W} - 0 \tag{2.8}
$$

and

$$
G(0) := \left\{ g : g = \sum_{h \in \mathbb{J}} u_h g^h,\ u_h \geq 0 \right\} = \mathbb{R}^d_+. \tag{2.9}
$$

Note that $\overline{\mathcal{W}} - \{0\} \subseteq \mathbb{R}^d_{++}$.

The allocation function P does not enter in the dynamics of the limiting process X; instead, it enters, in its parametric form, in the definition of the cone $\mathcal{W}$: see Theorem 4.1 and (41) in Kang et al. [12]. For $d = 2$, the cone $\mathcal{W}$ is a wedge in $\mathbb{R}^2$. For $d > 2$, the parameter α plays a crucial role in determining the shape of the cone $\mathcal{W}$: For $\alpha = 1$, $\mathcal{W}$ is a convex polyhedral cone; for $\alpha \neq 1$, $\mathcal{W}$ is a piecewise smooth cone with curved faces. From the example of Sect. 2.2, we can get an intuitive idea of the role of α: For $\alpha > 1$, the faces of $\mathcal{W}$ curve outward; for $\alpha < 1$, the faces of $\mathcal{W}$ curve inward and any two faces meet in a cusp (Section 5.6 of Kang et al. [12]). The parameter α also determines the degree of smoothness of each face of the cone. In particular, the faces are $\mathcal{C}^2$ smooth if and only if $\alpha \geq 2$.

In the $\alpha = 1$ case, Kang et al. [12] prove Conjecture 2.1 by using the results of Kang and Williams [11] (Theorem 5.4, which is an application of the invariance principle of Theorem 4.3) and the results of Dai and Williams [6] (Theorem 1.6). Roughly speaking, Theorem 4.3 of Kang and Williams [11] ensures that the sequence of processes $\{X^r\}$ is relatively compact and that every limit point (in distribution) is a solution of the stochastic differential equation with reflection (2.4). (Therefore, as a by-product, it ensures also existence of a solution). In order to have convergence, one then needs to use Theorem 1.6 of Dai and Williams [6] which proves uniqueness in distribution of the solution to (2.4) when $\mathcal{W}$ is a polyhedral cone.

In the $\alpha \neq 1$ case, the authors of Kang et al. [12] say that the above argument cannot be carried out, in particular due to the lack of a uniqueness result analogous to that of Dai and Williams [6] for a nonpolyhedral cone. In fact, as discussed in Introduction and at the beginning of Sect. 3, none of the results currently available in the literature applies to piecewise smooth, nonpolyhedral cones and there is no immediate extension to this case of the results for piecewise smooth domains or smooth cones. (Note that a piecewise smooth domain is defined as the intersection of a finite number of smooth domains, so that, in dimension higher than two, a piecewise smooth cone is not a piecewise smooth domain.)

Section 3 of this paper proves, under some conditions, that the solution of (2.4) in a piecewise $\mathcal{C}^2$, nonpolyhedral cone, is unique in distribution, thus providing an essential step toward proving the diffusion approximation in the $\alpha \geq 2$ case.

The example considered in Sect. 2.2 satisfies all conditions but one for arbitrary values of the parameters and the remaining condition for suitable values of the parameters. An important feature (which holds more generally for $\alpha > 1$) is that, for any values of the parameters, any solution of (2.4) starting from the origin leaves it immediately with probability one. Note, however, that this is not sufficient to prove uniqueness. The arguments used in Sect. 3, are illustrated below in the specific case of the example of Sect. 2.2. After describing the example, Sect. 2.2 states the uniqueness result and the properties of the solution precisely; Sect. 2.3 provides an outline of the proof of the uniqueness result; the main points of the proof are then dealt with each in a separate section.

2.2 The example

Consider the example where there are d resources, each resource has a route that only goes through that resource and there is one additional route that goes through all resources, that is: $\mathbb{J} := \{1, \dots, d\}$, $\mathbb{I} := \{1, \dots, d+1\}$ and

$$A_{ji} = \delta_{ji} \text{ for } j, i = 1, \dots, d, \qquad A_{j(d+1)} = 1, \text{ for } j = 1, \dots, d. \tag{2.10}$$

We will take

$$\nu_i = k_i = 1, \text{ for } i =, \dots, d+1, \qquad \mu_i = 1, \text{ for } i = 1, \dots, d, \quad \mu_{d+1} = \mu, \tag{2.11}$$

but the values of these parameters are relevant only in Sect. 2.5. In addition, for simplicity we take $d = 3$, but everything works in the higher-dimensional case. Then the cone $\mathcal{W}$, defined by (2.5), and the faces $\partial_h \mathcal{W}$, defined by (2.6), take the form

$$\mathcal{W} = \left\{ x \in \mathbb{R}^3_{++} : x_j = q_j^{1/\alpha} + \frac{1}{\mu^2} \left(q_1 + q_2 + q_3 \right)^{1/\alpha}, \ q \in \mathbb{R}^3_{++} \right\}, \tag{2.12}$$

$$\partial_h \mathcal{W} = \left\{ x \in \mathbb{R}^{3+}_{++} : x_h = \frac{1}{\mu^2} \Big(\sum_{l \neq h} q_l \Big)^{1/\alpha},\ x_j = q_j^{1/\alpha} + x_h,\ q_j \in \mathbb{R}_{++},\ j \neq h \right\}. \tag{2.13}$$

Straightforward computations show that each face $\partial_h \mathcal{W}$ is $\mathcal{C}^1$ smooth at each point if and only if $\alpha > 1$, and $\mathcal{C}^2$ smooth if and only if $\alpha \geq 2$. We assume $\alpha \geq 2$, but this degree of smoothness is needed only in Sect. 2.8 to be able to apply Theorem 3.20: in fact, in order to verify the assumptions of Theorem 3.20, one needs to show that, starting at a point different from the origin, any solution of (2.4) spends zero time on $\partial \mathcal{W} - \{0\}$ before hitting the origin, almost surely. This does not follow from the fact that $\partial \mathcal{W} - \{0\}$ has surface measure zero, but is ensured by the $\mathcal{C}^2$-smoothness of $\mathcal{W}_h$ and conditions on the directions of reflection that are satisfied in this example (see Proposition 2.12 of Kang and Ramanan [13] and Sect. 2.4). The other intermediate results, in particular the fact that any solution of (2.4) starting from the origin leaves it immediately with probability one, hold for $\alpha > 1$.

From (2.13), one can compute the unit inward normal vector $n^h(x)$ at $x \in \partial_h \mathcal{W}$:

$$\begin{aligned} n_j^h(x) &= -\, c_n^h(q) \frac{1}{\mu^2} \left(\sum_{l \neq h} q_l \right)^{(1-\alpha)/\alpha} q_j^{(\alpha-1)/\alpha}, \qquad j \neq h, \\ n_h^h(x) &= c_n^h(q) \left[1 + \frac{1}{\mu^2} \left(\sum_{l \neq h} q_l \right)^{(1-\alpha)/\alpha} \sum_{l \neq h} q_l^{(\alpha-1)/\alpha} \right], \end{aligned} \tag{2.14}$$

where c_n^h is the normalization constant,

$$c_n^h(q) := \left\{ \frac{1}{\mu^4} \left(\sum_{l \neq h} q_l \right)^{\frac{2(1-\alpha)}{\alpha}} \sum_{j \neq h} q_j^{\frac{2(\alpha-1)}{\alpha}} + \left[1 + \frac{1}{\mu^2} \left(\sum_{l \neq h} q_l \right)^{\frac{1-\alpha}{\alpha}} \sum_{l \neq h} q_l^{\frac{\alpha-1}{\alpha}} \right]^2 \right\}^{-\frac{1}{2}}$$

Since $\sum_{l \neq h} q_l > 0$ for all $x \in \overline{\partial_h \mathcal{W}} - \{0\}$, $0 < c_n^h(q) \leq 1$ for all $x \in \overline{\partial_h \mathcal{W}} - \{0\}$ and (2.14) extends to all $x \in \overline{\partial_h \mathcal{W}} - \{0\}$.

The matrix σ, defined by (2.15), in this case takes the form

$$\sigma = 2 \begin{bmatrix} \left(1 + \frac{1}{\mu^2}\right) & \frac{1}{\mu^2} & \frac{1}{\mu^2} \\ \frac{1}{\mu^2} & \left(1 + \frac{1}{\mu^2}\right) & \frac{1}{\mu^2} \\ \frac{1}{\mu^2} & \frac{1}{\mu^2} & \left(1 + \frac{1}{\mu^2}\right) \end{bmatrix}. \tag{2.15}$$

For this example, Theorem 2.3 determines a set of values of the parameter μ for which the solution to (2.4) is unique. Here a solution is meant in the weak sense and uniqueness is meant in distribution. More precisely:

Definition 2.2 A continuous process X is a solution of (2.4) if there exist a standard Brownian motion W, a continuous, nondecreasing process λ and a process γ with measurable paths, all defined on the same probability space as X, such that $W(t+\cdot)-W(t)$ is independent of $\mathcal{F}_t^{X,W,\lambda,\gamma}$ for all $t \geq 0$ and (2.4) is satisfied almost surely. If $X(0)=x$ almost surely for some $x \in \overline{\mathcal{W}}$, we say that X starts at x.

Given an initial distribution on $\overline{\mathcal{W}}$, *uniqueness in distribution* holds if any two solutions of (2.4) with the same initial distribution have the same distribution on $C_{\overline{\mathcal{W}}}[0,\infty)$.

For a solution X of (2.4), let

$$\vartheta^X := \inf\{t \geq 0 : X(t)=0\}, \tag{2.16}$$

$$\tau^{X,\delta} := \inf\{t \geq 0 : |X(t)|=\delta\}, \quad \delta > 0. \tag{2.17}$$

(When there is no risk of confusion, the superscript X will be omitted.)

Theorem 2.3 *If* $\mu > \sqrt{\frac{3}{\sqrt{2}-1}}$, *uniqueness in distribution holds for* (2.4).

If X *starts at* $x \in \overline{\mathcal{W}} - \{0\}$,

$$\mathbb{P}(\vartheta^X < \infty) = 0. \tag{2.18}$$

If X *starts at* 0,

$$\lim_{\delta \to 0} \tau^{X,\delta} = 0, \qquad a.s. \tag{2.19}$$

Remark 2.4 It may seem counterintuitive that the solution does not hit the origin for large enough values of μ, as large values of μ correspond to a small expected length of the document that uses all resources, but one has to take into account that, in the heavy traffic regime, (2.3) must be satisfied, so that, in some sense, to a smaller expected length of the document that uses all resources correspond smaller capacities.

Remark 2.5 The issue of the existence of solutions to (2.4) is not addressed in this paper. However, it is worth mentioning that for the example of this section existence of solutions can be proved by the same arguments as in Theorems 3.1 and 4.1 of Costantini and Kurtz [2].

2.3 Outline of the proof of Theorem 2.3

The proof of Theorem 2.3 requires several steps. This section provides an outline of these steps. For each step, the key points are proved in one of the following sections. Recall that B_δ denotes the ball of radius δ centered at the origin.

Step 1 By the localization results of Costantini and Kurtz [5], it is enough to prove uniqueness for (2.4) in each of the bounded domains $\{\mathcal{W}_n\}_{n\geq 0}$, where $\mathcal{W}_0 := \mathcal{W} \cap B_4$, $\mathcal{W}_n := \mathcal{W} \cap \left(\overline{B_{1+2(n-1)}}\right)^c \cap B_{6+2(n-1)}$, $n \geq 1$, taking the unit inward normal as the direction of reflection on the parts of the boundary of $\mathcal{W}_n$ that are not subsets of the boundary of $\mathcal{W}$.

Each domain of the form $\mathcal{W} \cap \left(\overline{B_\delta}\right)^c \cap B_{\delta'}$, $0 < \delta < \delta'$, is piecewise $\mathcal{C}^2$; therefore, taking the unit inward normal as the direction of reflection on $\mathcal{W} \cap \partial B_\delta$ and on $\mathcal{W} \cap \partial B_{\delta'}$, uniqueness is ensured by Dupuis and Ishii [9] (Sect. 2.4). In particular, uniqueness holds for each $\mathcal{W}_n$, $n \geq 1$; therefore, we only need to prove uniqueness in $\mathcal{W}_0$.

In addition, the results of Costantini and Kurtz [3] ensure that there exist strong Markov solutions of (2.4) in $\mathcal{W}_0$ and that uniqueness among strong Markov solutions is equivalent to uniqueness among all solutions. Therefore, in the sequel only strong Markov solutions are considered.

Step 2 Consider a solution X of (2.4) in $\mathcal{W}_0$ starting off the origin, that is, with initial distribution ν with compact support in $\overline{\mathcal{W}_0} - \{0\}$. For all δ small enough that B_δ has empty intersection with the support of ν, the localization results of Costantini and Kurtz [5] imply that X must agree with some solution of (2.4) in $\mathcal{W}_0 \cap \left(\overline{B_{\delta/2}}\right)^c$ (with direction of reflection on $\mathcal{W}_0 \cap \partial B_{\delta/2}$ the inward normal) up to the first time it hits ∂B_δ. Since Dupuis and Ishii [9] guarantee that uniqueness holds for solutions of (2.4) in $\mathcal{W}_0 \cap \left(\overline{B_{\delta/2}}\right)^c$, X is uniquely determined up to the first time it hits ∂B_δ for all δ sufficiently small; hence, it is uniquely determined up to the first time it hits the origin.

Moreover, every solution of (2.4) in $\mathcal{W}_0$ starting off the origin with probability one never reaches the origin (Sect. 2.5); hence, the solution is uniquely determined for all times. In particular, for every $x \in \overline{\mathcal{W}_0} - \{0\}$, any two solutions starting at x have the same distribution.

