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Merging of coherent upper conditional probabilities defined by Hausdorff outer measures

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ABSTRACT

Coherent upper conditional probabilities defined by Hausdorff measures on metric spaces are proven to represent merging opinions with increasing information when the metrics are bi-Lipschitz equivalent .

1. Introduction

Mutual absolute continuity is a property that two probability measures can have with respect to each other. Two probability measures, P and Q, defined on the same probability space are mutually absolutely continuous if the probability of an event according to P is zero, then the probability of the same event according to Q is also zero, and vice versa. Events with zero probability can be interpreted as unexpected events and can be represented by fractal sets, i.e. sets with a non-integer dimension which is less than the dimension of the probability space. Fractal sets are mathematical constructs with self-similarity at different scales, and they can be employed to model complex and irregular patterns. Absolute continuity of conditional probability measures plays a key role in reaching Bayesian consensus so in this paper, the role of fractal sets is highlighted. These sets are used to represent unexpected complex events in updating the partial knowledge of individuals and in achieving Bayesian consensus. Bayesian consensus is a concept rooted in Bayesian probability theory and decision theory. In the context of consensus, it refers to a process of combining individual beliefs or opinions to arrive at a group or collective decision in a rational and probabilistic manner. Bayesian methods are particularly useful when dealing with uncertainty and incomplete information. Each participant starts with their own beliefs, expressed as a probability distribution over possible outcomes or hypotheses. These initial beliefs are represented by a prior probability distribution. It reflects the individual's uncertainty about the true state of affairs before considering any new information. As new evidence or information becomes available, individuals update their beliefs using Bayes' theorem. Bayes' theorem allows the incorporation of new evidence into the existing beliefs to form a posterior probability distribution. The process of updating beliefs and reaching a consensus can be iterative, especially if more evidence becomes available over time. Participants continue to adjust their beliefs based on the most recent information. The Bayesian approach to consensus has several advantages. It allows for a formal and principled way to update beliefs in the face of new evidence. Additionally, it provides a clear framework for incorporating uncertainty and quantifying the degree of belief in different outcomes. The merging of opinions, often referred to as opinion aggregation or consensus building, involves combining the individual viewpoints or preferences of a group of people to form a collective decision or judgment. Applications of Bayesian consensus can be found in various fields, including statistics, artificial intelligence, decision analysis, and even in social sciences where opinions or judgments from different individuals need to be combined in a systematic and principled way. In [1], it is established that the distance between two conditional probabilities, denoted as $P(\cdot|G_n)$ and $Q(\cdot|G_n)$, defined on the same σ-field, converges to zero, except on a Q-probability zero set, under the condition that P and Q are mutually absolutely continuous. This result, grounded in the martingale convergence theorem, indicates a convergence or merging of opinions as information increases. The concept of weak merging has been explored in [2], while consensus among Bayesian decision-makers has been investigated in [3,4]. Absolute continuity has been examined in ergodic theory [5] and learning in a stationary process has been studied in [6]. This paper delves into the analysis of whether a similar result holds for a novel model of coherent upper conditional previsions defined in metric spaces using Hausdorff outer measures, introduced

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to represent partial knowledge [7–11]. Here, the conditioning event signifies the available information, and the complexity of information is expressed through the Hausdorff dimension of the conditioning event because the Hausdorff dimension of a set B is an aggregation operator since it is the Sugeno integral of $h^x(B)$ as function of x with respect to the Lebesgue measure as proven in [12]. So for any set B such that the Hausdorff dimension is known, the Sugeno integral of the Hausdorff measure $h^x(B)$ as function of x with respect to the Lebesgue measure can be compute. Examples are given in [13]. The central question revolves around understanding the relationship between partial knowledge generated in different metric spaces based on the same information. Hausdorff dimensions were introduced in probability theory [14] to calculate the dimensions of sets where the strong law of large numbers is violated in a Markov chain. The measures of subsets with respect to different metrics can vary significantly, and the same subset may have different Hausdorff dimensions. If the metrics are bi-Lipschitz equivalent, then a set possesses the same Hausdorff dimension in the two metric spaces, and the Hausdorff measures are proven to be mutually absolutely continuous.

This paper establishes that given a metric space (Ω, d), where Ω is a set with a positive and finite Hausdorff outer measure in its Hausdorff dimension, and given any metric d' bi-Lipschitz equivalent to d, then the coherent upper conditional probabilities defined by Hausdorff outer measures on (Ω, d) and on (Ω, d') are mutually absolutely continuous. Given a set B with a positive and finite Hausdorff outer measure in its Hausdorff dimension in the metric space (Ω, d), the class $\mathbf{K}_B$ of coherent upper conditional probabilities defined by Hausdorff outer measures on (Ω, d') is considered, and they are shown to be mutually absolutely continuous with respect to the upper conditional probability defined by Hausdorff outer measures with respect to the metric d.

