Theoretical Computer Science 990 (2024) 114404

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Finding dominating induced matchings in P_{10}-free graphs in polynomial time ☆

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ARTICLE INFO

Communicated by R. Klasing

Keywords:
Dominating induced matching
Efficient edge domination
P_{10}-free graphs
Polynomial time algorithm

ABSTRACT

Let $G = (V, E)$ be a finite undirected graph. An edge set $E' \subseteq E$ is a *dominating induced matching* (*d.i.m.*) in G if every edge in E is intersected by exactly one edge of E'. The *Dominating Induced Matching* (*DIM*) problem asks for the existence of a d.i.m. in G; this problem is also known as the *Efficient Edge Domination* problem; it is the Efficient Domination problem for line graphs.

The DIM problem is $\mathbb{NP}$-complete even for very restricted graph classes such as planar bipartite graphs with maximum degree 3 but is solvable in polynomial time for P_9-free graphs [and in linear time for P_7-free graphs] as well as for $S_{1,2,4}$-free, for $S_{2,2,2}$-free, and for $S_{2,2,3}$-free graphs. In this paper, combining two distinct approaches, we solve it in polynomial time for P_{10}-free graphs and introduce a partial result for the general case.

1. Introduction

Let $G = (V, E)$ be a finite undirected graph. A vertex $v \in V$ *dominates* itself and its neighbors. A vertex subset $D \subseteq V$ is an *efficient dominating set* (*e.d.s.* for short) of G if every vertex of G is dominated by exactly one vertex in D. The notion of efficient domination was introduced by Biggs [2] under the name *perfect code*. The EFFICIENT DOMINATION (ED) problem asks for the existence of an e.d.s. in a given graph G (note that not every graph has an e.d.s.)

A set M of edges in a graph G is an *efficient edge dominating set* (*e.e.d.s.* for short) of G if and only if it is an e.d.s. in its line graph $L(G)$. The EFFICIENT EDGE DOMINATION (EED) problem asks for the existence of an e.e.d.s. in a given graph G. Thus, the EED problem for a graph G corresponds to the ED problem for its line graph $L(G)$. Note that not every graph has an e.e.d.s. An efficient edge dominating set is also called *dominating induced matching* (*d.i.m.* for short), and the EED problem is called the DOMINATING INDUCED MATCHING (DIM) problem in various papers (see e.g. [3–7,10,12,13]); subsequently, we will use this notation instead of EED.

In [11], it was shown that the DIM problem is $\mathbb{NP}$-complete; see also [3,10,14–16]. However, for various graph classes, DIM is solvable in polynomial time. For mentioning some examples, we need the following notions:

Let P_k denote the chordless path P with k vertices, say $a_1, \dots, a_k$, and $k-1$ edges $a_i a_{i+1}$, $1 \le i \le k-1$; we also denote it as $P = (a_1, \dots, a_k)$.

☆ This article belongs to Section A: Algorithms, automata, complexity and games, Edited by Paul Spirakis.

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https://doi.org/10.1016/j.tcs.2024.114404
Received 2 November 2022; Received in revised form 14 October 2023; Accepted 16 January 2024

Available online 19 January 2024

For indices $i, j, k \geq 0$, let $S_{i,j,k}$ denote the graph H with vertices $u, x_1, \ldots, x_i$, $y_1, \ldots, y_j$, $z_1, \ldots, z_k$ such that the subgraph induced by $u, x_1, \ldots, x_i$ forms a P_{i+1} $(u, x_1, \ldots, x_i)$, the subgraph induced by $u, y_1, \ldots, y_j$ forms a P_{j+1} $(u, y_1, \ldots, y_j)$, and the subgraph induced by $u, z_1, \ldots, z_k$ forms a P_{k+1} $(u, z_1, \ldots, z_k)$, and there are no other edges in $S_{i,j,k}$; u is called the *center* of H. Thus, *claw* is $S_{1,1,1}$, and P_k is isomorphic to $S_{k-1,0,0}$.

For a set $\mathcal{F}$ of graphs, a graph G is called *$\mathcal{F}$-free* if no induced subgraph of G is contained in $\mathcal{F}$. If $|\mathcal{F}| = 1$, say $\mathcal{F} = \{H\}$, then instead of $\{H\}$-free, G is called H-free.