Note that the fact that, starting off the origin, the origin is never hit holds independently of the drift b: this is because very close to the origin the contribution of the drift is negligible with respect to that of the diffusion term and of the reflection (see (2.25)).

Step 3 Let X be an arbitrary solution of (2.4) in $\mathcal{W}_0$ starting at the origin. X immediately leaves the origin, that is (2.19) holds. In fact, the stronger statement

$$\lim_{\delta \to 0} \mathbb{E}[\tau^{X,\delta}] = 0.$$

holds (Sect. 2.6).

Step 4 A key consequence of Step 3 is the following: if any two solutions of (2.4) in $\mathcal{W}_0$ starting at the origin, X and $\widetilde{X}$, have the same hitting distributions on ∂B_δ, that is, $\mathcal{L}(X(\tau^{X,\delta})) = \mathcal{L}(\widetilde{X}(\tau^{\widetilde{X},\delta}))$, for all $\delta > 0$ sufficiently small, then any two solutions of (2.4) starting at the origin have the same one-dimensional distributions. Combined with Step 2, this yields that any two solutions with the same initial distribution have the same one-dimensional distributions, which gives uniqueness of their distribution on the path space (Sect. 2.7).

Step 5 By the previous step, we are reduced to proving that, for all $\delta > 0$ sufficiently small, for any solution X of (2.4) in $\mathcal{W}_0$ starting at the origin, the hitting distribution on ∂B_δ is always the same. Let $\tau^{X,2^{-2l}\delta}$ be the first time X hits $\partial B_{2^{-2l}\delta}$, $l \in \mathbb{Z}_+$. In the sequel of this step, in order to make formulas more readable, the following shorthand notation:

$$\tau^l := \tau^{X,2^{-2l}\delta}$$

is used. Note that, since X starts at the origin, $\tau^l < \tau^{l-1}$. $X(\tau^l + \cdot)$ is a solution of (2.4) with initial distribution supported in $\partial B_{2^{-2l}\delta}$, and hence, by Step 2, its distribution is the same for any X. In particular, since

$$X(\tau^{l-1}) = X(\tau^l + \tau^{X(\tau^l+\cdot),2^{-2(l-1)}\delta}),$$

we see that $\mathcal{L}\big(X(\tau^{l-1})\big|X(\tau^l) = x\big)$, $x \in \partial B_{2^{-2l}\delta}$, is the same for any X.

Since X is a strong Markov process, for $n \in \mathbb{N}$ $\{X_k\}_{0\le k\le n} := \{X(\tau^{n-k})\}_{0\le k\le n}$ is a (inhomogeneous) Markov chain and $X_n = X(\tau^{X,\delta})$. Moreover, since the origin is never hit by a solution of (2.4) between τ^{n-k} and τ^{n-k-1}, by Step 2 two Markov chains $\{X_k\}_{0\le k\le n}$ and $\{\widetilde{X}_k\}_{0\le k\le n}$ corresponding to two solutions X and $\widetilde{X}$ have the same transition kernels and differ only by their initial distributions. Then, by the reverse ergodic theorem of Costantini and Kurtz [4] (recalled as Theorem 3.20), $\mathcal{L}(X_n)$ and $\mathcal{L}(\widetilde{X}_n)$ must converge to the same limit as $n \to \infty$, that is, $\mathcal{L}(X(\tau^{X,\delta})) = \mathcal{L}(\widetilde{X}(\tau^{\widetilde{X},\delta}))$ (Sect. 2.8).

Remark 2.6 As mentioned above, in this example the probability transition kernels

$$Q_{n-k}(x, B) := \mathbb{P}\big(X(\tau^{n-k-1}) \in B\big|X(\tau^{n-k}) = x\big),$$

are the same for all solutions X, because the origin is never hit by a solution of (2.4) between τ^{n-k} and τ^{n-k-1}.

If the solutions of (2.4) could hit the origin between τ^{n-k} and τ^{n-k-1} with positive probability, in order to ensure a priori that the transition kernels are the same for all solutions, one would have to consider the killed Markov chains with subprobability transition kernels

$$Q_{n-k}(x, B) := \mathbb{P}\big(\tau^{n-k-1} < \vartheta^{n-k},\ X(\tau^{n-k-1}) \in B\big|X(\tau^{n-k}) = x\big),$$

where ϑ^{n-k} is the first time X hits the origin after τ^{n-k}.

However, the reverse ergodic theorem of Costantini and Kurtz [4] applies to killed Markov chains as well, under suitable assumptions (see Sect. 3.7).

For the proof of (2.18) and (2.19), see Sects. 2.5 and 2.6.

2.4 Uniqueness in a domain $\mathcal{W} \cap (\overline{B_\delta})^c \cap B_{\delta'}$

Let us verify that in each domain of the form $\mathcal{W}_{\delta,\delta'} := \mathcal{W}\cap\overline{B}_\delta^c\cap B_{\delta'}$, $0 < \delta < \delta'$, taking the unit inward normal as the direction of reflection on $\partial B_\delta \cap \mathcal{W}$ and on $\partial B_{\delta'} \cap \mathcal{W}$, the assumptions of Dupuis and Ishii [9] are satisfied. These assumptions are satisfied in particular if, at each point $x \in \partial\mathcal{W}_{\delta,\delta'}$, the set of normal vectors is linearly independent and a certain matrix (the matrix defined by (2.22)) has spectral radius strictly less than 1. Denote

$\partial_0\mathcal{W}_{\delta,\delta'} := \partial B_\delta \cap \mathcal{W},\ \partial_4\mathcal{W}_{\delta,\delta'} := \partial B_{\delta'} \cap \mathcal{W},$

$$\partial_h \mathcal{W}_{\delta,\delta'} := \partial_h \mathcal{W} \cap \left(\overline{B_\delta}\right)^c \cap B_{\delta'} \; h \in \{1,2,3\},$$
$$g^0(x) := n^0(x) := \frac{x}{|x|}, \quad g^4(x) := n^4(x); = -\frac{x}{|x|}, \quad g^h(x) := g^h, \quad h \in \{1,2,3\},$$
$$\mathcal{J}(x) := \left\{h \in \{0,1,2,3,4\} : x \in \overline{\partial_h \mathcal{W}_{\delta,\delta'}}\right\}.$$

Note that, by (2.14),

$$g^h \cdot n^h(x) = n^h_h(x) > 0, \quad \text{for} x \in \overline{\partial_h \mathcal{W}_{\delta,\delta'}}, \quad h \in \{1,2,3\},$$

and obviously the same holds for $h \in \{0,4\}$. Let us check that, at each point on the boundary, all normal vectors are linearly independent. Clearly,

$$\begin{aligned} n^0(x)\cdot n^h(x) = 0 \qquad \forall x \in \overline{\partial_h \mathcal{W}_{\delta,\delta'}} \cap \overline{\partial_0 \mathcal{W}_{\delta,\delta'}}, \quad h \in \{1,2,3\}, \\ n^4(x)\cdot n^h(x) = 0 \qquad \forall x \in \overline{\partial_h \mathcal{W}_{\delta,\delta'}} \cap \overline{\partial_4 \mathcal{W}_{\delta,\delta'}}, \quad h \in \{1,2,3\}. \end{aligned} \tag{2.20}$$

Let $x \in \overline{\partial_h \mathcal{W}_{\delta,\delta'}} \cap \overline{\partial_k \mathcal{W}_{\delta,\delta'}} \cap \mathcal{W}_{\delta,\delta'}$, $h,k \in \{1,2,3\}$, $h \neq k$. Then, by (2.14),

$$n^h_h(x) > 0, \quad n^h_k(x) = 0, \tag{2.21}$$

so that, of course, $n^h(x)$ and $n^k(x)$ are linearly independent. Finally at a point $x \in \overline{\partial_h \mathcal{W}_{\delta,\delta'}} \cap \overline{\partial_k \mathcal{W}_{\delta,\delta'}} \cap \overline{\partial_0 \mathcal{W}_{\delta,\delta'}}$, $h,k \in \{1,2,3\}$, $h \neq k$, $n^0(x)$ is orthogonal to both $n^h(x)$ and $n^k(x)$, which are linearly independent among themselves, so the three vectors are linearly independent, and similarly at $x \in \overline{\partial_h \mathcal{W}_{\delta,\delta'}} \cap \overline{\partial_k \mathcal{W}_{\delta,\delta'}} \cap \overline{\partial_4 \mathcal{W}_{\delta,\delta'}}$.

The main assumption of Dupuis and Ishii [9] is (3.7) of Dupuis and Ishii [9]. A sufficient condition is the condition stated in Remark 3.1 of Dupuis and Ishii [9], namely that, at every point $x \in \partial \mathcal{W}_{\delta,\delta'}$, $\mathcal{J}(x) = \{h_1, \cdots h_k\}$, the spectral radius of the matrix of elements

$$\frac{|g^{h_i}(x)\cdot n^{h_j}(x)|}{g^{h_i}(x)\cdot n^{h_i}(x)} - \delta_{h_i h_j}, \quad i,j = 1,\dots,k, \tag{2.22}$$

is strictly less than 1. But, for each $h \in \mathcal{J}(x)$,

$$\begin{aligned} &g^0(x)\cdot n^h(x) = n^0(x)\cdot n^h(x), \quad g^4(x)\cdot n^h(x) = n^4(x)\cdot n^h(x), \\ &g^k(x)\cdot n^h(x) = n^h_k(x), \end{aligned}$$

and we see, by (2.20) and (2.21), that 0 is the only eigenvalue of the above matrix, so the condition is trivially satisfied.

2.5 Starting off the origin, the origin is never reached

Let $\mathbb{A}$ be the operator

$$\mathbb{A} f(x) := b\cdot \nabla f(x) + \frac{1}{2}\mathrm{tr}\left(\left(\sigma\sigma^T\right) D^2 f(x)\right), \tag{2.23}$$

and consider the function

$$V(x) := |x|^{\beta}, \quad x \in \mathbb{R}^3 - \{0\}, \qquad \text{with} \;\; 1 - \frac{2}{\left(1 + \frac{3}{\mu^2}\right)^2} < \beta < 0. \tag{2.24}$$

Recalling (2.8) and (2.9), we see that V satisfies

$$\nabla V(x) \cdot g \le 0, \quad \forall g \in G(x), \quad \forall x \in \partial \mathcal{W}_0 - \{0\}, \;\; |x| < 4.$$

In addition, straightforward computations give

$$\mathbb{A}V(x) = \beta |x|^{\beta-2} \left\{ b \cdot x + \frac{1}{2} \mathrm{tr}\left(\sigma\sigma^T\right) + \frac{1}{2}\frac{\beta - 2}{|x|^2}|\sigma^T x|^2 \right\}, \tag{2.25}$$

and

$$\mathrm{tr}\left(\sigma\sigma^T\right) = 4\left(2 + \left(1 + \frac{3}{\mu^2}\right)^2\right), \quad \left|\sigma^T x\right|^2 \le 4\left(1 + \frac{3}{\mu^2}\right)^2 |x|^2,$$

so that, setting

$$c_V := 4 + 2(\beta - 1)\left(1 + \frac{3}{\mu^2}\right)^2,$$
$$\mathbb{A}V(x) \;\le\; \beta|x|^{\beta-2}\Big\{ b \cdot x + c_V \Big\} \le 0, \quad \forall x \in \overline{\mathcal{W}_0} - \{0\}, \;\; |x| \le \frac{c_V}{|b| + 1}.$$

Let δ be any positive number such that $0 < \delta \le \frac{c_V}{|b|+1}$, $0 < \widetilde{\delta} < \delta$ and $0 < \epsilon < \widetilde{\delta}$. Also let $\tau^{X,\delta}$ and $\tau^{X,\epsilon}$ be defined as in (2.19). In the sequel, the superscript X is omitted. By applying Ito's formula to the function V (actually to a function in $\mathcal{C}_b^2(\mathbb{R}^3)$ that agrees with V for $|x| \ge \epsilon$), we see that, for any solution X of (2.4) in $\mathcal{W}_0$, with $\mathbb{P}\big(|X(0)| = \widetilde{\delta}\big) = 1$,

$$\begin{aligned}
&\mathbb{E}\left[V\left(X\left(\tau^{\epsilon} \wedge \tau^{\delta} \wedge t\right)\right)\right] \\
&= \left(\widetilde{\delta}\right)^{\beta} + \mathbb{E}\left[\int_0^{\tau^{\epsilon} \wedge \tau^{\delta} \wedge t} \mathbb{A}V(X(s))\,\mathrm{d}s\right] + \mathbb{E}\left[\int_0^{\tau^{\epsilon} \wedge \tau^{\delta} \wedge t} \nabla V(X(s)) \cdot \gamma(s)\,\mathrm{d}\lambda(s)\right] \\
&\le \left(\widetilde{\delta}\right)^{\beta}.
\end{aligned}$$