In summary, this paper establishes that the distance between two coherent upper conditional previsions defined by Hausdorff outer measures, with respect to bi-Lipschitz equivalent metrics, converges to zero as the information increases. Two conditional probabilities are considered mutually absolutely continuous if they share the same null sets, i.e., $P(A|G_n) = 0 \Leftrightarrow Q(A|G_n)$. It is crucial to note that the concept of null sets, representing sets with zero probability, is integral to the support of a probability measure, and the existence of the support depends on the properties of the underlying topological space. Specifically, if the topological space is not second countable, meaning there is no countable collection $\mathcal{V}$ of open sets such that any open subset of the topological space can be expressed as a union of elements of $\mathcal{V}$, then the support may not exist. However, in the case of a separable metric space, which possesses a countable dense subset, it is second countable.

In general, across topological spaces or metric spaces, the support of a measure may or may not exist. One advantageous aspect of defining coherent upper conditional probability in a metric space using Hausdorff outer measures is that if the metric space is separable, the support always exists. Conversely, if the metric space is not separable, then the Hausdorff outer measure of any sets is infinite, and coherent upper conditional probabilities are defined, according to the proposed model, by a 0–1 valued finitely additive but not countably additive probability. This ensures that events with zero probability are consistently defined when representing opinions in a metric space using coherent upper conditional probabilities.

The model considered in the paper and based on Hausdorff outer measures is able to represent unexpected events which are represented by sets with zero probability. The advantage to define coherent upper conditional probability in a metric space by Hausdorff outer measures is that if the metric space is separable the support always exists and if the metric space is not separable then the Hausdorff outer measure of any set is infinity. So in any case the events with zero probability can be determined. We can also observe that the more general models proposed by Walley [15], may be not able to represent events with zero probability different from the empty set as it occurs for the vacuous upper and lower probabilities, defined by $\overline{P}(A) = max(I_A)$ and $\underline{P}(A) = min(I_A)$.

It important to note the role of the Hausdorff outer measures to prove the convergence of the opinions; all findings given in the paper hold because coherent conditional upper bounded defined with respect to Hausdorff outer measures satisfy the disintegration property on every Borel partition. It occurs because Hausdorff outer measures are metric outer measures and so all Borelian sets are measurable with respect to Hausdorff outer measures. Disintegration property does not hold for a general model of upper conditional prevision.

2. Coherent conditional upper bounds

Let $\mathbf{B}$ be a partition of a non-empty set Ω and let $\wp(\Omega)$, the class of all subsets of Ω. A random variable is a function $X : \Omega \to \hat{\Re} = \Re \cup \{-\infty, +\infty\}$ and the class of all random variables is denoted by $\mathcal{R}(\Omega)$ and it is not a linear space; in fact, if random variables take values $-\infty, +\infty$ then the sum between two of them can be not defined (when for the same ω one takes value $+\infty$ and the others $-\infty$).

Denote by $L(\Omega) \subset \mathcal{R}(\Omega)$ be the linear space of all bounded random variables defined on Ω; for every $B \in \mathbf{B}$ denote by $X|B$ the restriction of X to B and by $\sup(X|B)$ the supremum of values that X assumes on B. Let $L(B)$ be the linear space of all bounded random variables $X|B$. Denote by I_A the indicator function of any event $A \in \wp(B)$, i.e. $I_A(\omega) = 1$ if $\omega \in A$ and $I_A(\omega) = 0$ if $\omega \in A^c$. For every $B \in \mathbf{B}$ let $\mathcal{K}(B)$ be a linear space of random variables $X|B$ with $X \in \mathcal{R}(\Omega)$. Coherent conditional upper bounds $\overline{CCP}(\cdot|B)$ are real valued functionals defined on a linear space $\mathcal{K}(B)$.

Definition 1. Coherent conditional upper bounds are functionals $\overline{CCP}(\cdot|B)$ defined on a linear space $\mathcal{K}(B)$ with values in the real number, such that the following axioms of coherence hold for every X and Y in $\mathcal{K}(B)$ and every strictly positive constant λ:

(1) $\overline{CCP}(X|B) \leq \sup(X|B)$;
(2) $\overline{CCP}(\lambda X|B) = \lambda \overline{CCP}(X|B)$ (positive homogeneity);
(3) $\overline{CCP}(X + Y|B) \leq \overline{CCP}(X|B) + \overline{CCP}(Y|B)$ (subadditivity).

Definition 1 is the definition of coherent upper conditional prevision given in Walley [15,16] if $\mathcal{K}(B)$ coincides with $L(B)$. The conjugate coherent conditional lower bound of a coherent conditional upper bound $\overline{CCP}(X|B)$ on $\mathcal{K}(B)$ is defined by the conjugacy property $\underline{CCP}(X|B) = -\overline{CCP}(-X|B)$. If $CCP(X|B) = \underline{CCP}(X|B) = \overline{CCP}(X|B)$ for every X belonging to $\mathcal{K}(B)$ then $CCP(X|B)$ is called a coherent *linear* conditional prevision and if $\mathcal{K}(B) = L(B)$ then $CCP(X|B)$ is a linear, positive and positively homogeneous functional ([17–20] [15, Corollary 2.8.5]).

From axioms (1)–(3) and by the conjugacy property we have that

$$\underline{CCP}(I_B|B) = \overline{CCP}(I_B|B) = 1$$

In Walley [15] the restrictions of the functionals $\overline{CCP}(X|B)$ defined for $B \in \mathbf{B}$ and $X \in L(B)$ satisfying axioms (1)–(3) and such that $\overline{CCP}(I_B|B) = 1$ are called *separately coherent*.