The following results are known:

Theorem 1. *DIM is solvable in polynomial time for*

(i) *$S_{1,1,1}$-free graphs [10],*
(ii) *$S_{1,2,3}$-free graphs [13],*
(iii) *$S_{2,2,2}$-free graphs [12],*
(iv) *$S_{1,2,4}$-free graphs [6],*
(v) *$S_{2,2,3}$-free graphs [7],*
(vi) *$S_{1,1,5}$-free graphs [8],*
(vii) *P_7-free graphs [4] (in this case even in linear time),*
$(viii)$ *P_8-free graphs [5],*
(ix) *P_9-free graphs [9].* □

In [12], it is conjectured that for every fixed i, j, k, DIM is solvable in polynomial time for $S_{i,j,k}$-free graphs (actually, an even stronger conjecture is mentioned in [12]); this includes P_k-free graphs for $k \geq 10$.

Based on the two distinct approaches described in [5] and in [12,13], we show in this paper that DIM can be solved in polynomial time for P_{10}-free graphs (generalizing the corresponding results for P_7-free, for P_8-free, and for P_9-free graphs).

2. Definitions and basic properties

2.1. Basic notions

Let G be a finite undirected graph without loops and multiple edges. Let $V(G)$ or V denote its vertex set and $E(G)$ or E its edge set; let $|V| = n$ and $|E| = m$. For $v \in V$, let $N(v) := \{u \in V : uv \in E\}$ denote the *open neighborhood of* v, and let $N[v] := N(v) \cup \{v\}$ denote the *closed neighborhood of* v. For $U, W \subseteq V$, with $U \cap W = \emptyset$, let us say that U *contacts* W if some vertex of U is adjacent to some vertex of W; in particular, if $U = \{u\}$, then let us simply say that u *contacts* W. For $U, W \subseteq V$, with $U \cap W = \emptyset$, let us say that U *has a join with* W and let us write $U \textcircled{1} W$ if every vertex of U is adjacent to every vertex of W; in particular, if $U = \{u\}$, then let us simply write $u \textcircled{1} W$.

A vertex set S is *independent* in G if for every pair of vertices $x, y \in S$, $xy \notin E$. A vertex set Q is a *clique* in G if for every pair of vertices $x, y \in Q$, $x \neq y$, $xy \in E$. For $uv \in E$ let $N(uv) := N(u) \cup N(v) \setminus \{u, v\}$ and $N[uv] := N[u] \cup N[v]$.

For $U \subseteq V$, let $G[U]$ denote the subgraph of G induced by vertex set U. Clearly $xy \in E$ is an edge in $G[U]$ exactly when $x \in U$ and $y \in U$; thus, $G[U]$ can simply be denoted by U (if understandable).

For graphs H_1, H_2 with disjoint vertex sets, $H_1 + H_2$ denotes the disjoint union of H_1, H_2, and for $k \geq 2$, kH denotes the disjoint union of k copies of H. For example, $2P_2$ is the disjoint union of two edges.

As already mentioned, a *chordless path* P_k, $k \geq 2$, has k vertices, say $v_1, \ldots, v_k$, and $k-1$ edges $v_i v_{i+1}$, $1 \leq i \leq k-1$; the *length of* P_k is $k-1$.

A *chordless cycle* C_k, $k \geq 3$, has k vertices, say $v_1, \ldots, v_k$, and k edges $v_i v_{i+1}$, $1 \leq i \leq k-1$, and $v_k v_1$; the *length of* C_k is k.

Let K_i, $i \geq 1$, denote the clique with i vertices. Let $K_4 - e$ or *diamond* be the graph with four vertices, say v_1, v_2, v_3, u, such that (v_1, v_2, v_3) forms a P_3 and $u \textcircled{1} \{v_1, v_2, v_3\}$; its *mid-edge* is the edge uv_2.

A *butterfly* has five vertices, say, v_1, v_2, v_3, v_4, u, such that v_1, v_2, v_3, v_4 induce a $2P_2$ with edges v_1v_2 and v_3v_4 (the *peripheral edges* of the butterfly), and $u \textcircled{1} \{v_1, v_2, v_3, v_4\}$.