By sending $t \to \infty$, we get

$$\mathbb{E}\left[V\left(X\left(\tau^{\epsilon} \wedge \tau^{\delta}\right)\right)\right] \le \left(\widetilde{\delta}\right)^{\beta},$$

which yields

$$\mathbb{P}(\tau^\epsilon < \tau^\delta) = \epsilon^{-\beta}\big(\mathbb{E}[V(X(\tau^\epsilon \wedge \tau^\delta))] - \mathbb{E}\big[V(X(\tau^\epsilon \wedge \tau^\delta))\mathbf{1}_{\tau^\delta < \tau^\epsilon}\big]\big) \le \big(\widetilde{\delta}\big)^\beta \epsilon^{-\beta},$$

and hence, sending $\epsilon \to 0$,

$$\mathbb{P}(\vartheta^X < \tau^{X,\delta}) = 0. \tag{2.26}$$

Now let $\widetilde{\delta} = \delta/2$ and

$$\begin{aligned}
&\tau_0^{\delta/2} := 0, \quad \vartheta_0 := \vartheta, \quad \tau_1^\delta := \tau^\delta, \quad \tau_1^{\delta/2} := \inf\left\{t \ge \tau_1^\delta : |X(t)| = \frac{\delta}{2}\right\},\\
&\vartheta_1 := \inf\left\{t \ge \tau_1^{\delta/2} : X(t) = 0\right\},\\
&\tau_2^\delta := \inf\left\{t \ge \tau_1^{\delta/2} : |X(t)| = \delta\right\}, \quad \tau_2^{\delta/2} := \inf\left\{t \ge \tau_2^\delta : |X(t)| = \frac{\delta}{2}\right\}, \quad \text{etc.}
\end{aligned}$$

Then

$$\{\vartheta < \infty\} = \bigcup_{l=0}^{\infty}\{\tau_l^{\delta/2} < \vartheta_l < \tau_{l+1}^\delta\}.$$

However, for $l \ge 0$, $X^l := X(\tau_l^{\delta/2} + \cdot)$ is a solution of (2.4) in $\mathcal{W}_0$, with $\mathbb{P}\big(|X^l(0)| = \frac{\delta}{2}\big) = 1$, and $\vartheta_l = \vartheta^{X^l}$, $\tau_{l+1}^\delta = \tau^{X^l,\delta}$; therefore, by (2.26),

$$\mathbb{P}\big(\tau_l^{\delta/2} < \vartheta_l < \tau_{l+1}^\delta\big) = \mathbb{P}\big(\vartheta^{X^l} < \tau^{X^l,\delta}\big) = 0, \qquad l \ge 0,$$

and we can conclude that

$$\mathbb{P}\big(\vartheta < \infty\big) = 0.$$

Now let X be any solution of (2.4) in $\mathcal{W}_0$ with initial distribution $\mathcal{L}(X(0))$ supported in $\overline{\mathcal{W}_0} - \{0\}$, and let δ be small enough that B_δ has empty intersection with the support of $\mathcal{L}(X(0))$. $X^{\delta/2} := X(\tau^{\delta/2} + \cdot)$ is a solution of (2.4) in $\mathcal{W}_0$, with $\mathbb{P}\big(|X^{\delta/2}(0)| = \frac{\delta}{2}\big) = 1$, and $\vartheta^X = \vartheta^{X^{\delta/2}}$; therefore,

$$\mathbb{P}\big(\vartheta^X < \infty\big) = \mathbb{P}\big(\vartheta^{X^{\delta/2}} < \infty\big) = 0.$$

Exactly the same argument shows (2.18).

2.6 Starting at the origin, the origin is immediately left

Let X be a solution of (2.4) in $\mathcal{W}_0$, that is X satisfies (2.4) with $\mathcal{W}$ replaced by $\mathcal{W}_0$,

$$\partial_4\mathcal{W}_0 := \partial B_4 \cap \mathcal{W}, \quad g^4(x) := n^4(x) = -\frac{x}{|x|}, \tag{2.27}$$

and $G(x)$ is given by (2.8), where now h ranges in $\{1, 2, 3, 4\}$, and (2.9).

Suppose $X(0) = 0$ almost surely. Here X is fixed, so we will omit the superscript X in $\tau^{X,\delta}$. Let

$$e := \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \quad f(x) = \frac{1}{2}(e \cdot x)^2.$$

By Ito's formula,

$$\mathbb{E}\left[f\left(X\left(t \wedge \tau^{\delta}\right)\right)\right] = \mathbb{E}\left[\int_0^{t\wedge\tau^{\delta}} \mathbb{A}f(X(s))\mathrm{d}s + \int_0^{t\wedge\tau^{\delta}} (e \cdot X(s))(e \cdot \gamma(s))\, \mathrm{d}\lambda(s)\right].$$

Now, for every $x \in \partial\mathcal{W}_0$, $|x| \leq \delta < 4$,

$$e \cdot x \geq 0 \,\text{and} e \cdot g \geq 0, \quad \forall g \in G(x).$$

In addition,

$$\underline{\lim}_{x\in\overline{\mathcal{W}_0}-\{0\},\, x\to 0} \mathbb{A}f(x) = \underline{\lim}_{x\in\overline{\mathcal{W}_0}-\{0\},\, x\to 0} \left((e \cdot x)\,(e \cdot b) + \frac{1}{2}e^T\left(\sigma\sigma^T\right)e\right) = \frac{1}{2}\left|\sigma^T e\right|^2.$$

Therefore, for δ sufficiently small,

$$\frac{1}{2}\delta^2 \geq \mathbb{E}\left[f\left(X\left(t \wedge \tau_{\delta}\right)\right)\right] \geq \frac{1}{4}\left|\sigma^T e\right|^2 \mathbb{E}\left[t \wedge \tau^{\delta}\right],$$

and, by taking the limit as $t \to \infty$,

$$\mathbb{E}\left[\tau^{\delta}\right] \leq \frac{2\delta^2}{|\sigma^T e|^2},$$

which of course implies also (2.19). Exactly the same argument applies to an arbitrary solution of (2.4) in $\mathcal{W}$ starting at the origin.

2.7 If any two solutions starting at the origin have the same hitting distributions on ∂B_δ, uniqueness holds for solutions starting at the origin

Let X be a strong Markov solution of (2.4) starting at the origin. Recall that

$$\tau^{X,\delta} := \inf\{t \geq 0 : |X(t)| = \delta\}.$$

Since there is no risk of confusion, the superscript X in $\tau^{X,\delta}$ will be dropped.

Let f be a continuous function on $\overline{\mathcal{W}_0}$ vanishing in a neighborhood of the origin and fix $\delta > 0$ sufficiently small that f vanishes on B_δ. For any $\eta > 0$, any solution X

of (2.4) and each $n \in \mathbb{N}$, we have

$$\mathbb{E}\left[\int_0^\infty e^{-\eta t} f(X(t))\mathrm{d}t\right]$$
$$= \mathbb{E}\left[\int_{\tau^{\delta/n}}^\infty e^{-\eta t} f(X(t))\mathrm{d}t\right] = \mathbb{E}\left[e^{-\eta\tau^{\delta/n}} \int_{\tau^{\delta/n}}^\infty e^{-\eta(t-\tau^{\delta/n})} f(X(t))\,\mathrm{d}t\right].$$

$\lim_{n\to\infty} e^{-\eta\tau^{\delta/n}} = 1$ almost surely by Step 3; therefore,

$$\mathbb{E}\left[\int_0^\infty e^{-\eta t} f(X(t))\mathrm{d}t\right]$$
$$= \lim_{n\to\infty} \mathbb{E}\left[\int_{\tau^{\delta/n}}^\infty e^{-\eta(t-\tau^{\delta/n})} f(X(t))\,\mathrm{d}t\right] = \lim_{n\to\infty} \mathbb{E}\left[\int_0^\infty e^{-\eta t} f(X(\tau^{\delta/n}+t))\,\mathrm{d}t\right]. \tag{2.28}$$

On the other hand, the time-shifted process $X(\tau^{\delta/n}+\cdot)$ is a solution of (2.4) with initial distribution $\mathcal{L}(X(\tau^{\delta/n}))$; hence, by Step 2, its distribution is uniquely determined by $\mathcal{L}\big(X(\tau^{\delta/n})\big)$. But, by assumption, $\mathcal{L}\big(X(\tau^{\delta/n})\big)$ is the same for all solutions X starting at 0. Therefore, for all n, $\mathbb{E}\left[\int_0^\infty e^{-\eta t} f(X(\tau^{\delta/n}+t))\,dt\right]$ is independent of the specific solution X, and hence, so is the first line of (2.28).

Then, by the uniqueness of the Laplace transform and the continuity of X, it follows that, for all $t \geq 0$, $\mathbb{E}[f(X(t))]$ is the same for all solutions X starting at the origin, for each function f vanishing in a neighborhood of the origin. Since $\mathbf{1}_{\overline{\mathcal{W}_0}-\{0\}}$ can be approximated pointwise by a uniformly bounded sequence of continuous functions vanishing in a neighborhood of the origin, we see that, for each continuous f,

$$\mathbb{E}[f(X(t))] = \mathbb{E}[f(X(t))\mathbf{1}_{\overline{\mathcal{W}_0}-\{0\}}(X(t))] + f(0)\big(1 - \mathbb{E}[\mathbf{1}_{\overline{\mathcal{W}_0}-\{0\}}(X(t))]\big)$$

is the same for all solutions X starting at the origin, that is, all solutions starting at the origin have the same one-dimensional distributions. By Step 2, the same obviously holds starting at every $x \in \overline{\mathcal{W}_0} - \{0\}$. By a standard conditioning argument, this implies that any two solutions with the same initial distribution have the same one-dimensional distributions, and in turn, this yields uniqueness in distribution for (2.4) (see Theorem 3.12 and Lemma 3.24 of Costantini and Kurtz [4], recalling that we are considering only strong Markov solutions).

2.8 The ergodic argument and conclusion of the proof of Theorem 2.3

For each $n \in \mathbb{N}$, the inhomogeneous Markov chain $\{X_k\}_{0\leq k\leq n} := \{X(\tau^{n-k})\}_{0\leq k\leq n}$, where $\tau^{n-k} := \tau^{X,2^{-2(n-k)}\delta}$ is the first time X hits $\partial B_{2^{-2(n-k)}\delta}$, has transition kernels

$$Q_{n-k}(x,B) := \mathbb{P}\big(X(\tau^{n-k-1}) \in B \big| X(\tau^{n-k}) = x\big),$$
$$B \in \mathcal{B}\big(\partial B_{2^{-2(n-k-1)}\delta}\big),\ x \in \partial B_{2^{-2(n-k)}\delta},\ 0 \leq k \leq n-1. \tag{2.29}$$

Then, denoting by Q_{n-k} the integral operators corresponding to the kernels (2.29), for every $f \in \mathcal{C}(\partial B_\delta)$,

$$\mathbb{E}\big[f(X(\tau^\delta))\big] = \mathbb{E}\big[f(X_n)\big] = \int_{\partial B_{2^{-2n}\delta}} Q_n \cdots Q_1 f(x)\, \mathrm{d}\nu_n(x), \tag{2.30}$$

where $\nu_n; = \mathcal{L}(X_0) = \mathcal{L}(X(\tau^n))$. As discussed in Step 5) of Sect. 2.3, for another solution $\widetilde{X}$ of (2.4) starting at the origin, the transition kernels of the corresponding Markov chain $\{\widetilde{X}_k\}_{0\le k\le n}$ are the same as for X, so that

$$\mathbb{E}\big[f(\widetilde{X}(\widetilde{\tau}^\delta))\big] = \mathbb{E}\big[f(\widetilde{X}_n)\big] = \int_{\partial B_{2^{-2n}\delta}} Q_n \cdots Q_1 f(x)\, \mathrm{d}\widetilde{\nu}_n(x),$$

where $\widetilde{\nu}_n; = \mathcal{L}(\widetilde{X}_0) = \mathcal{L}(\widetilde{X}(\widetilde{\tau}^n))$.

By the reverse ergodic theorem of Costantini and Kurtz [4] (recalled as Theorem 3.20), the limit

$$\lim_{n\to\infty} \int_{\partial B_{2^{-2n}\delta}} Q_n \cdots Q_1 f(x)\, d\nu_n(x)$$

exists and is independent of the sequence of probability measures $\{\nu_n\}_{n\ge 1}$; therefore,

$$\mathbb{E}\big[f(X(\tau^\delta)\big] = \mathbb{E}\big[f(\widetilde{X}(\widetilde{\tau}^\delta)\big],$$

for every $f \in \mathcal{C}(\partial B_\delta)$, that is, the hitting distributions of X and $\widetilde{X}$ coincide.

3 Uniqueness for semimartingale ORBM in a piecewise $\mathcal{C}^2$ cone

The study of semimartingale obliquely reflecting Brownian motion (ORBM) in a piecewise smooth cone, i.e., the intersection of a finite number of smooth cones, cannot be reduced to the study of semimartingale ORBM in a piecewise smooth domain. Therefore, the results of Dupuis and Ishii [9] or Dai and Williams [6] do not apply. Moreover, the arguments used in these papers do not carry over to piecewise smooth cones.

Kwon and Williams [19] have studied semimartingale ORBM in a smooth cone $\mathcal{W}$, with radially constant, smooth direction of reflection g. The results of this paper use the smoothness of the cone in two ways:
- to find a function Φ that solves

$$\begin{cases} \Delta\Phi = 0, & \text{in} \mathcal{W}, \\ \nabla\Phi \cdot g = 0, & \text{on} \partial\mathcal{W} - \{0\}, \end{cases} \tag{3.1}$$

where 0 is the vertex of the cone; the function Φ determines whether the vertex is reached;

- to prove the support theorem (Theorem 3.1 of Kwon and Williams [19]), which allows to use the Krein–Rutman theorem for strongly positive operators to uniquely characterize the ORBM.

Here we seek to uniquely characterize the semimartingale ORBM in a piecewise smooth cone $\mathcal{W}$. The main difference between this work and Kwon and Williams [19] is that the present work relies not on the Krein–Rutman theorem, but on a new reverse, ergodic theorem for killed Markov chains proved in Costantini and Kurtz [4] and recalled as Theorem 3.20. The assumptions of Theorem 3.20 have a clear probabilistic meaning and may be verified in a wider range of situations: piecewise smooth cones—the case of this paper; directions of reflection that are not radially constant; variable coefficients of the reflecting process; domains that are not cones but can be locally approximated by cones (see Costantini and Kurtz [4] and Costantini and Kurtz [5]), etc. Moreover, verifying the assumptions of Theorem 3.20 does not require the delicate oscillation estimates needed to apply the Krein–Rutman theorem (see Theorem 3.2 of Kwon and Williams [19]).