The unconditional coherent upper bounded, denoted as $\overline{CCP} = \overline{CCP}(\cdot|\Omega)$ emerges as a specific instance when the conditioning event is Ω. Coherent upper conditional probabilities are specifically derived when considering only 0–1 valued random variables.

Definition 2. Given a partition $\mathbf{B}$ and a random variable $X \in L(\Omega)$, a coherent conditional upper bound $\overline{CCP}(X|\mathbf{B})$ is introduced as a random variable on Ω taking the value equal to $\overline{CCP}(X|B)$ if ω belongs to the element B of the partition.

Definition 3. A bounded random variable $X \in L(\Omega)$ is designates ***B**-measurable* (or, measurable with respect to a partition $\mathbf{B}$ of Ω) if it is constant on the atoms of the partition.

The following necessary condition for coherence holds [15, p. 292]:

Proposition 1. *If $CCP(X|B)$ is a coherent linear prevision for every B that belongs to a partition $\mathbf{B}$ of Ω then $CCP(X|\mathbf{B}) = X$ for all random variables $X \in L(\Omega)$ that are $\mathbf{B}$-measurable.*

2.1. Coherent conditional upper bounds defined with respect to hausdorff outer measures

In the axiomatic approach, as outlined in Section 34 of [21], the concept of conditional expectation is established in relation to a σ-field representing conditioning events, denoted as $\mathbf{G}$, through the Radon–Nikodym derivative. In [22] it is demonstrated that when the σ-field $\mathbf{G}$ is properly encompassed within the σ-field of the probability space and includes all individual points in the interval $[0, 1]$, the conditional expectation determined by the Radon–Nikodym derivative lacks coherence.

This lack of coherence arises due to a conflict with one of the fundamental properties of the Radon–Nikodym derivative, which necessitates measurability concerning the σ-field of conditioning events. This requirement contradicts the essential condition for the coherence of a linear conditional prevision, as reiterated in Proposition 1.

A new model of coherent upper conditional probability based on Hausdorff outer measures on a metric space has been introduced for bounded and unbounded random variables [11].

Hausdorff outer measures [23] [24] are examples of outer measures defined on a metric space.

In the context of a metric space (Ω, d) with the induced topology $\mathcal{T}$ generated by the metric d, open sets in the topology are defined to be the empty set and countable or finite unions of sets $D_r(x) = \left\{\omega \in \Omega : d(\omega, x) < r\right\}$, where $r \geq 0$ and $x \in \Omega$. This topology is the basis for defining the Borel σ-field $\mathcal{B}$, which is the smallest σ-field containing all open sets of Ω.

The diameter of a non-empty set U in Ω is denoted by $|U| = \sup\left\{d(x, y) : x, y \in U\right\}$. A subset A of Ω is considered to have a δ-cover $\left\{U_i\right\}$ if $A \subseteq \bigcup_i U_i$ and $0 \leq |U_i| < \delta$ for each i.

For $s \geq 0$ and $\delta > 0$, the expression $h_{s,\delta}(A) = \inf \sum_{i=1}^{\infty} |U_i|^s$ is defined, where the infimum is taken over all δ-covers $\left\{U_i\right\}$; the Hausdorff s-dimensional outer measure of A, denoted by $h^s(A)$. is obtained by :

$$h^s(A) = \lim_{\delta \to 0} h_{s,\delta}(A)..$$

This limit exists but may be infinite since $h_{s,\delta}(A)$ increases as δ decreases.

A subset F of Ω is considered measurable with respect to the outer measure h^s defined on $\wp(\Omega)$ if it decomposes every subset of Ω additively, meaning:

$$h^s(E) = h^s(F \cap E) + h^s(F^c \cap E)$$

for all sets $E \subseteq \Omega$.

The property of being a metric outer measure ensures that if sets E and F are positively separated (i.e., $d(E, F) = \inf\left\{d(x, y) : x \in E, y \in F\right\} > 0$), then:

$$h^s(E \cup F) = h^s(F) + h^s(F)$$

According to Falconer's Theorem 1.5 [24], as Hausdorff outer measures are metric outer measures, all Borel subsets of Ω are measurable.

For any set E, the Hausdorff outer measure $h^s(E)$ is non-increasing as s increases from 0 to $+\infty$. The Hausdorff dimension of a set A, denoted $dim_H(A)$, is defined as the unique value such that:

$$h^s(A) = \infty \text{ if } 0 \leq s < dim_H(A),$$

$$h^s(A) = 0 \text{ if } dim_H(A) < s < \infty.$$

Two distinct notions of equivalence can be applied to metrics: Bi-Lipschitz equivalence [25] and topological equivalence.

Definition 4. Given a metric space (Ω, d), a metric d' on Ω is bi-Lipschitz equivalent to the metric d if there exist two positive real constants α, β such that

$$\alpha d'(x, y) \leq d(x, y) \leq \beta d'(x, y)$$

Definition 5. Given a metric space (Ω, d) and a metric d' on Ω then d and d' are topological equivalent if they induce the same topology.