We often consider an edge $e = uv$ to be a set of two vertices; then it makes sense to say, for example, $u \in e$ and $e \cap e' \neq \emptyset$, for an edge e'. For two vertices $x, y \in V$, let $dist_G(x, y)$ denote the *distance between* x *and* y *in* G, i.e., the length of a shortest path between x and y in G. The *distance between a vertex* z *and an edge* xy is the length of a shortest path between z and x, y, i.e., $dist_G(z, xy) = \min\{dist_G(z, v) : v \in \{x, y\}\}$. The *distance between two edges* $e, e' \in E$ is the length of a shortest path between e and e', i.e., $dist_G(e, e') = \min\{dist_G(u, v) : u \in e, v \in e'\}$. In particular, this means that $dist_G(e, e') = 0$ if and only if $e \cap e' \neq \emptyset$.

An edge subset $M \subseteq E$ is an *induced matching* if the pairwise distance between its members is at least 2, that is, M is isomorphic to kP_2 for $k = |M|$. Obviously, if M is a d.i.m. then M is an induced matching.

Clearly, G has a d.i.m. if and only if every connected component of G has a d.i.m.; from now on, connected components are mentioned as *components*.

Note that if G has a d.i.m. M, and $V(M)$ denotes the vertex set of M then $V \setminus V(M)$ is an independent set, say I, i.e.,

$$V \text{ has the partition } V = V(M) \cup I. \quad (1)$$

From now on, all vertices in I are colored white and all vertices in $V(M)$ are colored black. According to [12], we also use the following notions: A partial black-white coloring of $V(G)$ is *feasible* if the set of white vertices is an independent set in G and every black vertex has at most one black neighbor. A complete black-white coloring of $V(G)$ is *feasible* if the set of white vertices is an independent set in G and every black vertex has exactly one black neighbor. Clearly, M is a d.i.m. of G if and only if the black vertices $V(M)$ and the white vertices $V \setminus V(M)$ form a complete feasible coloring of $V(G)$.

2.2. Reduction steps, forbidden subgraphs, forced edges, and excluded edges

Various papers on this topic introduced and applied some *forcing rules* for reducing the graph G to a subgraph G' such that G has a d.i.m. if and only if G' has a d.i.m., based on the condition that for a d.i.m. M, V has the partition $V = V(M) \cup I$ such that all vertices in $V(M)$ are black and all vertices in I are white (recall (1)).

A vertex $v \in V$ is *forced to be black* if for every d.i.m. M of G, $v \in V(M)$. Analogously, a vertex $v \in V$ is *forced to be white* if for every d.i.m. M of G, $v \notin V(M)$.

An edge $e \in E$ is a *forced* edge of G if for every d.i.m. M of G, $e \in M$. Analogously, an edge $e \in E$ is an *excluded* edge of G if for every d.i.m. M of G, $e \notin M$.

For the correctness of the reduction steps, we have to argue that G has a d.i.m. if and only if the reduced graph G' has one (provided that no contradiction arises in the vertex coloring, i.e., it is feasible).

Then let us introduce two reduction steps which will be applied later.

Vertex Reduction. Let $u \in V(G)$. If u is forced to be white, then

(i) color black all neighbors of u, and
(ii) remove u from G.

Let G' be the reduced subgraph. Clearly, Vertex Reduction is correct, i.e., G has a d.i.m. if and only if G' has a d.i.m.

Edge Reduction. Let $uv \in E(G)$. If u and v are forced to be black, then

(i) color white all neighbors of v and of v (other than u and v), and
(ii) remove u and v (and the edges containing u or v) from G.

Again, clearly, Edge Reduction is correct, i.e., G has a d.i.m. if and only if the reduced subgraph G' has a d.i.m.

The subsequent notions and observations lead to some possible reductions (some of them are mentioned e.g. in [3–5]).