As in Kwon and Williams [19], here too we employ a function (or two functions) to analyze the time till the vertex is reached, but we use the observation that, for our purposes, the functions do not need to satisfy the equalities in (3.1), but only corresponding inequalities (Condition 3.5). Thus, one does not need to solve the boundary value problem (3.1), but only to find functions that satisfy Condition 3.5, and one can avoid requiring smoothness of the cone (as an example, see the function in Sect. 2.5). On the other hand, while Kwon and Williams [19] gives a precise classification of the possible behaviors of the time till the vertex is reached, here we only obtain upper and lower bounds on $\mathbb{P}(\tau^{X,\delta} < \vartheta^X)$ (where X is a semimartingale ORBM starting at x, $0 < |x| < \delta$, $\tau^{X,\delta} := \inf\{t \geq 0 : \ |X(t)| = \delta\}$, $\vartheta^X := \inf\{t \geq 0 : \ X(t) = 0\}$). However, these bounds are enough to satisfy the first assumption of Theorem 3.20. In particular, one can give a sufficient condition for $\mathbb{P}(\tau^{X,\delta} < \vartheta^X) = 1$, which implies that, starting at $x \neq 0$, the origin is never reached (this is the case of the example of Sect. 2: see Sect. 2.5).

The second assumption of Theorem 3.20 replaces the assumption of the Krein–Rutman theorem that the operator (in applications to ORBM, the transition operator of a killed Markov chain) be a compact, strongly positive operator. This second assumption is verified by a coupling lemma proved in Costantini and Kurtz [2] (Lemma 5.3 of Costantini and Kurtz [2]) and by a scaling limit and a nontrivial extension of the support theorem of Kwon and Williams [19] to piecewise $\mathcal{C}^2$ cones (Lemma 3.22). A careful inspection shows that the proof of the support theorem of Kwon and Williams [19] carries over if one can prove that the semimartingale ORBM obtained from the scaling limit spends zero time on $\partial\mathcal{W} - \{0\}$ before hitting the origin. This follows from a result of Kang and Ramanan [13] (Proposition 2.12 of Kang and Ramanan [13]) if $\mathcal{W}$ is piecewise $\mathcal{C}^2$, and this is why $\mathcal{C}^1$ smoothness of the faces of $\mathcal{W}$ is not sufficient.

The problem, the assumptions and the main result of the paper (Theorem 3.6) are formulated precisely in Sect. 3.1. At the end of Sect. 3.1, the outline of the proof of Theorem 3.6 is discussed and compared with that of Theorem 2.3. The proof is then carried out in Sects. 3.3 and 3.9. Section 3.2 contains a preliminary result and the Appendix contains the proof of a technical result.

3.1 Formulation of the problem and main result

Let $\mathcal{W} \subseteq \mathbb{R}^d$ be a piecewise $\mathcal{C}^2$ cone with vertex at the origin, i.e.,

$$\mathcal{W} := \bigcap_{j=1}^{m} \mathcal{W}_j, \quad \mathcal{W}_j := \{x = rz,\ z \in \mathcal{S}_j,\ r > 0\}, \tag{3.2}$$

where $\mathcal{S}_j$ is a nonempty domain in the unit sphere S^{d-1} with $\mathcal{C}^2$ boundary. More precisely

$$\begin{aligned} \mathcal{S}_j = \{z \in S^{d-1} : \varphi_j(z) > 0\}, \quad \partial\mathcal{S}_j = \{z \in S^{d-1} : \varphi_j(z) = 0\}, \\ \varphi_j \in \mathcal{C}^2(S^{d-1}), \qquad \inf_{z \in \partial\mathcal{S}_j} |\nabla_{S^{d-1}} \varphi_j(z)| > 0. \end{aligned} \tag{3.3}$$

Clearly

$$\mathcal{W} = \{x = rz,\ z \in \mathcal{S},\ r > 0\}, \quad \text{where} \mathcal{S} := \bigcap_{j=1}^{m} \mathcal{S}_j. \tag{3.4}$$

Let g^j be a radially constant direction of reflection on the face

$$\partial_j \mathcal{W} := \partial\mathcal{W}_j \cap \bigcap_{i \neq j} \mathcal{W}_i,$$

that is, g^j is a unit vector field defined on $\partial\mathcal{W}_j - \{0\}$ and, for $x = rz, z \in \partial\mathcal{S}_j, r > 0$,

$$g^j(x) = g^j(z). \tag{3.5}$$

A semimartingale ORBM can be defined as a solution of the following stochastic differential equation with reflection:

$$\begin{aligned} X(t) = X(0) + b\,t + \sigma\, W(t) + \int_0^t \gamma(s)\, \mathrm{d}\lambda(s), \quad t \geq 0, \\ X(t) \in \overline{\mathcal{W}}, \quad \lambda(t) = \int_0^t \mathbf{1}_{\partial\mathcal{W}}(X(s))\mathrm{d}\lambda(s), \quad t \geq 0, \\ \gamma(t) \in G(X(t)), \quad |\gamma(t)| = 1, \quad d\lambda - a.e., \quad t \geq 0, \end{aligned} \tag{3.6}$$

where, for $x \in \partial\mathcal{W} - 0$, $G(x)$ is the cone of possible directions of reflection at x:

$$G(x) := \left\{ g : g = \sum_{j \in \mathcal{J}(x)} u_j g^j(x),\ u_j \geq 0 \right\}, \quad \mathcal{J}(x) := \{j : x \in \overline{\partial_j \mathcal{W}}\}, \tag{3.7}$$

and $G(0)$ is the cone of possible directions of reflection at 0, that is, the closed convex cone generated by

$$\left\{g^j(x),\ x \in \overline{\partial_j \mathcal{W}} - \{0\},\ j = 1, \cdots, m\right\}. \tag{3.8}$$

Definition 3.1 A continuous process X is a solution of (3.6) if there exist a standard Brownian motion W, a continuous, nondecreasing process λ and a process γ with measurable paths, all defined on the same probability space as X, such that $W(t+\cdot) - W(t)$ is independent of $\mathcal{F}_t^{X,W,\lambda,\gamma}$ for all $t \geq 0$ and (3.6) is satisfied a.s.

Weak uniqueness or *uniqueness in distribution* holds if any two solutions of (3.6) with the same initial distribution have the same distribution on $C_{\overline{\mathcal{W}}}[0, \infty)$.

A stochastic process $\widetilde{X}$ (for example, a solution of an appropriate martingale problem or submartingale problem) is a weak solution of (3.6) if there is a solution X of (3.6) such that $\widetilde{X}$ and X have the same distribution.

Remark 3.2 There is not a unique representation (3.2)–(3.3) of $\mathcal{W}$, as it would be for a polyhedral cone, because each function φ_j can be extended from $\overline{\mathcal{S}}$ do S^{d-1} in infinitely many ways. However, we only need that there exists at least one representation and a set of directions of reflection $\{g^j\}_{j=1,\ldots,m}$ that satisfy Conditions 3.3 and 3.5.

The goal of this section is to provide conditions on $\mathcal{W}$, G and σ under which uniqueness in distribution holds for (3.6). Since the focus is on uniqueness, existence of solutions to (3.6) for every initial distribution $\nu \in \mathcal{P}(\overline{\mathcal{W}})$ will be assumed throughout.

For $x \in \partial\mathcal{W}_j - \{0\}$, $n^j(x)$ will denote the unit inward normal to $\overline{\mathcal{W}_j}$ and, for each $x \in \partial\mathcal{W} - \{0\}$, $N(x)$ will denote the closed, convex cone generated by $\left\{n^j(x)\right\}_{j \in \mathcal{J}(x)}$. Note that, for $x \in \partial\mathcal{W}_j - \{0\}$, $x = rz$, $z \in \partial\mathcal{S}_j$, $r > 0$,

$$n^j(x) = n^j(z), \tag{3.9}$$

and $n^j(z)$ coincides with the inward unit normal to $\overline{\mathcal{S}_j}$ in S^{d-1}; in particular, $n^j(z)$ lies in the tangent space to S^{d-1} at z. Also note that, for $x \in \partial\mathcal{W} - \{0\}$, $x = rz$, $z \in \partial\mathcal{S}$, $r > 0$,

$$\mathcal{J}(x) = \mathcal{J}(z) := \{j : z \in \partial\mathcal{S}_j\}. \tag{3.10}$$

For each $\zeta \in \partial\mathcal{S}$, if $j \notin \mathcal{J}(\zeta)$, ζ has positive distance from $\partial\mathcal{S}_j$; hence, there is $\delta_j(\zeta) > 0$ such that, for $z \in \partial\mathcal{S} \cap B_{\delta_j(\zeta)}(\zeta)$, z has positive distance from $\partial\mathcal{S}_j$ as well, so that $j \notin \mathcal{J}(z)$; therefore, setting $\delta_{\mathcal{J}}(\zeta) := \min_{j \notin \mathcal{J}(\zeta)} \delta_j(\zeta)$,

$$\mathcal{J}(z) \subseteq \mathcal{J}(\zeta), \quad \forall z \in \partial\mathcal{S} \cap B_{\delta_{\mathcal{J}}(\zeta)}(\zeta). \tag{3.11}$$

The main assumptions are formulated in the following two conditions.

Condition 3.3 (i) *For $j = 1, \ldots, m$, $g^j : S^{d-1} \to \mathbb{R}^d$ is a Lipschitz continuous vector field of unit length on $\partial\mathcal{S}_j$ such that*

$$\inf_{z \in \partial\mathcal{S}_j} g^j(z) \cdot n^j(z) > 0.$$

(ii) *For every* $z \in \partial\mathcal{S}$*, the vectors* $\{n^j(z)\}_{j\in\mathcal{J}(z)}$ *are linearly independent. In particular,* $|\mathcal{J}(z)| \le d-1$.
(iii) *For* $z \in \partial\mathcal{S}$, $\mathcal{J}(z) = \{h_1, \dots, h_k\}$*, the matrix of elements*

$$\frac{|g^{h_i}(z)\cdot n^{h_j}(z)|}{g^{h_i}(z)\cdot n^{h_i}(z)} - \delta_{ij}, \quad i, j = 1, ..., k,$$

has spectral radius strictly less than 1.
(iv) *Let*

$$N(0) := \{n \in \mathbb{R}^d : n\cdot x \ge 0, \ \forall x \in \overline{\mathcal{W}}\}, \tag{3.12}$$

and suppose that $\overset{\circ}{N}(0) \neq \emptyset$.
There exists a unit vector $e \in N(0)$ *such that*

$$e\cdot g > 0, \quad \forall g \in G(0), \ |g| = 1.$$

Without loss of generality, one can suppose $e \in \overset{\circ}{N}(0)$.

Remark 3.4 For each $z \in \partial\mathcal{S}$, let

$$N(z) := \left\{ n : n = \sum_{j\in\mathcal{J}(z)} u_j n^j(z), \ u_j \ge 0 \right\}.$$

As in the remark at page 160 of Dupuis and Ishii [8], perturbing the matrix of Condition 3.3 (iii) by a matrix with small positive entries and using the Perron–Frobenius theorem, we see that for each $z \in \partial\mathcal{S}$ there is a unit vector $e(z) \in N(z)$ and $\delta(z) > 0$, such that

$$e(z)\cdot g \ge c_e > 0, \qquad \forall g \in G(\zeta), \ |g| = 1, \ \forall \zeta \in \partial\mathcal{S}, \ |\zeta - z| \le \delta(z), \tag{3.13}$$

where c_e is a positive constant independent of z.

In Condition 3.3, assumptions (i) to (iii) guarantee that, at every point of the boundary other than the origin, the assumptions of Dupuis and Ishii [9] are satisfied. In particular, (3.13) ensures that, if a solution of (3.6) reaches $\partial\mathcal{W} - \{0\}$, it leaves the boundary immediately. Assumption (iv) ensures that the same holds at the origin.

Denote by $\mathbb{A}$ the operator of the form

$$\mathbb{A}f(x) := b\cdot\nabla f(x) + \frac{1}{2}\mathrm{tr}((\sigma\sigma^T)D^2 f(x)). \tag{3.14}$$

σ is assumed to be nonsingular.