Proposition 2. *Let (Ω, d) be a metric space and d' be a metric on Ω bi-Lipschitz equivalent to d. Then d and d' are topological equivalent.*

The following example shows that the converse is not true.

Example 1. Let $(\Re^n, d)$ be the Euclidean metric space and let d' be a metric defined $\forall \overline{x}, \overline{y} \in \Re^n$ as follows

$$d'(\overline{x}, \overline{y}) = \frac{d(\overline{x}, \overline{y})}{1 + d(\overline{x}, \overline{y})}..$$

Then d' is topological equivalent to the Euclidean metric d, but it is not bi-Lipschitz equivalent to d since there are not two positive real constants α, β such that $\alpha d'(x, y) \leq d(x, y) \leq \beta d'(x, y)$

In [7] the following result has been proved.

Theorem 1. *Let (Ω, d) be a metric space and let $\mathbf{B}$ be a partition of Ω. For $B \in \mathbf{B}$ denote by s the Hausdorff dimension of the conditioning event B and by h^s the Hausdorff s-dimensional outer measure. Let m_B be a 0–1 valued finitely additive, but not countably additive, probability on $\wp(B)$. Thus, for each $B \in \mathbf{B}$, the function defined on $\wp(B)$ by*

$$\overline{\mu}(A|B) = \begin{cases} \frac{h^s(A\cap B)}{h^s(B)} & if \quad 0 < h^s(B) < +\infty \\ m_B(A \cap B) & if \quad h^s(B) \in \{0, +\infty\} \end{cases}$$

is a coherent upper conditional probability.

If $B \in \mathbf{B}$ is a set with a positive and finite Hausdorff outer measure, the coherent upper conditional probability exhibits submodularity and continuity from below. Additionally, its limitation to the category of all measurable sets conforms to being a Borel regular, countably additive probability.

If $B \in \mathbf{B}$ is such that $h^s(B) \in \{0, +\infty\}$ then the coherent upper conditional probability is defined by a 0–1 valued finitely additive, but not countably additive, probability m_B on $\wp(B)$. The existence of m_B is a consequence of the prime ideal theorem and any m_B is coherent. 0–1 valued finitely additive probabilities are in correspondence one-to-one with ultrafilters.

If the conditioning set B is characterized by a Hausdorff outer measure equal to zero or infinity, the coherent upper conditional probability is established as a finite, 0–1 valued measure that is finitely, rather than countably, additive. This ensures that, under this condition, the confinement of the conditional probability to the Borel σ-field constitutes a complete conditional probability according to Dubins' definition [26]. Specifically, it adheres to the comprehensive compound rule for all Borelian sets A, B, C.

$$P(A \cap B|C) = P(A|B \cap C)P(B|C)..$$

The class of all absolutely Choquet integrable random variables [27, 28] on B, i.e. the random variables X such that

$$-\infty < \frac{1}{h^s(\Omega)} \int_B |X| dh^s < +\infty \text{ if } 0 < h^s(B) < +\infty$$

is a linear space [29] denoted by $L^*(B)$ In [11] the following theorem has been proven:

Theorem 2. *Let (Ω, d) be a metric space and let $\mathbf{B}$ be a partition of Ω. For $B \in \mathbf{B}$ denote by s the Hausdorff dimension of the conditioning event B and by h^s the Hausdorff s-dimensional outer measure. Let m_B be a 0–1 valued* *finitely additive, but not countably additive, probability on $\wp(B)$. Then for each $B \in \mathbf{B}$ the functional $\overline{CCP}(X|B)$ defined on the linear space $L^*(B)$ by*

$$\overline{CCP}(X|B) = \begin{cases} \frac{1}{h^s(B)} \int_B X dh^s & if \quad 0 < h^s(B) < +\infty \\ \int_B X dm_B & if \quad h^s(B) \in \{0, +\infty\} \end{cases}$$

is a coherent conditional upper bound if B has positive and finite Hausdorff measure in its Hausdorff dimension. Moreover it is a linear prevision whose restriction to events assumes only the values $0-1$ if B has Hausdorff outer measure in its Hausdorff dimension equal to zero or infinity.

In Theorem 7 of [11] it is proven that coherent conditional upper bounds defined by Hausdorff outer measures as in Theorem 2 satisfy the disintegration property

$$\overline{CCP}(\overline{CCP}(X|\mathbf{B})) = \overline{CCP}(X)$$

for every random variable $X \in L^*(\Omega)$ and for every partition, whose atoms are h^s-measurable where s is the Hausdorff dimension of Ω.

3. Absolute continuity of coherent upper conditional probability measures defined by hausdorff outer measures

In this section, we introduce the concept of absolute continuity for coherent upper conditional probabilities. We demonstrate that coherent upper conditional probabilities, defined on a metric space (Ω, d) through Hausdorff outer measures as presented in Theorem 1, are absolutely continuous concerning any coherent upper conditional probability defined by Theorem 1 in a metric space (Ω, d'), where d' is a bounded metric that is bi-Lipschitz equivalent to the metric d. This phenomenon arises due to the fact that events with zero Hausdorff measure in a metric space also possess Hausdorff measure equal to zero in a metric space with a bi-Lipschitz equivalent metric.