Observation 1 *([3–5]). Let M be a d.i.m. of G.*

(i) *M contains at least one edge of every odd cycle C_{2k+1} in G, $k \geq 1$, and exactly one edge of every odd cycle C_3, C_5, C_7 in G.*
(ii) *No edge of any C_4 can be in M.*
(iii) *For each C_6 either exactly two or none of its edges are in M.*

Proof. See e.g. Observation 2 in [4]. □

Since by Observation 1 (i), every triangle contains exactly one M-edge, and the pairwise distance of M-edges is at least 2, we have:

Corollary 1. *If G has a d.i.m. then G is K_4-free.* □

Assumption 1. From now on, by Corollary 1, we assume that the input graph is K_4-free (else it has no d.i.m.).

Clearly, it can be checked (directly) in polynomial time whether the input graph is K_4-free.

By Observation 1 (i) with respect to C_3 and by the distance property, we have the following:

Observation 2. *The mid-edge of any diamond in G and the two peripheral edges of any induced butterfly are forced edges of G.* □

Assumption 2. From now on, by Observation 2, we assume that the input graph is (diamond, butterfly)-free.

In particular, we can apply the Edge Reduction to each mid-edge of any induced diamond and to each peripheral edge of any induced butterfly; that can be done in polynomial time.

2.3. The distance levels of an M-edge xy in a P_3

Based on [5], we first describe some general structure properties for the distance levels of an edge in a d.i.m. M of G. Since G is (K_4, diamond, butterfly)-free, we have:

Observation 3. *For every vertex v of G, $N(v)$ is the disjoint union of isolated vertices and at most one edge. Moreover, for every edge $xy \in E$, there is at most one common neighbor of x and y.* □

Since it is trivial to check whether G has a d.i.m. M with exactly one edge, from now on *we can assume that* $|M| \geq 2$. In particular, since G is connected and butterfly-free, we have (see also [5]):

Observation 4. *If $|M| \geq 2$ then there is an edge in M which is contained in a P_3 of G.* □

Proof. Let $xy \in M$ and assume that xy is not part of an induced P_3 of G. Since G is connected and $|M| \geq 2$, $(N(x) \cup N(y)) \setminus \{x, y\} \neq \emptyset$, and since we assume that xy is not part of an induced P_3 of G and G is K_4- and diamond-free, there is exactly one neighbor of xy, namely a common neighbor, say z of x and y. Again, since $|M| \geq 2$, z has a neighbor $a \notin \{x, y\}$, and since G is K_4- and diamond-free, a, x, y, z induce a paw. Clearly, the edge za is excluded and has to be dominated by a second M-edge, say $ab \in M$ but now, since G is butterfly-free, $zb \notin E$. Thus, z, a, b induce a P_3 in G, and Observation 4 is shown. □

Remark 1. In what follows, let us fix an edge xy in the solution, i.e. let us fix $xy \in M$. In particular, according to the assumption that $|M| \geq 2$ and to Observation 4, let us fix an edge xy in order that there is a vertex r such that $\{r, x, y\}$ induce a P_3 with edge $rx \in E$. Then, we have that x and y are black, and that could lead to a feasible black-white coloring of $V(G)$ [with vertices x and y black] if no contradiction arises. ⋄

Let us write $N_0 = N_0(xy) = \{x, y\}$, and for $i \geq 1$ let

$$N_i = N_i(xy) = \{z \in V : dist_G(z, xy) = i\}$$

denote the *distance levels of xy*.

Then we start by considering the partition of V into N_i, $i \geq 0$, with respect to the edge xy (under the assumption that $xy \in M$).

If an edge $x'y' \in E$ is contained in **every** d.i.m. M of G with $xy \in M$, we say that $x'y'$ is an *xy-forced M-edge*.

If a vertex $v \in V$ is contained in **no** d.i.m. M of G with $xy \in M$, we say that v is a *xy-excluded M-vertex*.

In the following description which is based on the assumption of Remark 1:

— whenever a xy-forced M-edge, say $x'y'$, is detected, we re-define $N_0 := N_0 \cup \{x'y'\}$ and consequently re-define the distance levels with respect to N_0;

— whenever a xy-excluded M-vertex, say v, is detected, we apply the Vertex Reduction to v *only if* such a reduction does not disconnect the graph (in fact since we have fixed $xy \in M$, if such a reduction disconnects the graphs, then the approach of "fixing an edge in the solution" could not be iterated in an efficient way).