Condition 3.5 *For some* $\delta_{\mathcal{W}} > 0$*, either of the following conditions is satisfied*

(i) *There exists a function* $V \in \mathcal{C}^2(\overline{\mathcal{W}} - \{0\})$ *such that*

$$\lim_{x\in\overline{\mathcal{W}},\, x\to 0} V(x) = \infty, \tag{3.15}$$

$$\begin{aligned} \nabla V(x)\cdot g \le 0, \quad & \forall g \in G(x),\ x \in \big(\partial\mathcal{W} - \{0\}\big) \cap \overline{B_{\delta_{\mathcal{W}}}(0)}, \\ \mathbb{A}V(x) \le 0, \quad & \forall x \in \big(\overline{\mathcal{W}} - \{0\}\big) \cap \overline{B_{\delta_{\mathcal{W}}}(0)}. \end{aligned} \tag{3.16}$$

(ii) *There exist two functions* $V_+, V_- \in \mathcal{C}^2(\overline{\mathcal{W}} - \{0\})$ *such that*

$$\begin{aligned} & V_+(x) > 0, \quad V_-(x) > 0, \quad \text{for} x \in \big(\overline{\mathcal{W}} - \{0\}\big) \cap \overline{B_{\delta_{\mathcal{W}}}(0)}, \\ & \lim_{x\in\overline{\mathcal{W}},\, x\to 0} V_+(x) = \lim_{x\in\overline{\mathcal{W}},\, x\to 0} V_-(x) = 0, \end{aligned} \tag{3.17}$$

$$\inf_{0<\delta\le\delta_{\mathcal{W}}} \frac{\inf_{|x|=\delta} V_+(x)}{\sup_{|x|=\delta} V_-(x)} > 0, \qquad \inf_{0<\delta\le\delta_{\mathcal{W}}} \frac{\inf_{|x|=\delta} V_-(x)}{\sup_{|x|=\delta} V_+(x)}, > 0 \tag{3.18}$$

$$\begin{aligned} & \nabla V_+(x)\cdot g \ge 0, \quad \nabla V_-(x)\cdot g \le 0, \quad \forall g \in G(x),\ x \in \big(\partial\mathcal{W} - \{0\}\big) \cap \overline{B_{\delta_{\mathcal{W}}}(0)} \\ & \mathbb{A}V_+(x) \ge 0, \quad \mathbb{A}V_-(x) \le 0, \quad \forall x \in \big(\overline{\mathcal{W}} - \{0\}\big) \cap \overline{B_{\delta_{\mathcal{W}}}(0)}. \end{aligned} \tag{3.19}$$

For a solution X of (3.6), let

$$\vartheta^X := \inf\{t \ge 0 : X(t) = 0\}, \tag{3.20}$$

$$\tau^{X,\delta} := \inf\{t \ge 0 : |X(t)| = \delta\}, \quad \delta > 0. \tag{3.21}$$

(When there is no risk of confusion, the superscript X will be omitted.) In addition, denote by X^x a solution of (3.6) starting at x. Condition 3.5 (i) is a sufficient condition for the following: for $\delta \le \delta_{\mathcal{W}}$ and $0 < |x| < \delta$,

$$\mathbb{P}(\tau^{X^x,\delta} < \vartheta^{X^x}) = 1. \tag{3.22}$$

In particular, this yields

$$\mathbb{P}(\vartheta^{X^x} < \infty) = 0, \qquad \forall x \ne 0$$

(see Sect. 2.5).

The meaning of Condition 3.5 (ii) instead is the following. If Condition 3.5 (ii) is satisfied, for $\delta \le \delta_{\mathcal{W}}$,

$$\lim_{x\to 0} \mathbb{P}(\tau^{X^x,\delta} < \vartheta^{X^x}) = 0, \tag{3.23}$$

but the above probability vanishes at the same rate for all points that have the same distance from the origin, that is, the following uniform lower bound holds:

$$\inf_{x,\tilde{x}\in\overline{\mathcal{W}}:\, 0<|x|=|\tilde{x}|<\delta} \frac{\mathbb{P}(\tau^{X^x,\delta} < \vartheta^{X^x})}{\mathbb{P}(\tau^{X^{\tilde{x}},\delta} < \vartheta^{X^{\tilde{x}}})} \ge c_0 > 0.$$

Of course, this trivially holds also if Condition 3.5 (i) is satisfied. In the context of this work, the above uniform lower bound is the first assumption of Theorem 3.20.

Note that (3.22) and (3.23) imply that only one between Conditions 3.5 (i) and (ii) can be verified.

The main result of this work is the following.

Theorem 3.6 *Under Conditions 3.3 and 3.5, for every $\nu \in \mathcal{P}(\overline{\mathcal{W}})$ uniqueness in distribution holds for (3.6).*

The solution, X, spends zero time at the origin, that is,

$$\int_0^\infty \mathbf{1}_{\{0\}}(X(t))\,dt = 0, \qquad a.s. \tag{3.24}$$

(3.24) *implies in particular that, if X starts at* 0*,*

$$\lim_{\delta\to 0} \tau^{X,\delta} = 0, \qquad a.s. \tag{3.25}$$

Remark 3.7 Assuming that the directions of reflection are radially constant (that is (3.5) holds) simplifies proofs. However, it can be shown, by the same arguments as in Costantini and Kurtz [4], that Theorem 3.6 carries over to radially variable directions of reflection, as long as

$$|g^j(rz) - \bar{g}^j(z)| \le c_{\mathcal{W}} r, \qquad \forall z \in \partial\mathcal{S}_j,\ 0 < r \le r_{\mathcal{W}},$$

for some unit vector field $\bar{g}^j \in \mathcal{C}^2(S^{d-1}, \mathbb{R}^d)$ of unit length on $\partial\mathcal{S}_j$ and some $c_{\mathcal{W}}, r_{\mathcal{W}} > 0$.

Theorem 3.6 is a general result which includes Theorem 2.3 as a special case. The proof follows the outline given in Sect. 2.3, but Step 5 is more complex. In fact, in general, the solutions of (3.6) starting at the origin can hit the origin between the hitting time of $\partial B_{2^{-2(n-k)}\delta}$ and the hitting time of $\partial B_{2^{-2(n-k-1)}\delta}$, so that, as mentioned in Remark 2.6, in order to have transition kernels that are independent of the specific solution, one has to consider Markov chains killed when the origin is reached. However, the reverse ergodic theorem of Costantini and Kurtz [4] applies to killed Markov chains as well, under a suitable assumption which is ensured by Condition 3.5 (see Sect. 3.5). Step 3 is also partly different, because, in order to carry out Step 4 in the general case, one needs to prove also that any solution of (3.6) spends zero time at the origin,

Each of the steps of the proof of Theorem 3.6 is developed in one of the Sects. 3.3 and 3.9. While in Sect. 2 only key points were proved, but those were proved in detail, here proofs are complete but concise and often refer to previous works. Section 3.2 contains some preliminary results.

3.2 Constrained martingale problems

ORBMs (and more generally stochastic processes with boundary conditions) can be characterized also as *natural solutions of constrained martingale problems*. Characterizing the process as the natural solution of a constrained martingale problem allows to verify tightness in a much simpler way, whenever an approximation or convergence argument is needed. The present work builds on some previous results which are stated in terms of constrained martingale problems; therefore, definitions and some of the results are recalled here. Definitions and results are formulated for the specific constrained martingale problem corresponding to (3.6). The reader is referred to Kurtz [17], Kurtz [18], Costantini and Kurtz [1], Costantini and Kurtz [3] for a complete treatment. In these papers, the state space is compact, but the definitions extend to a locally compact state space as well as most of the results.

Let E_0 be an open subset of $\mathbb{R}^d$,

$$\mathcal{D} := \mathcal{C}_b^2(\overline{E_0}), \tag{3.26}$$

and $\mathbb{A}$ be given by (3.14) with domain

$$\mathcal{D}(\mathbb{A}) = \mathcal{D}. \tag{3.27}$$

To each $x \in \partial E_0$, associate a convex cone $G(x)$ in such a way that the set

$$\Xi := \{(x, u) \in \partial E_0 \times S^{d-1} : u \in G(x)\}, \tag{3.28}$$

is closed and define

$$\mathbb{B} : \mathcal{D} \to \mathcal{C}(\Xi), \quad \mathbb{B} f(x, u) := \nabla f(x) \cdot u \tag{3.29}$$

Define $\mathcal{L}_\Xi$ to be the space of measures μ on $[0, \infty) \times \Xi$ such that $\mu([0, t] \times \Xi) < \infty$ for all $t > 0$. $\mathcal{L}_\Xi$ is topologized so that $\mu_n \in \mathcal{L}_\Xi \to \mu \in \mathcal{L}_\Xi$ if and only if

$$\int_{[0,\infty)\times\Xi} f(s, u)\mu_n(\mathrm{d}s \times \mathrm{d}u) \to \int_{[0,\infty)\times\Xi} f(s, u)\mu(\mathrm{d}s \times \mathrm{d}u)$$

for all continuous f with compact support in $[0, \infty) \times \Xi$. It is possible to define a metric on $\mathcal{L}_\Xi$ that induces the above topology and makes $\mathcal{L}_\Xi$ into a complete, separable metric space. Also define $\mathcal{L}_{S^{d-1}}$ in the same way.

An $\mathcal{L}_{S^{d-1}}$-valued random variable Λ_1 is adapted to a filtration $\{\mathcal{G}_t\}$ if

$$\Lambda_1([0, \cdot] \times B) \, \mathrm{is} \{\mathcal{G}_t\} - \mathrm{adapted}, \quad \forall B \in \mathcal{B}(S^{d-1}).$$

An adapted $\mathcal{L}_\Xi$-valued random variable is defined analogously.

Definition 3.8 Let $\mathbb{A}$, Ξ and $\mathbb{B}$ be as in (3.14), (3.27), (3.28) and (3.29). A process X in $D_{\overline{E_0}}[0, \infty)$ is a solution of the *constrained martingale problem* for $(\mathbb{A}, E_0, \mathbb{B}, \Xi)$ if there exists a random measure Λ with values in $\mathcal{L}_\Xi$ and a filtration $\{\mathcal{F}_t\}$ such that X and Λ are $\{\mathcal{F}_t\}$-adapted and for each $f \in \mathcal{D}$,

$$f(X(t)) - f(X(0)) - \int_0^t \mathbb{A} f(X(s))\mathrm{d}s - \int_{[0,t]\times\Xi} \mathbb{B} f(x, u)\Lambda(\mathrm{d}s \times \mathrm{d}x \times \mathrm{d}u) \tag{3.30}$$

is a $\{\mathcal{F}_t\}$-local martingale.

A *natural solution* of a constrained martingale problem is a solution obtained by time-changing a solution of the corresponding *controlled martingale problem*, which is a "slowed down" version of the constrained martingale problem.

Definition 3.9 Let $\mathbb{A}$, Ξ and $\mathbb{B}$ be as in (3.14), (3.27), (3.28) and (3.29). $(Y, \lambda_0, \Lambda_1)$ is a solution of the *controlled martingale problem* for $(\mathbb{A}, E_0, \mathbb{B}, \Xi)$, if Y is a process in $\mathcal{D}_{\overline{E_0}}[0, \infty)$, λ_0 is nonnegative and nondecreasing, Λ_1 is a random measure with values in $\mathcal{L}_{S^{d-1}}$ such that

$$\begin{aligned} \lambda_1(t) := \Lambda_1([0, t] \times S^{d-1}) = \int_{[0,t]\times S^{d-1}} \mathbf{1}_\Xi(Y(s), u)\Lambda_1(\mathrm{d}s \times \mathrm{d}u), \\ \lambda_0(t) + \lambda_1(t) = t, \end{aligned} \tag{3.31}$$

and there exists a filtration $\{\mathcal{G}_t\}$ such that Y, λ_0 and Λ_1 are $\{\mathcal{G}_t\}$-adapted and

$$f(Y(t)) - f(Y(0)) - \int_0^t \mathbb{A}f(Y(s))\mathrm{d}\lambda_0(s) - \int_{[0,t]\times S^{d-1}} \mathbb{B}f(Y(s), u)\Lambda_1(\mathrm{d}s \times \mathrm{d}u) \tag{3.32}$$

is a $\{\mathcal{G}_t\}$-martingale for all $f \in \mathcal{D}$.

For every nondecreasing path $\lambda_0 \in \mathcal{D}_{[0,\infty)}[0, \infty)$, define

$$\lambda_0^{-1}(t) = \inf\{s : \lambda_0(s) > t\}, \quad t \geq 0. \tag{3.33}$$

Definition 3.10 Let $\mathbb{A}$, Ξ and $\mathbb{B}$ be as in (3.14), (3.27), (3.28) and (3.29). A solution, X, of the constrained martingale problem for $(\mathbb{A}, E_0, \mathbb{B}, \Xi)$ is called *natural* if, for some solution $(Y, \lambda_0, \Lambda_1)$ of the controlled martingale problem for $(\mathbb{A}, E_0, \mathbb{B}, \Xi)$ with filtration $\{\mathcal{G}_t\}$,

$$\begin{aligned} X(t) = Y(\lambda_0^{-1}(t)), \mathcal{F}_t = \mathcal{G}_{\lambda_0^{-1}(t)}, \\ \Lambda([0, t] \times C) = \int_{[0,\lambda_0^{-1}(t)]\times S^{d-1}} \mathbf{1}_C(Y(s), u)\Lambda_1(\mathrm{d}s \times \mathrm{d}u), \quad C \in \mathcal{B}(\Xi), \end{aligned} \tag{3.34}$$

Uniqueness holds for natural solutions of the constrained martingale problem for $(\mathbb{A}, E_0, \mathbb{B}, \Xi)$ if any two natural solutions with the same initial distribution have the same distribution on $D_{\overline{E_0}}[0, \infty)$.

The two characterizations of a semimartingale ORBM as a solution of (3.6) and as a natural solution of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}, \mathbb{B}, \Xi)$, with the cone $G(x)$ given by (3.7) and (3.8), are equivalent. In particular, taking into account

Lemma 3.11, if X is a natural solution of the constrained martingale problem for $(\mathbb{A}, E_0, \mathbb{B}, \Xi)$, by (3.34) and (3.31) the corresponding Λ satisfies

$$\int_{[0,t]\times\Xi} h(x,u)\,\Lambda(\mathrm{d}r\times\mathrm{d}x\times du) = \int_{[0,\lambda_0^{-1}(t)]\times S^{d-1}} h(Y(r),u)\,\Lambda_1(\mathrm{d}r\times\mathrm{d}u)$$
$$= \int_{[0,t]\times\Xi} h(X(r),u)\,\Lambda(\mathrm{d}r\times\mathrm{d}x\times\mathrm{d}u)$$

Therefore, Λ is concentrated on the set

$$\big\{(r,x,u)\in[0,t]\times\Xi : x = X(r)\big\}$$
$$= \big\{(r,x,u) : 0\le r\le t,\ x = X(r)\in\partial\mathcal{W},\ u\in G(X(r)),\ |u|=1\big\},$$

which corresponds to the conditions in the second and third lines of (3.6) (except $X(t)\in\overline{\mathcal{W}}$, which holds by definition for any solution).