Theorem 3. *Let (Ω, d) be a metric space, let d and d' be two metrics on Ω bi-Lipschitz equivalent and let h^s and h_1^s be the s-dimensional Hausdorff measures defined respectively in the metric space (Ω, d) and (Ω, d'). Then there exist two positive real constants α, β such that*

$$\alpha h_1^s(E) \leq h^s(E) \leq \beta h_1^s(E)$$

Proof. The result follows by the definition of Hausdorff outer measures and by the fact that the metrics are bi-Lipschitz equivalent (see Lemma 1.8 of [24]).

Theorem 4. *Let (Ω, d) be a metric space and let d' be a metric on Ω bi-Lipschitz equivalent to d. Then the Hausdorff dimension of any set $A \in \wp(\Omega)$ is invariant in the two metric spaces (Ω, d) and (Ω, d').*

The Hausdorff dimension of any set $A \in \wp(\Omega)$ is not invariant with respect to two topological equivalent metrics which are not bi-Lipschitz equivalent.

Example 2. Let $\Omega = [0, 1]$ and let d be the Euclidean metric

$$d(\omega_1, \omega_2) = |\omega_1 - \omega_2|$$

and let d' the *discrete distance*,i.e.

$$d'(\omega_1, \omega_2) = \begin{cases} 0 & if \quad \omega_1 = \omega_2 \\ 1 & \quad otherwise. \end{cases}$$

Clearly d and d' are not topologically equivalent; in fact all subsets of Ω are open sets in the topology induced by d' since $D_r(x) = \{\omega \in \Omega : d(\omega, x) < r\} = \{x\}$ if $r < 1$ and $D_r(x) = \{\omega \in \Omega : d(\omega, x) < r\} = \Omega$ if $r \geq 1$, while singletons are not open sets in the topology induced by the Euclidean metric.

Example 3. Let $(\Re^2, d)$ be the Euclidean metric space and let d'' be the metric defined by

$$d''(x, y) = max\{|x_1 - y_1|; |x_2 - y_2|\}.$$

Then d and d'' are topologically equivalent.

The notion of boundedness of a set depends on the metric.

Definition 6. A metric on Ω is bounded if $diam(\Omega)$ is bounded. A metric space (Ω, d) is bounded if d is bounded.

Proposition 3. *Let (Ω, d) be a metric space and suppose Ω is a set with Hausdorff outer measure, in its Hausdorff dimension, positive and finite. Then (Ω, d) be a bounded metric space.*

Definition 7. Let μ and ν be two coherent upper probabilities measures on the same σ-field $\mathcal{F}$. Then ν is absolutely continuous with respect to μ, ($\mu \ll \nu$, for short) if $\mu(A) = 0 \Rightarrow \nu(A) = 0$ for every $A \in \mathcal{F}$.

The vacuous coherent upper prevision $\overline{CCP}(X) = sup\{I_A(\omega) : \omega \in \Omega\}$ is not absolutely continuous with respect to any coherent upper conditional prevision.

Theorem 5. *Let (Ω, d) be a bounded metric space where Ω is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension. Consider $\mathbf{B}$, a partition of Ω, and let d' be a bounded metric on Ω that is bi-Lipschitz equivalent to d. For every $B \in \mathbf{B}$ with positive and finite Hausdorff outer measures in its dimensions in both metric spaces, the coherent upper conditional probabilities μ_B and ν_B, defined respectively in (Ω, d) and (Ω, d') as in Theorem 1, are mutually absolutely continuous.*

Proof. The given statement implies that since d' is a bounded metric, according to Theorem 3, Ω has a positive and finite Hausdorff measure in its Hausdorff dimension, also in the metric space (Ω, d'). Let s be the Hausdorff dimension of B, let h^s and h_1^s be the s-dimensional Hausdorff measure in the two metric spaces and let μ_B and ν_B be the two upper conditional probabilities on $\wp(B)$ defined by

$$\mu_B(A) = \frac{h^s(A \cap B)}{h^s(B)} \text{ and } \nu_B(A) = \frac{h_1^s(A \cap B)}{h_1^s(B)}.$$

Since d' is bi-Lipschitz equivalent to d by Theorem 3 we have that there exist two positive real constants α and β such that

$$\alpha\nu(A) = \alpha\frac{h_1^s(A)}{h^s(\Omega)} \leq \mu(A) = \frac{h^s(A)}{h^s(\Omega)} \leq \beta\frac{h_1^s(A)}{h_1^s(\Omega)} = \beta\nu(A)$$

so that $\nu(A) = 0$ implies $\mu(A) = 0$ and $\mu(A) = 0$ implies $\nu(A) = 0$.

4. Credal sets of coherent countably additive conditional probabilities defined by hausdorff measures with respect to bounded bi-lipschitz equivalent metrics

In this section, we establish a proof demonstrating that the distance between coherent conditional probabilities, defined by Hausdorff measures with respect to metrics that are bi-Lipschitz equivalent, converges to zero as the amount of information increases. Consequently, the credal set, as defined in [30], encompassing all these coherent conditional probabilities, symbolizes opinions that converge or merge with the accumulation of information, aligning with the concept introduced in [1].