Clearly, by Observation 4 and since G is P_{10}-free, we have:

$$N_k = \emptyset \text{ for every } k \geq 8. \tag{2}$$

Recall that by (1), $V = V(M) \cup I$ is a partition of V where $V(M)$ is the set of black vertices and I is the set of white vertices which is independent.

Since we assume that $xy \in M$ (and is an edge in a P_3), clearly, $N_1 \subseteq I$ and thus:

$$N_1 \text{ is an independent set of white vertices.} \tag{3}$$

Moreover, no edge between N_1 and N_2 is in M. Since $N_1 \subseteq I$ and all neighbors of vertices in I are in $V(M)$, we have $N_2 \subseteq V(M)$ and thus:

$$G[N_2] \text{ is the disjoint union of edges and isolated vertices.} \tag{4}$$

Let M_2 denote the set of edges $uv \in E$ with $u, v \in N_2$ and let $S_2 = \{u_1, \dots, u_k\}$ denote the set of isolated vertices in N_2; $N_2 = V(M_2) \cup S_2$ is a partition of N_2. Obviously:

$$M_2 \subseteq M \text{ and } S_2 \subseteq V(M). \tag{5}$$

Obviously, by (5), we have:

$$\text{Every edge in } M_2 \text{ is an } xy\text{-forced } M\text{-edge.} \tag{6}$$

Thus, from now on, as one can re-define N_0 by involving M_2-edges, we can assume that $V(M_2) = \emptyset$, i.e., $N_2 = S_2 = \{u_1, \dots, u_k\}$. For every $i \in \{1, \dots, k\}$, let $u'_i \in N_3$ denote the *M-mate* of u_i (i.e., $u_iu'_i \in M$). Let $M_3 = \{u_iu'_i : 1 \leq i \leq k\}$ denote the set of M-edges with one endpoint in S_2 (and the other endpoint in N_3). Obviously, by (5) and the distance condition for a d.i.m. M, the following holds:

$$\text{No edge with both ends in } N_3 \text{ and no edge between } N_3 \text{ and } N_4 \text{ is in } M. \quad (7)$$

As a consequence of (7) and the fact that every triangle contains exactly one M-edge (recall Observation 1 (i)), we have:

$$\text{For every } C_3 \; abc \text{ with } a \in N_3, \text{ and } b,c \in N_4, \; bc \in M \text{ is an } xy\text{-forced } M\text{-edge}. \quad (8)$$

This means that for the edge bc, one can re-define N_0 by involving edge bc, and from now on, we can assume that there is no such triangle abc with $a \in N_3$ and $b,c \in N_4$, i.e., for every edge $uv \in E$ in N_4:

$$N(u) \cap N(v) \cap N_3 = \emptyset. \quad (9)$$

According to (5) and the assumption that $V(M_2) = \emptyset$ (recall $N_2 = \{u_1, \dots, u_k\}$), let:

$T_{one} := \{t \in N_3 : |N(t) \cap N_2| = 1\}$,
$T_i := T_{one} \cap N(u_i)$, $1 \le i \le k$, and
$S_3 := N_3 \setminus T_{one}$.

By definition, T_i is the set of *private* neighbors of $u_i \in N_2$ in N_3 (note that $u'_i \in T_i$), $T_1 \cup \ldots \cup T_k$ is a partition of T_{one}, and $T_{one} \cup S_3$ is a partition of N_3.

Let us report from [5] the following lemma.

Lemma 1 *([5]). The following statements hold:*

- (i) *For all $i \in \{1, \dots, k\}$, $T_i \cap V(M) = \{u'_i\}$.*
- (ii) *For all $i \in \{1, \dots, k\}$, T_i is the disjoint union of vertices and at most one edge.*
- (iii) *$G[N_3]$ is bipartite.*
- (iv) *$S_3 \subseteq I$, i.e., S_3 is an independent subset of white vertices.*
- (v) *If a vertex $t_i \in T_i$ sees two vertices in T_j, $i \ne j$, $i,j \in \{1, \dots, k\}$, then $u_i t_i \in M$ is an xy-forced M-edge.* □

Then let us introduce the following forcing rules (which are correct).