Theorem 3.12 formulates the equivalence precisely.

Lemma 3.11 *For every solution* $(Y, \lambda_0, \Lambda_1)$ *of the controlled martingale problem for* $(\mathbb{A}, \mathcal{W}, \mathbb{B}, \Xi)$, $\lambda_0(t) > 0$ *for all* $t > 0$, *almost surely. Moreover,* λ_0 *is strictly increasing almost surely.*

Proof The proof that $\lambda_0(t) > 0$ for all $t > 0$ is analogous to that of Lemma 6.8 of Costantini and Kurtz [3] and Lemma 3.1 of Dai and Williams [6] and is based on Remark 3.4 and Condition 3.3 (iv). The second statement follows by Lemma 3.4 of Costantini and Kurtz [3]. (Note that Lemma 3.4 of Costantini and Kurtz [3] holds in locally compact spaces as well.) □

Theorem 3.12 *Let* Ξ *be as in* (3.28) *with* $G(x)$, $x\in\partial\mathcal{W}$, *given by* (3.7) *and* (3.8), $\mathbb{A}$, $\mathbb{B}$ *be as in* (3.14), (3.29) *and assume Conditions* 3.3 *and* 3.5.

Every solution of (3.6) *is a natural solution of the constrained martingale problem for* $(\mathbb{A}, \mathcal{W}, \mathbb{B}, \Xi)$.

Every natural solution of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}, \mathbb{B}, \Xi)$ *is a weak solution of* (3.6).

Proof The proof relies on Lemma 3.11 and is the same as for Theorem 6.12 of Costantini and Kurtz [3]. □

3.3 Localization

This section and the next one carry out Step 1 of the outline in Sect. 2.3. Specifically this section shows that one can reduce from proving uniqueness in $\mathcal{W}$ to proving uniqueness in each of a family of bounded domains $\{\mathcal{W}_n\}_{n\ge0}$.

By Theorem 3.12, uniqueness in distribution for (3.6) is equivalent to uniqueness for the constrained martingale problem for $(\mathbb{A}, \mathcal{W}, \mathbb{B}, \Xi)$. Let

$$\mathcal{W}_0 := \mathcal{W}\cap B_4,\quad \partial_j\mathcal{W}_0 := \partial_j\mathcal{W}\cap B_4,\ j=1,\ldots,m,\quad \partial_{m+1}\mathcal{W}_0 := \partial B_4\cap\mathcal{W},$$

$$g^{m+1}(x) := n^{m+1}(x) = -\frac{x}{|x|}, \text{ for} x \in \partial B_4,$$

and $G_0(x)$, Ξ_0 be defined by (3.7), (3.8) and (3.28) with $\mathcal{W}$ replaced by $\mathcal{W}_0$, g^j, $j = 1, \ldots, m$ as in Condition 3.3 (i) and g^{m+1} as above. For $n \geq 1$, let

$$\begin{aligned}
\mathcal{W}_n &:= \mathcal{W} \cap \left(\overline{B_{1+2(n-1)}}\right)^c \cap B_{6+2(n-1)}, \\
\partial_j \mathcal{W}_n &:= \partial_j \mathcal{W} \cap \left(\overline{B_{1+2(n-1)}}\right)^c \cap B_{6+2(n-1)}, \\
\partial_0 \mathcal{W}_n &:= \partial B_{1+2(n-1)} \cap \mathcal{W}, \quad \partial_{m+1} \mathcal{W}_n := \partial B_{6+2(n-1)} \cap \mathcal{W}, \\
g^0(x) &:= n^0(x) = \frac{x}{|x|}, \text{ for} x \in \partial B_{1+2(n-1)}, \\
g^{m+1}(x) &:= n^{m+1}(x) = -\frac{x}{|x|}, \text{ for} x \in \partial B_{6+2(n-1)},
\end{aligned}$$

and $G_n(x)$, Ξ_n be defined by (3.7) and (3.28) with $\mathcal{W}$ replaced by $\mathcal{W}_n$, g^j, $j = 1, \ldots, m$ as in Condition 3.3 (i) and g^0, g^{m+1} as above.

Proposition 3.13 *Assume Conditions 3.3 and 3.5. If uniqueness holds for natural solutions of the constrained martingale problem for* $(\mathbb{A}, \mathcal{W}_n, \mathbb{B}, \Xi_n)$ *for each* $n \geq 0$, *then uniqueness holds for natural solutions of the constrained martingale problem for* $(\mathbb{A}, \mathcal{W}, \mathbb{B}, \Xi)$.

Proof See Appendix. □

3.4 Uniqueness in a domain $\mathcal{W} \cap \left(\overline{B_\delta}\right)^c \cap B_{\delta'}$

Completing Step 1 of the outline in Sect. 2.3, let us show that uniqueness holds in each domain of the form

$$\mathcal{W}_{\delta,\delta'} := \mathcal{W} \cap \left(\overline{B_\delta}\right)^c \cap B_{\delta'} \qquad 0 < \delta < \delta', \tag{3.35}$$

Let

$$\begin{aligned}
&\partial_j \mathcal{W}_{\delta,\delta'} := \partial_j \mathcal{W} \cap \left(\overline{B_\delta}\right)^c \cap B_{\delta'}, \quad \partial_0 \mathcal{W}_{\delta,\delta'} := \partial B_\delta \cap \mathcal{W}, \quad \partial_{m+1} \mathcal{W}_{\delta,\delta'} := \partial B_{\delta'} \cap \mathcal{W}, \\
&g^0(x) := n^0(x) = \frac{x}{|x|}, \text{ for} x \in \partial B_\delta, \quad g^{m+1}(x) := n^{m+1}(x) = -\frac{x}{|x|}, \text{ for} x \in \partial B_{\delta'},
\end{aligned} \tag{3.36}$$

and $G_{\delta,\delta'}(x)$, $\Xi_{\delta,\delta'}$ be defined by (3.7) and (3.28) with $\mathcal{W}$ replaced by $\mathcal{W}_{\delta,\delta'}$, g^j, $j = 1, \ldots, m$ as in Condition 3.3 (i) and g^0, g^{m+1} as above.

Lemma 3.14 *For every* $0 < \delta < \delta'$, *uniqueness holds for natural solutions of the constrained martingale problem for* $(\mathbb{A}, \mathcal{W}_{\delta,\delta'}, \mathbb{B}, \Xi_{\delta,\delta'})$.

In particular, uniqueness holds for natural solutions of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}_n, \mathbb{B}, \Xi_n)$ *for each* $n \geq 1$.

Proof By the analog of Theorem 3.12, it is equivalent to prove uniqueness in distribution for (3.6) in $\mathcal{W}_{\delta,\delta'}$ with cone of directions of reflection $G_{\delta,\delta'}$. Conditions 3.3 (i), (ii) and (iii) imply that the assumptions of Case 2 of Dupuis and Ishii [9] are verified (see Remark 3.1 of Dupuis and Ishii [9]). Therefore, by Corollary 5.2 of Dupuis and Ishii [9], the solution to (3.6) is pathwise unique, and hence unique in distribution. □

The next lemma carries out the first part of Step 2 of the outline in Sect. 2.3, namely proves that the distribution of a natural solution of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}_0, \mathbb{B}, \Xi_0)$ is uniquely determined till the first time the solution hits the origin. The second statement in Step 2 that, starting off the origin, with probability one the origin is never reached, holds if Condition 3.5 (i) is verified, as in the example of Sect. 2. In this case, Step 4 has a simpler proof. However, Step 4 can be carried out as well if the origin can be reached (see Sect. 3.5).

For a solution X of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}_0, \mathbb{B}, \Xi_0)$ and $0 < \delta < 4$, let ϑ^X and $\tau^{X,\delta}$ be defined by (3.20) and (3.21), respectively. When there is no risk of confusion, the superscript X will be omitted.

Lemma 3.15 *For every initial distribution ν with support in $\mathcal{W}_0 - \{0\}$, $X(\cdot \wedge \vartheta^X)$ has the same distribution for all natural solutions X of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}_0, \mathbb{B}, \Xi_0)$ with initial distribution ν.*

Proof For every natural solution X of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}_0, \mathbb{B}, \Xi_0)$,

$$X(\cdot \wedge \vartheta^X) = \lim_{k\to\infty} X\left(\cdot \wedge \tau^{X,\mathcal{W}_0 \cap \left(\overline{B_{1/k}}\right)^c}\right),$$

where $\tau^{X,\mathcal{W}_0 \cap \left(\overline{B_{1/k}}\right)^c} := \inf\{t \geq 0 : X(t) \in \overline{B_{1/k}}\}$.

However, $X(\cdot \wedge \tau^{X,\mathcal{W}_0 \cap \left(\overline{B_{1/k}}\right)^c})$ is a solution of the stopped constrained martingale problem for $(\mathbb{A}, \mathcal{W}_{1/(2k),4}, \mathbb{B}, \Xi_{1/(2k),4}; \mathcal{W}_{1/(2k),4} \cap \left(\overline{B_{1/k}}\right)^c)$, which is unique by Lemma 3.14 and Theorem 2.11 of Costantini and Kurtz [5]. □

3.5 Exit time from B_δ and occupation time of the origin

This section carries out Step 3 of the outline in Sect. 2.3. As mentioned in Sect. 3.4, in general a natural solution of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}_0, \mathbb{B}, \Xi_0)$ can reach the origin with positive probability. However, Step 4 can still be carried out, provided one shows that every solution spends zero time at the origin.

Lemma 3.16 *There exists $\bar{\delta} > 0$, $\bar{c} > 0$, depending only on the data of the problem, such that, for $\delta \leq \bar{\delta}$, for every natural solution X of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}_0, \mathbb{B}, \Xi_0)$ starting at 0,*

$$\mathbb{E}[\tau^{X,\delta}] \leq \bar{c}\delta^2.$$

For every natural solution X of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}_0, \mathbb{B}, \Xi_0)$,

$$\int_0^\infty \mathbf{1}_{\{0\}}(X(t))\,\mathrm{d}t = 0, \quad a.s. \tag{3.37}$$

(3.37) *implies in particular that, if X starts at* 0,

$$\lim_{\delta\to 0} \tau^{X,\delta} = 0, \qquad a.s.$$

Proof The first assertion follows from Condition 3.3 (iv) as in Sect. 2.6. The proof is analogous to that of Lemma 4.2 of Costantini and Kurtz [2] and Lemma 6.4 of Taylor and Williams [22].

For the second assertion, the proof of Lemma 3.21 of Costantini and Kurtz [4] carries over. The argument is similar to that of Lemma 2.1 of Taylor and Williams [22]. It relies on the fact that every natural solution X of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}_0, \mathbb{B}, \Xi_0)$ is a continuous semimartingale with $< X > (t) = \sigma\sigma^T t$ and on Tanaka's formula for the local time.

The third assertion follows by observing that

$$\begin{aligned} 0 \le \lim_{\delta\to 0} \left(\tau^{X,\delta} \wedge 1\right) &= \lim_{\delta\to 0} \int_0^{\tau^{X,\delta}\wedge 1} \mathbf{1}_{\overline{B_\delta(0)}}(X(t))\,\mathrm{d}t \\ &\le \lim_{\delta\to 0} \int_0^1 \mathbf{1}_{\overline{B_\delta(0)}}(X(t))\,\mathrm{d}t = \int_0^1 \mathbf{1}_{\{0\}}(X(t))\,\mathrm{d}t = 0. \end{aligned}$$

□

3.6 Uniqueness and hitting distributions

The first goal of this section is to prove the claim of Step 1 of the outline in Sect. 2.3 that, in order to prove uniqueness among natural solutions of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}_0, \mathbb{B}, \Xi_0)$, one can reduce to proving uniqueness among strong Markov, natural solutions (Lemma 3.17). Next Step 4 of the outline is carried out (Lemma 3.18).

Lemma 3.17 *There exist strong Markov natural solutions of the constrained martingale problem for* $(\mathbb{A}, \mathcal{W}_0, \mathbb{B}, \Xi_0)$.

If uniqueness holds among strong Markov solutions, then uniqueness holds for natural solutions of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}_0, \mathbb{B}, \Xi_0)$.

Proof Existence of solutions of the controlled martingale problem for $(\mathbb{A}, \mathcal{W}_0, \mathbb{B}, \Xi_0)$ can be proved as in the first step of the proof of Theorem 3.23 of Costantini and Kurtz [4]; hence, Conditions 3.5 a) and b) of Costantini and Kurtz [3] are satisfied. By Lemma 3.11 and Lemma 3.3 of Costantini and Kurtz [3], Condition 3.5 c) of Costantini and Kurtz [3] is also satisfied. Then the assertion follows from Corollaries 4.12 and 4.13 of Costantini and Kurtz [3]. □

Lemma 3.18 *Suppose that, for each $\delta > 0$ sufficiently small, the hitting distribution*

$$\mathcal{L}(X(\tau^{X,\delta}))(B) = \mathbb{P}\{X(\tau^{X,\delta}) \in B\}, \quad B \in \mathcal{B}\big(\partial B_\delta\big),$$

is the same for all strong Markov, natural solutions X of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}_0, \mathbb{B}, \Xi_0)$ starting at 0.

Then uniqueness holds among strong Markov natural solutions of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}_0, \mathbb{B}, \Xi_0)$.