Definition 8. Let (Ω, d) and (Ω, d_i) be two metric spaces and let $\mathbf{B}$ be a partition of Ω. Let $B \in \mathbf{B}$ be a set with positive and finite Hausdorff outer measures in its dimensions in both metric spaces and denote by μ_B and ν_B^i the coherent conditional probabilities defined on the Borel σ-field $\mathcal{B}$ by Theorem 1 in the two metric spaces. The distance between μ_B and ν_B^i is defined by

$$\sup|\mu_B(D) - \nu_B^i(D)|$$

where the supremum is taken over $D \in \mathcal{B}$

In the paper of Blackwell and Dubins [1] it is shown that, given a monotone increasing or monotone decreasing sequence of σ-fields $\{\mathbf{G}_n\}$, the distance between two conditional probabilities defined in the axiomatic way on the same σ-field, $P(\cdot|\mathbf{G}_n)$ and $Q(\cdot|\mathbf{G}_n)$, goes to zero as n goes to $+\infty$ except on a Q-probability zero set, if Q is absolutely continuous with respect to P. This result is an application of the Radon–Nikodym derivative and the generalized martingale convergence theorem. In the next section martingales with respect to coherent conditional (upper) bounds defined by Hausdorff outer measures are introduced.

4.1. Martingales with respect to coherent conditional upper bounds defined by hausdorff outer measures

Martingales are defined in the context where coherent conditional bounds are established using Hausdorff measures rather than the Radon–Nikodym derivative. Several generalized martingale convergence theorems are proven within this framework.

Consider the σ-field $\mathbf{F}$ generated by a finite or countable partition $\mathbf{B}$ of Ω. This σ-field contains sets that are finite or countable unions of the atoms of the partition. It is the smallest σ-field that encompasses the partition $\mathbf{B}$. In this setting, the coherent conditional upper bound $\overline{CCP}(X|\mathbf{B})$ is a random variable defined on Ω that associates with each $\omega \in \Omega$ the value $\overline{CCP}(X|\mathbf{B}) = \overline{CCP}(X|B)$ if ω belongs to B.

Definition 9. Let (Ω, d) be a metric space, where Ω is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension s. Let $\{\mathbf{B}_n\}$ be a sequence of Borel finite or countable partitions of Ω and let $\mathbf{F}_n$ be the σ-field generated by $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_n$. We have that $\mathbf{F}_n \subseteq \mathbf{F}_{n+1}$ for all $n \in N$ and if $\mathcal{F} = \bigcup_n \mathbf{F}_n$ is the σ-field generated by all $\mathbf{F}_n$ then $\mathcal{F} = \mathcal{B}$. Let $X_1, X_2, \ldots$ be a sequence random variables in $L^*(\Omega)$. The sequence $\{(X_n, \mathbf{F}_n) : n = 1, 2, \ldots\}$ is a martingale if

$$\overline{CCP}(X_{n+1}|\mathbf{F}_n) = X_n..$$

Example 4. Let $Z \in L^*(\Omega)$ and $\mathbf{F}_n$ non-decreasing Borel σ-fields. Then

$$\{(X_n, \mathbf{F}_n) : n = 1, 2, \ldots\} = \left\{\overline{CCP}(Z|\mathbf{F}_n), n = 1, 2, \ldots\right\}$$

is a martingale relative to $\{\mathbf{F}_n.n = 1, 2, \ldots\}$. In fact, since $\mathbf{F}_n \subset \mathbf{F}_{n+1}$ and $\overline{CCP}$ satisfies the disintegration property on every Borel partition, we have

$$\overline{CCP}(X_{n+1}|\mathbf{F}_n) = \overline{CCP}(\overline{CCP}(Z|\mathbf{F}_{n+1})|\mathbf{F}_n) = \overline{CCP}(Z|\mathbf{F}_n) = X_n.$$

Remark 1. The difference with the axiomatic definition of martingales (see for example [21, Section 35]) is that in Definition 9 the random variables X_n are not required to be measurable with respect to the σ-field of the conditioning events $\mathbf{F}_n$. It occurs because coherent upper conditional probabilities are defined on $\wp(\Omega)$ so no measurability condition is required for a random variable.

4.2. Merging for coherent upper conditional probabilities defined by hausdorff outer measures

In this section we investigate if coherent upper conditional probabilities assigned by Hausdorff outer measures in different metric spaces, whose metrics are bi-Lipschitz, merge with each other. The following results hold since Hausdorff outer measures are metric outer measures and so all Borelian sets are measurable with respect to Hausdorff outer measures.

Denote by $H_n(\omega)$ the atom of the σ-field $\mathbf{F}_n$ containing ω.