Proof The proof of Lemma 3.27 of Costantini and Kurtz [4] carries over. The argument is more complex but similar to that in Sect. 2.7. Instead of (2.28), we now have, for any strong Markov natural solution of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}_0, \mathbb{B}, \Xi_0)$, X,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty e^{-\eta t} f(X(t))\mathrm{d}t\right] \\
&\quad = \mathbb{E}\left[\int_0^\vartheta e^{-\eta t} f(X(t))\mathrm{d}t\right] \\
&\qquad + \lim_{n\to\infty} \mathbb{E}\left[\sum_{l=1}^{\infty}\prod_{m=0}^{l-1} e^{-\eta(\vartheta_m^n - \tau_m^n)} \int_{\tau_l^n}^{\vartheta_l^n} e^{-\eta(t-\tau_l^n)} f(X(t))\mathrm{d}t\right],
\end{aligned} \tag{3.38}$$

where, for each $n \geq 0$,

$$\begin{aligned}
&\vartheta_0^n := \vartheta := \inf\{t \geq 0 : X(t) = 0\}, \\
&\tau_l^n := \inf\{t \geq \vartheta_{l-1}^n : |X(t)| = \delta\, 2^{-2n}\}, \quad \vartheta_l^n := \inf\{t \geq \tau_l^n : X(t) = 0\}, \qquad l \geq 1,
\end{aligned}$$

and with the convention that $e^{-\infty} = 0$. The limit holds by (3.37). □

3.7 The ergodic argument

This section and Sects. 3.8 and 3.9 carry out Step 5 of the outline in Sect. 2.3. By Sect. 3.6, we are reduced to proving that, for all $\delta > 0$ sufficiently small, for any strong Markov, natural solution X of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}_0, \mathbb{B}, \Xi_0)$ starting at the origin, the hitting distribution on ∂B_δ is always the same.

Le X be a strong Markov, natural solution of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}_0, \mathbb{B}, \Xi_0)$ starting at the origin and define, for $0 < \delta < 4$,

$$\tau^l := \inf\{t \geq 0 : |X(t)| = 2^{-2l}\delta\}, \quad l \in \mathbb{Z}_+. \tag{3.39}$$

Note that, since X starts at the origin, $\tau^l < \tau^{l-1}$. As discussed in Remark 2.6, in the general case a solution can hit the origin between τ^l and τ^{l-1} with positive probability. Then, since the distribution of a solution starting on $\partial B_{2^{-2l}\delta}$ is uniquely determined only until it hits the origin, in order to have transition kernels that are the same for all solutions, we cannot consider the transition kernels (2.29), but we have to consider the subprobability transition kernels

$$Q_{n-k}(x, B) := \mathbb{P}\big(\tau^{n-k-1} < \vartheta^{n-k},\ X(\tau^{n-k-1}) \in B \big| X(\tau^{n-k}) = x\big),$$
$$B \in \mathcal{B}\big(\partial B_{2^{-2(n-k-1)}\delta}\big),\ x \in \partial B_{2^{-2(n-k)}\delta},\ 0 \le k \le n-1,\ n \in \mathbb{N}, \tag{3.40}$$

where

$$\vartheta^{n-k} := \inf\{t \ge \tau^{n-k} :\ X(t) = 0\}. \tag{3.41}$$

However, Lemma 3.19 shows that one can generalize (2.30) and still employ an ergodic type argument as in Sect. 2.8.

Lemma 3.19 *For $n \in \mathbb{N}$, $0 \le k \le n-1$, let Q_{n-k} be the integral operator corresponding to* (3.40)*, where X is a strong Markov, natural solution of the constrained martingale problem for* $(\mathbb{A}, \mathcal{W}_0, \mathbb{B}, \Xi_0)$ *starting at the origin; Q_{n-k} is uniquely defined by Lemma* 3.15*.*

Then, for every strong Markov, natural solution X of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}_0, \mathbb{B}, \Xi_0)$ *starting at* 0*,*

$$\mathbb{E}[f(X(\tau^{X,\delta}))] = \frac{\int Q_n \cdots Q_1 f(x)\nu_n(dx)}{\int Q_n \cdots Q_1 1(x)\nu_n(dx)}, \quad \forall f \in \mathcal{C}\big(\partial B_\delta\big), \quad \forall n \ge 1, \tag{3.42}$$

where ν_n is defined by

$$\nu_n(C) := \mathbb{P}\{X(\tau^n) \in B\}, \quad B \in \mathcal{B}\big(\partial B_{2^{-2n}\delta}\big), \quad n \ge 0, \tag{3.43}$$

and τ^n is as in (3.39)*.*

Proof The proof is the same as for Lemma 3.27 of Costantini and Kurtz [4]. It relies on the strong Markov property. □

Note that (3.42) reduces to (2.30) if

$$Q_n \cdots Q_1 1(x) = \mathbb{P}\big(\tau^\delta < \vartheta^n \big| X(\tau^n) = x\big) = 1, \quad \forall x \in \partial B_{2^{-2n}\delta},\ \forall n \ge 1,$$

as in the case of the example of Sect. 2.

The ergodic theorem that is used to prove both Theorem 2.3 and Theorem 3.6 is the following *reverse ergodic theorem for inhomogeneous killed Markov chains*, which is proved in Section 2 of Costantini and Kurtz [4].

Theorem 3.20 *Let $E_0, E_1, E_2, \ldots$ be a sequence of compact metric spaces and $Q_1, Q_2, \ldots$ be a sequence of subprobability transition kernels, with Q_l governing transitions from E_l to E_{l-1}. For $x, \widetilde{x} \in E_l$, let*

$$f_{l,\widetilde{x}}(x, y) := \frac{\mathrm{d}Q_l(x, \cdot)}{\mathrm{d}\big(Q_l(x, \cdot) + Q_l(\widetilde{x}, \cdot)\big)}(y) \tag{3.44}$$

and

$$\epsilon_l(x,\widetilde{x}) := \int \left(f_{l,\widetilde{x}}(x,y) \wedge f_{l,x}(\widetilde{x},y)\right)(Q_l(x,\mathrm{d}y) + Q_l(\widetilde{x},dy)). \tag{3.45}$$

Assume $\sup_x Q_l(x, E_{l-1}) > 0$*, for all* l*, and there exist* $c_0 > 0$ *and* $\epsilon_0 > 0$ *such that*

(i)

$$\inf_{x\in E_n} Q_n \cdots Q_1 1(x) \geq c_0 \sup_{x\in E_n} Q_n \cdots Q_1 1(x), \quad \forall n \geq 1,$$

(ii)

$$\inf_n \inf_{x,\widetilde{x}\in E_n} \epsilon_n(x,\widetilde{x}) \geq \epsilon_0.$$

Then

$$\sup_{x\in E_n} Q_n \cdots Q_1 1(x) > 0, \quad \forall k,$$

and, for any sequence $\{\nu_n\}$*,* ν_n *a probability measure on* E_n*,*

$$\lim_{n\to\infty} \frac{\int Q_n Q_{n-1}\cdots Q_1 f(x)\,\nu_n(\mathrm{d}x)}{\int Q_n Q_{n-1}\cdots Q_1 1(x)\,\nu_n(\mathrm{d}x)} = C(f), \qquad \forall f \in \mathcal{C}(E_0),$$

where $C(f)$ *is independent of the sequence* $\{\nu_n\}$.

Sections 3.8 and 3.9 show that, under Conditions 3.3 and 3.5, assumptions (i) and (ii), respectively, are verified by the transition kernels (3.40) for δ sufficiently small, so that Theorem 3.20 can be applied. Then the fact that, for any strong Markov, natural solution X of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}_0, \mathbb{B}, \Xi_0)$ starting at the origin, the hitting distribution on ∂B_δ is always the same will follow from Lemma 3.19.

Finally, the proof of Theorem 3.6 is summarized at the end of Sect. 3.9.

3.8 Uniform bound on hitting times

For the transition kernels (3.40), assumption (i) of Theorem 3.20 translates into the following uniform lower bound. Let X be a strong Markov, natural solution of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}_0, \mathbb{B}, \Xi_0)$ starting at the origin and let τ^δ, τ^n and ϑ^n be defined by (3.21), (3.39) and (3.41) (with $n-k$ replaced by n), respectively. Supposing

$$\mathbb{P}\left(\tau^{X,\delta} < \vartheta^n \middle| X(\tau^n) = x\right) > 0, \quad \forall x \in \overline{\mathcal{W}} \cap \partial B_{2^{-2n}\delta}, \ \forall n \geq 1,$$

there exists $c_0 > 0$ such that

$$\inf_{n\geq 1} \inf_{x,\widetilde{x}\in\overline{\mathcal{W}}\cap\partial B_{2^{-2n}\delta}} \frac{\mathbb{P}\left(\tau^{X,\delta} < \vartheta^n \middle| X(\tau^n) = x\right)}{\mathbb{P}\left(\tau^{X,\delta} < \vartheta^n \middle| X(\tau^n) = \widetilde{x}\right)} \geq c_0. \tag{3.46}$$

Recall that, by Lemma 3.15, the numerator in (3.46) is the same for all strong Markov, natural solutions of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}_0, \mathbb{B}, \Xi_0)$ starting at 0 and is the same as

$$\mathbb{P}\left(\tau^{X^x,\delta} < \vartheta^{X^x}\right),$$

where X^x is a solution starting at $x \in \overline{\mathcal{W}}$, $|x| = 2^{-2n}\delta$, and analogously for the denominator.

Lemma 3.21 (i) *If Condition 3.5 (i) is verified,*

$$\mathbb{P}\left(\tau^{X^x,\delta} < \vartheta^{X^x}\right) = 1, \qquad \forall x \in \overline{\mathcal{W}},\ 0 < |x| < \delta \leq \delta_{\mathcal{W}}. \tag{3.47}$$

(ii) *If Condition 3.5 (ii) is verified,*

$$\mathbb{P}\left(\tau^{X^x,\delta} < \vartheta^{X^x}\right) > 0, \qquad \forall x \in \overline{\mathcal{W}},\ 0 < |x| < \delta \leq \delta_{\mathcal{W}}. \tag{3.48}$$

Moreover, there exists $c_0 > 0$ such that, for all $0 < \delta \leq \delta_{\mathcal{W}}$,

$$\inf_{x,\widetilde{x}\in\overline{\mathcal{W}}:\, 0<|x|=|\widetilde{x}|<\delta} \frac{\mathbb{P}\left(\tau^{X^x,\delta} < \vartheta^{X^x}\right)}{\mathbb{P}\left(\tau^{X^{\widetilde{x}},\delta} < \vartheta^{X^x}\right)} \geq c_0. \tag{3.49}$$

Of course, if (3.47) holds (3.48) and (3.49) hold too.

Proof The proof of Lemma 3.29 of Costantini and Kurtz [4] carries over.

If Condition 3.5 (i) is verified, the assertion follows as in Sect. 2.5 by applying Ito's formula to the function V between times 0 and $\tau^{X,\epsilon} \wedge \tau^{X,\delta} \wedge t$, $0 < \epsilon < |x|$, and taking limits as $t \to \infty$ first, and $\epsilon \to 0$ next.

If Condition 3.5 (ii) is verified, the first assertion follows by applying Ito's formula to the functions V_+ an V_- between the same times as above and taking the same limits, but using (3.17). The second assertion then follows by (3.18). □

3.9 Uniform bound on hitting distributions of ∂B_δ and proof of Theorem 3.6

Both for the example of Sect. 2 and in the general case, in order to be able to apply Theorem 3.20 we still need to verify the lower bound (ii). This is done in the following two lemmas. Finally, at the end of this section, the proof of Theorem 3.6 is summarized.

Lemma 3.22 *For any sequence $\{x^n\} \subseteq \overline{\mathcal{W}}$ such that $\{2^{2n}x^n\}$ converges to some $\bar{x} \in \overline{\mathcal{W}} - \{0\}$, let X^{x^n} be a natural solution of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}, \mathbb{B}, \Xi)$ starting at x^n.*

Then the sequence of processes $\left\{2^{2n} X^{x^n}(2^{-4n}\cdot)\right\}$ *is relatively compact and any of its limit points,* $\bar{X}^{\bar{x}}$*, is a solution of* (3.6) *with* $b = 0$ *and* $\bar{X}^{\bar{x}}(0) = \bar{x}$*.*

In particular, for every open set $\mathcal{O}$ *such that* $\mathcal{O} \cap \mathcal{W} \cap \partial B_{2\delta}(0) \neq \emptyset$*, there exists a constant* $\eta_0 > 0$ *such that, for* $|x^n| = 2^{-2n}\delta$*,* $\{2^{2n}x^n\}$ *converging to* $\bar{x}$*, and* $\mathcal{O}^n := \{x : 2^{2n}x \in \mathcal{O}\}$*,*

$$\liminf_{n\to\infty} \mathbb{P}(\sigma^n < \vartheta^{X^{x^n}}, \quad X^{x^n}(\sigma^n) \in \mathcal{O}^n) \geq \eta_0, \tag{3.50}$$

where

$$\sigma^n := \inf\{t \geq 0 : |X^{x^n}(t)| = 2^{-2n+1}\delta\}.$$

Proof The first two assertions can be proved by the same time change and compactness arguments as in Lemma 4.5 and Theorem 4.1 of Costantini and Kurtz [2].

Let us prove the last assertion. As in Lemma 3.30 of Costantini and Kurtz [4], let $\psi : [0, t_0] \to \mathbb{R}^d$ be a continuous function such that

$$\psi(0) = \quad \bar{x}, \quad |\psi(s)| = |\bar{x}| = \delta, \ \psi(s) \in \mathcal{W}, \text{ for } 0 < s \leq \frac{t_0}{2}, \quad 2\psi\left(\frac{t_0}{2}\right) \in \mathcal{O} \cap \mathcal{W},$$
$$\frac{\psi(s)}{|\psi(s)|} = \frac{\psi\left(\frac{t_0}{2}\right)}{|\psi\left(\frac{t_0}{2}\right)|}, \quad |\psi(s)| = \delta\frac{2}{t_0}(t_0 - s) + \left(2\delta + \frac{\epsilon}{2}\right)\frac{2}{t_0}\left(s - \frac{t_0}{2}\right), \text{ for } \frac{t_0}{2} < s \leq t_0$$

and

$$\epsilon < \delta, \quad B_\epsilon\left(2\psi\left(\frac{t_0}{2}\right)\right) \subseteq \mathcal{O} \cap \mathcal{W}.$$

Then

$$\liminf_{n\to\infty} \mathbb{P}\big(\sigma^n < \vartheta^{X^{x^n}}, \quad X^{x^n}(\sigma^n) \in \mathcal{O}^n\big) \geq \liminf_{n\to\infty} \mathbb{P}\Big(\sup_{0\leq s\leq t_0} \big|X^{x^n}(s) - \psi(s)\big| < \frac{\epsilon}{2}\Big)$$
$$\geq \mathbb{P}\Big(\sup_{0\leq s\leq t_0} \big|\bar{X}^{\bar{x}}(s) - \psi(s)\big| < \frac{\epsilon}{2}\Big) = P^{\bar{X}^{\bar{x}}}\Big\{x \in \mathcal{C}_{[0,\infty)}(\overline{\mathcal{W}}) : \big|x(s) - \psi(s)\big| < \frac{\epsilon}{2}\Big\},$$

where $P^{\bar{X}^{\bar{x}}}$ is the law of $\bar{X}^{\bar{x}}$. Since $\delta \leq |\psi(s)|$ for all $0 \leq s \leq t_0$, the right-hand side equals

$$P^{\bar{X}^{\bar{x}}(\cdot\wedge\vartheta^{\bar{X}^{\bar{x}}})}\Big\{x \in \mathcal{C}_{[0,\infty)}(\overline{\mathcal{W}}) : \big|x(s) - \psi(s)\big| < \frac{\epsilon}{2}\Big\},$$

where $P^{\bar{X}^{\bar{x}}(\cdot\wedge\vartheta^{\bar{X}^{\bar{x}}})}$ is the law of $\bar{X}^{\bar{x}}(\cdot\wedge\vartheta^{\bar{X}^{\bar{x}}})$ and is uniquely determined by Lemma 3.15. In Lemma 3.30 of Costantini and Kurtz [4], the assertion then follows from Theorem 3.1 of Kwon and Williams [19]. Here that theorem cannot be applied immediately, because the cone is not smooth. However, a careful inspection of Theorem 3.1 of Kwon and Williams [19] shows that it extends to a cone with piecewise smooth boundary, provided that

$$\mathbb{E}\bigg[\int_0^{\vartheta^{\bar{X}^{\bar{x}}}} \mathbf{1}_{\partial\mathcal{W}-\{0\}}(\bar{X}^{\bar{x}}(t))\,\mathrm{d}t\bigg] = 0.$$

However,

$$\mathbb{E}\bigg[\int_0^{\vartheta^{\bar{X}^{\bar{x}}}} \mathbf{1}_{\partial\mathcal{W}-\{0\}}(\bar{X}^{\bar{x}}(t))\,\mathrm{d}t\bigg] = \lim_{n\to\infty} \mathbb{E}\bigg[\int_0^{\tau^{\bar{X}^{\bar{x}},\,\delta/n}} \mathbf{1}_{\partial\mathcal{W}}(\bar{X}^{\bar{x}}(t))\,\mathrm{d}t\bigg]$$
$$= \lim_{n\to\infty} \mathbb{E}\bigg[\int_0^{\tau^{\widetilde{X}^{n,\bar{x}},\,\delta/n}} \mathbf{1}_{\partial\mathcal{W}}(\widetilde{X}^{n,\bar{x}})(t))\,\mathrm{d}t\bigg],$$

where $\widetilde{X}^{n,\bar{x}}$ is the natural solution of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}_{\delta/(2n),4}, \mathbb{B}, \Xi_{\delta/(2n),4})$, with $b = 0$ and $\mathcal{W}_{\delta/(2n),4}$, $\Xi_{\delta/(2n),4}$ defined by (3.35), (3.36), (3.28), and the last term on the right-hand side is zero by Proposition 2.12 of Kang and Ramanan [13]. (Note that every solution of a constrained martingale problem is a solution of the corresponding submartingale problem.) □

Lemma 3.23 *The transition kernels* (3.40) *verify assumption (ii) of Theorem* 3.20*.*

Proof The proof is the same as for Lemma 3.31 of Costantini and Kurtz [4]. Let $x, \widetilde{x} \in \partial B_{2^{-2n}\delta}$. Lemma 5.3 of Costantini and Kurtz [2] and (3.50) allow to construct two strong Markov, natural solutions of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}_0, \mathbb{B}, \Xi_0)$, starting at x and $\widetilde{x}$, respectively, that couple before reaching $\partial B_{2^{-2(n-1)}\delta}$. This yields that

$$\|Q_n(x,\cdot) - Q_n(\widetilde{x},\cdot)\|_{TV} \le Q_n(x, \partial B_{2^{-2(n-1)}\delta}) \vee Q_n(\widetilde{x}, \partial B_{2^{-2(n-1)}\delta}) - \epsilon_0,$$

for some positive constant ϵ_0. On the other hand,

$$Q_n(x, \partial B_{2^{-2(n-1)}\delta}) \vee Q_n(\widetilde{x}, \partial B_{2^{-2(n-1)}\delta}) \le \|Q_n(x,\cdot) - Q_n(\widetilde{x},\cdot)\|_{TV} + \epsilon_n(x, \widetilde{x}).$$

□

Proof of Theorem 3.6 By Theorem 3.13 and Lemma 3.14, it is sufficient to prove uniqueness for natural solutions of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}_0, \mathbb{B}, \Xi_0)$. To this end, by Lemma 3.17 and Lemma 3.18, it is enough to show that the hitting distribution on ∂B_δ, $\mathcal{L}(X(\tau^\delta))$, is the same for all strong Markov, natural solutions of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}_0, \mathbb{B}, \Xi_0)$ starting at the origin. By Lemma 3.19, this will be proved if we can apply Theorem 3.20 to the transition kernels (3.40).

In order to be able to apply Theorem 3.20, one needs to verify the two assumptions (i) and (ii). Assumption (i) is verified by Lemma 3.21, while assumption (ii) is verified by Lemma 3.23, □

A Appendix

Proof of Theorem 3.13 The proof consists essentially in verifying the assumptions of Theorem 2.13 (ii) of Costantini and Kurtz [5].

Let

$$U_0 =: \overline{\mathcal{W}} \cap B_3, \qquad U_n := \overline{\mathcal{W}} \cap \left(\overline{B_{2+2(n-1)}}\right)^c \cap B_{5+2(n-1)}, \quad n \geq 1.$$

A natural solution of the stopped constrained martingale problem for $(\mathbb{A}, \mathcal{W}, \mathbb{B}, \Xi; U_n)$ is a solution that can be obtained by a solution of the stopped controlled martingale problem for $(\mathbb{A}, \mathcal{W}, \mathbb{B}, \Xi; U_n)$ by a time change analogous to that in 3.10, that is the process first is stopped and then time changed (see Definitions 2.1, 2.4 and 2.5 of Costantini and Kurtz [5]). The time change does not always produce a solution of the stopped constrained martingale problem; in fact, this is one of the requirements of Theorem 2.13 (ii) of Costantini and Kurtz [5]. Theorem 2.13 (ii) of Costantini and Kurtz [5] requires the following: For every solution $(Y, \lambda_0, \Lambda_1)$ of the controlled martingale problem for $(\mathbb{A}, \mathcal{W}, \mathbb{B}, \Xi)$, $\lambda_0(t) > 0$ for all $t > 0$; for every solution $(Y^{U_n}, \lambda_0^{U_n}, \Lambda_1^{U_n})$ of the stopped controlled martingale problem for $(\mathbb{A}, \mathcal{W}, \mathbb{B}, \Xi; U_n)$, $Y^{U_n} \circ \left(\lambda_0^{U_n}\right)^{-1}$ is a natural solution of the stopped constrained martingale problem for $(\mathbb{A}, \mathcal{W}, \mathbb{B}, \Xi; U_n)$; and uniqueness holds for natural solutions of the stopped constrained martingale problem for $(\mathbb{A}, \mathcal{W}, \mathbb{B}, \Xi; U_n)$. Under these conditions, uniqueness holds for natural solutions of the constrained martingale problem for $(\mathbb{A}, \mathcal{W}, \mathbb{B}, \Xi)$.

The first assumption is verified by Lemma 3.11.

As far as the second assumption is concerned, let $(Y^{U_n}, \lambda_0^{U_n}, \Lambda_1^{U_n})$ be a solution of the stopped controlled martingale problem for $(\mathbb{A}, \mathcal{W}, \mathbb{B}, \Xi; U_n)$ with initial distribution supported in $\overline{\mathcal{W}_n}$. Then it can be immediately verified that $(Y^{U_n}, \lambda_0^{U_n}, \Lambda_1^{U_n})$ is a solution of the stopped controlled martingale problem for $(\mathbb{A}, \mathcal{W}_n, \mathbb{B}, \Xi_n; U_n)$. By Lemma 3.11 and Lemmas 3.3, 3.4 and Corollary 3.9 b) of Costantini and Kurtz [3], $Y^{U_n} \circ \left(\lambda_0^{U_n}\right)^{-1}$ is a natural solution of the stopped constrained martingale problem for $(\mathbb{A}, \mathcal{W}_n, \mathbb{B}, \Xi_n; U_n)$ and hence a natural solution of the stopped constrained martingale problem for $(\mathbb{A}, \mathcal{W}, \mathbb{B}, \Xi; U_n)$. If $(Y^{U_n}, \lambda_0^{U_n}, \Lambda_1^{U_n})$ is a solution of the stopped controlled martingale problem for $(\mathbb{A}, \mathcal{W}, \mathbb{B}, \Xi; U_n)$ with initial distribution not supported in $\overline{\mathcal{W}_n}$, let

$$\begin{aligned} \widetilde{Y}^{U_n}(t) &:= Y^{U_n}(t)\mathbf{1}_{\overline{\mathcal{W}_n}}(Y^{U_n}(0)) + y_n^0 \mathbf{1}_{\overline{\mathcal{W}} - \overline{\mathcal{W}_n}}(Y^{U_n}(0)), \quad t \geq 0, \\ \left(\widetilde{\lambda}_0^{U_n}, \widetilde{\Lambda}_1^{U_n}\right) &:= \left(\lambda_0^{U_n}, \Lambda_1^{U_n}\right)\mathbf{1}_{\overline{\mathcal{W}_n}}(Y_n(0)), \end{aligned}$$

where y_n^0 is some fixed point in $\overline{\mathcal{W}_n}$. Then $(\widetilde{Y}^{U_n}, \widetilde{\lambda}_0^{U_n}, \widetilde{\Lambda}_1^{U_n})$ is a solution of the stopped controlled martingale problem for $(\mathbb{A}, \mathcal{W}_n, \mathbb{B}, \Xi_n; U_n)$ with initial distribution supported in $\overline{\mathcal{W}_n}$ and $\widetilde{Y}^{U_n} \circ \left(\widetilde{\lambda}_0^{U_n}\right)^{-1}$ is a natural solution of the stopped constrained martingale problem for $(\mathbb{A}, \mathcal{W}, \mathbb{B}, \Xi; U_n)$. Since $Y^{U_n} \circ \left(\lambda_0^{U_n}\right)^{-1}(t) = Y^{U_n}(0)$ for all $t \geq 0$ if $Y^{U_n}(0) \in \overline{\mathcal{W}} - \overline{\mathcal{W}_n}$, and $Y^{U_n} \circ \left(\lambda_0^{U_n}\right)^{-1}(t) = \widetilde{Y}^{U_n} \circ \left(\widetilde{\lambda}_0^{U_n}\right)^{-1}(t)$ for all $t \geq 0$

if $Y^{U_n}(0) \in \overline{\mathcal{W}_n}$, $Y^{U_n} \circ \big(\lambda_0^{U_n}\big)^{-1}$ is indeed a natural solution of the stopped constrained martingale problem for $(\mathbb{A}, \mathcal{W}, \mathbb{B}, \Xi; U_n)$.

Finally, for every natural solution X^{U_n} of the stopped constrained martingale problem for $(\mathbb{A}, \mathcal{W}, \mathbb{B}, \Xi; U_n)$, setting

$$\widetilde{X}^{U_n}(t) = X^{U_n}(t)\mathbf{1}_{\overline{\mathcal{W}_n}}(X^{U_n}(0)) + x_n^0\mathbf{1}_{\overline{\mathcal{W}} - \overline{\mathcal{W}_n}}(X^{U_n}(0)), \quad t \geq 0,$$

where x_n^0 is a fixed point in $\overline{\mathcal{W}_n}$, the distribution of X^{U_n} is uniquely determined by its initial distribution and by the distribution of $\widetilde{X}^{U_n}$ and $\widetilde{X}^{U_n}$ is a natural solution of the stopped constrained martingale problem for $(\mathbb{A}, \mathcal{W}_n, \mathbb{B}, \Xi_n; U_n)$. On the other hand, Lemma 3.11 and Proposition 2.9 of Costantini and Kurtz [5] yield that $(\mathbb{A}, \mathcal{W}_n, \mathbb{B}, \Xi_n)$ and U_n verify Condition 2.7 of Costantini and Kurtz [5]. Therefore, by Corollary 2.12 of Costantini and Kurtz [5], uniqueness holds for natural solutions of the stopped constrained martingale problem for $(\mathbb{A}, \mathcal{W}_n, \mathbb{B}, \Xi_n; U_n)$. □

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