Definition 10. Let (Ω, d) and (Ω, d_i) be two metric spaces where Ω is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension s. Let $\{\mathbf{B}_n\}$ be a sequence of Borel finite or countable partitions of Ω and let $\mathbf{F}_n$ be the σ-field generated by $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_n$. Let $H_n \in \mathbf{F}_n$ be a set with positive and finite s-Hausdorff outer measures in both metric spaces and denote by μ_{H_n} and $\nu^i_{H_n}$ the coherent conditional probabilities defined on $\mathcal{B}$ by Theorem 2 in the two metric spaces. Then μ_{H_n} merges to $\mu^i_{H_n}$ along $\{\mathbf{F}_n\}_{n=1}^{\infty}$ if for all $\epsilon > 0$ there exists $N = N(\epsilon, \omega)$ such that for all $n > N$ such that $H_n \in \mathbf{F}_n$ is a set with positive and finite s-Hausdorff outer measures in both metric spaces and all $\omega \in \Omega$

$$\left|\mu(A|H_n(\omega)) - \mu^i(A|H_n(\omega))\right| < \epsilon \text{ for all } A \in \mathcal{B}.$$

Next theorem shows that the Doob's martingale convergence theorem holds for martingales defined by coherent conditional upper bounds as in Theorem 2

Suppose that $\mathbf{F}_n$ are σ-fields satisfying $\mathbf{F}_1 \subset \mathbf{F}_2 \subset \ldots \subset \mathbf{F}_n$. If the union $\bigcup_{n=1}^{\infty} \mathbf{F}_n$ generates the σ-field $\mathbf{F}_\infty$, this is expressed by $\mathbf{F}_n \uparrow \mathbf{F}_\infty$.

In the sequel we prove that

- $X_n = \overline{CCP}(Z|\mathbf{F}_n)$ are uniformly integrable (Theorem 6)
- $X_n = \overline{CCP}(Z|\mathbf{F}_n)$ converges to X (Theorem 7)
- if X_n converges to X then by the uniform integrability we prove that $\int_H X d\mu_\Omega = \int_H Z d\mu_\Omega = \int_H \overline{CCP}(Z|\mathbf{F}_\infty) d\mu_\Omega$ for all atoms H of $\mathbf{F}_\infty$ (Theorem 8).

Definition 11. A sequence $\{X_n\}$ is uniformly integrable if

$$\lim_{\alpha\to\infty} \sup_n \int_{|X_n|\geq\alpha} |X_n| d\mu_\Omega = 0..$$

Theorem 6. *If Z is a random variable in $L^*(\Omega)$, (i.e. Z is a Choquet integrable random variable) and $\mathbf{F}_n$ are non-decreasing Borel-σ-fields then the random variables $\overline{CCP}(Z|\mathbf{F}_n)$ are uniformly integrable.*

Proof. Since $|\overline{CCP}(Z|\mathbf{F}_n)| \leq \overline{CCP}(|Z||\mathbf{F}_n)$, Z may be assumed nonnegative. Let $A_{\alpha,n} = \left\{\overline{CCP}|Z||\mathbf{F}_n \geq \alpha\right\}$ then, since the disintegration property holds for coherent conditional upper bounds defined by Hausdorff outer measures on Borel partitions, we have

$$\int_{A_{\alpha,n}} \overline{CCC}(Z|\mathbf{F}_n) d\mu_\Omega = \int_{A_{\alpha,n}} Z d\mu_\Omega.$$

Then by Lemma 2 p. 491 of [21] the random variables $\overline{CCP}(Z|\mathbf{F}_n)$ are uniformly integrable.

Theorem 7. *Let Ω be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension and let $X_n = \overline{CCP}(Z|\mathbf{F}_n)$ then $X_n \to X$ with probability 1, where X is a random variable such that $\overline{CCP}(|X|) < \infty$.*

Proof. Since $X_n = \overline{CCP}(Z|\mathbf{F}_n)$ is a martingale and Z is Choquet integrable then X_n is a bounded non-decreasing sequence then $X_n \to X$ with probability 1, where X is a random variable such that $\overline{CCP}(|X|) < \infty$.

Theorem 8. *Let Ω be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension and let Z be a random variable belonging to $L^*(\Omega)$. Let $\mathbf{F}_n$ be non-decreasing Borel-σ-fields such that $\mathbf{F}_n \uparrow \mathbf{F}_\infty$, then $\overline{CCP}(Z|\mathbf{F}_n) \to \overline{CCP}(Z|\mathbf{F}_\infty)$ with probability 1 with respect to μ_Ω.*

Proof. Since Z is a random variable in $L^*(\Omega)$, by Example 4 the random variables $X_n = \overline{CCP}(Z|\mathbf{F}_n)$ form a martingale, according to Definition 9. Since $\overline{CCP}(|X_n|) \leq \overline{CCP}(|Z|) < \infty$ then the X_n converge to an integrable X. We have to identify X with $\overline{CCP}(Z|\mathbf{F}_\infty)$. Let H be an atom of the σ-field F_∞ with positive and finite Hausdorff measure in its Hausdorff dimension s equal to the Hausdorff dimension of Ω. By the uniform integrability (Theorem 6) it is possible to integrate to the limit so that $\int_H X d\mu_\Omega = \lim_{n\to+\infty} \int_\omega X_n d\mu_\Omega$; since the atoms H of the σ-fields F_n are Borel-measurable the coherent conditional bounds satisfy the disintegration property and we obtain

$$\lim_{n\to+\infty} \int_H X_n d\mu_\Omega = \int_H P(Z|\mathbf{F}_n) d\mu_\Omega = \int_H Z d\mu_\Omega$$

Therefore $\int_H X d\mu_\Omega = \int_H Z d\mu_\Omega$ for all atoms H of $\mathbf{F}_\infty$ with positive Hausdorff measure μ_Ω. By the disintegration property

$$\int_H X d\mu_\Omega = \int_H Z d\mu_\Omega = \int_H \overline{CCP}(Z|\mathbf{F}_\infty) d\mu_\Omega$$

so that

$$X = \overline{CCP}(Z|\mathbf{F}_\infty)..$$

Remark 2. The previous theorem holds for every Choquet integrable random variable with respect to μ_Ω when the increasing information is represented by a sequence of Borel σ-fields; all results hold because coherent conditional upper bounds defined with respect Hausdorff outer measures satisfy the disintegration property on every Borel partition. It occurs because Hausdorff outer measures are metric outer measures and so all Borelian sets are measurable with respect to Hausdorff outer measures. In the proof of Theorem 2 of the paper of Blackwell and Dubins [1] a similar result is proven for $\mathbf{F}_\infty$-measurable random variables to assure that conditional probabilities, defined by probability measures which are absolutely continuous, merge for $\mathbf{F}_n$, not only containing Borel sets.

Theorem 9. *Let (Ω, d) be a metric space where Ω is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension s and let d_i be a metric bi-Lipschitz equivalent to d. Let $\mathcal{B}$ the Borel σ-field of the metric spaces (Ω, d) and (Ω, d_i). Let $H \in \mathbf{F}_\infty$ with Hausdorff dimension equal to s. Denote by h^s and by h_i^s respectively the s-dimensional Hausdorff measure with respect to the metric d and d_i. Let H be an atom of positive and finite s-dimensional Hausdorff outer measure in the two metric spaces and $\mu_H(A)$ and $\mu_H^i(A)$ the coherent upper conditional probabilities defined by*

$$\mu_H(A) = \frac{h^s(A)}{h^s(H)} \text{ and } \mu_H^i(A) = \frac{h_i^s(A)}{h_i^s(H)}.$$

then μ_H^i merges to μ_H for all $A \in \wp(\Omega)$.

Proof. Since H has positive and finite Hausdorff outer measure in its Hausdorff dimension s equal to the Hausdorff dimension of Ω, the coherent upper conditional probabilities μ_H^i and μ_H are defined by

$$\mu_H(A) = \frac{\overline{CCP}(I_{A\cap H})}{h^s(H)} \text{ and } \mu_H^i(A) = \frac{\overline{CCP}(I_{A\cap H})}{h_i^s(H)}$$

so that by Theorem 8

$$\mu_H(A) = \frac{\overline{CCP}(I_{A\cap H})}{h^s(H)} = \overline{CCP}((I_{A\cap H})|\mathbf{F}_\infty) = \frac{\overline{CCP}(I_{A\cap H})}{h_i^s(H)} = \mu_H^i(A)$$

for all $\omega \in H$, that is μ_H^i merges to μ_H.

Remark 3. It can be noted that when the Hausdorff measure of set H equals zero or infinity, such as in the case of a countable set, Theorem 2 defines a coherent conditional probability through a 0–1 valued finitely additive probability that lacks countable additivity. In such scenarios, Theorem 8 is not applicable. In fact, there are instances where 0–1 valued conditional probabilities exist, displaying mutual absolute continuity but failing to converge. An illustrative example can be found in Section 6 of [1].

For each H atom of $\mathbf{F}_\infty$ with positive and finite Hausdorff outer measure in its Hausdorff dimension s let μ_H be the coherent upper conditional probability defined by Theorem 2 in (Ω, d) and let $\mathbf{K}_H = [\mu_H^1, \mu_H^2, \ldots]$ be the credal set of all coherent upper conditional probabilities μ_i, which are defined by a distance which is bi-Lipschitz with respect to d. So each μ_B^i is absolutely continuous with respect to μ_B. Then the credal set $\mathbf{K}_H$ represents the class of the opinions of the individuals, which agree when the information increases.

5. Conclusions

Conditional probability can be naturally understood as representing an individual's belief or opinion regarding an event, considering information provided by a σ-field or a partition. The findings in [4] suggest that if two individuals' opinions, expressed through conditional probabilities, align on events with positive probabilities under the first probability measure, then these events will also have positive probabilities under the second probability measure, leading to convergence when the number of observations increases.

When coherent upper conditional probabilities are defined within a metric space, as proposed in this paper, different individuals may establish coherent upper conditional probabilities in distinct metric spaces. The presented results demonstrate that when diverse opinions are captured by coherent upper conditional probabilities defined by Hausdorff outer measures in various metric spaces, with metrics that are bi-Lipschitz equivalent, the distance between these upper conditional probabilities approaches zero as the cardinality of the σ-field of conditioning events increases. This phenomenon exemplifies the "merging of opinions with increasing information".

However, it is important to note that the above result does not hold if the conditioning event has a probability of zero or infinity. In such cases, the conditional probability is defined by a 0–1 valued finitely additive but not countably additive probability and there are instances in literature of 0–1 valued probabilities that are mutually absolutely continuous but do not converge.

CRediT authorship contribution statement

Serena Doria: Conceptualization, Formal analysis, Methodology, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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