Since no edge in N_3 is in M (recall (7)), we have:

(R1) For any vertex $v \in N_3$, if v is black (white), then all vertices of $N(v) \cap (N_3 \cup N_4)$ must be colored white (black).

Moreover, by Lemma 1, we have:

(R2) Every T_i, $i \in \{1, \dots, k\}$, should contain exactly one vertex which is black. Thus, if $t \in T_i$ is black then all the remaining vertices in $T_i \setminus \{t\}$ must be colored white.
(R3) If all but one vertices of T_i, $1 \le i \le k$, are white and the final vertex $t \in T_i$ is not yet colored, then t must be colored black.

2.4. The main body of the solution method

Let us say that, for any graph $G = (V,E)$, a *central vertex* of G is any vertex $v \in V$ such that $max\{dist_G(v,u) : u \in V\} \le max\{dist_G(v',u) : u \in V\}$ for every $v' \in V$.

Theorem 2 *([1]). Every connected P_t-free graph $G = (V,E)$ admits a vertex $v \in V$ such that $dist_G(v,u) \le \lfloor t/2 \rfloor$ for every $u \in V$.* □

Let $G = (V,E)$ be a connected P_{10}-free graph. Then let $v \in V$ be any central vertex of G. Note that, by Theorem 2, $dist_G(v,u) \le 5$ for every $u \in V$.

From one hand, for every edge vu of G, with $u \in N(v)$, one has $N_k(vu) = \emptyset$ for any $k \ge 5$; furthermore, by the choice of v, by Assumption 1 and by Assumption 2, one has that [unless G is a triangle] edge vu is contained in an induced P_3 of G: in fact, if vu is contained in no induced P_3 of G, then $N(vu)$ contains exactly one vertex (else an induced K_4 or an induced diamond arises), say vertex z, but then $max\{dist_G(z,\bar{v}) : \bar{v} \in V\} < max\{dist_G(v,\bar{v}) : \bar{v} \in V\}$, which contradicts the fact that v is a central vertex of G; it follows that, by Remark 1, *all properties introduced in the previous subsection for edge xy hold for edge vu as well.*

From the other hand, if one could check, for any edge vu, with $u \in N(v)$, whether there is a d.i.m. M' of G with $vu \in M'$, then one could conclude that: either G has a d.i.m. [with $v \in V(M)$], or G has no d.i.m. M with $v \in V(M)$; in particular, in the latter case, one can apply the Vertex Reduction to v and thus remove v from G.

Then let us introduce the following recursive algorithm which formalizes the approach we will adopt to check if G has a d.i.m.

Algorithm DIM(G)
Input. A connected P_{10}-free graph $G=(V,E)$ which enjoys Assumption 1 and Assumption 2.
Output. A d.i.m. of G or the proof that G has no d.i.m.

(Step 1) Compute any central vertex, say x, of G.
(Step 2) For each edge xy, with $y \in N(x)$, of G (recall that xy is contained in a P_3 of G) do:
(2.1) compute the distance levels N_i with respect to xy and re-define iteratively N_0 by involving those xy-forced edges as shown above;
— if no contradiction arose, according to (3)-(4) or to Lemma 1(ii)-(iv) or to forcing rules (R1)-(R3), then go to Step (2.2);
— else consider the next edge;
(2.2) check if G has a d.i.m. M with $xy \in M$; if *yes*, then return it, and STOP;
(Step 3) Apply the Vertex Reduction to x and thus remove x from G; let G' denote the resulting graph, where the neighbors of x in G are colored by black; if G' is disconnected, then execute Algorithm DIM(H) for each component H of G'; else, go to Step 2, with $G := G'$.
(Step 4) Return "G has no d.i.m." and STOP. □

Then, by the above, Algorithm DIM(G) is correct and can be executed in polynomial time as soon as Step (2.2) can be so.

Then in what follows let us show that Step (2.2) can be solved in polynomial time, with the agreement that G enjoys Assumption 1 and Assumption 2, and that no contradiction arose, according to (3)-(4) or to Lemma 1(ii)-(iv) or to forcing rules (R1)-(R3).

For that we consider the cases $N_4 = \emptyset$ and $N_4 \neq \emptyset$.

3. The case $N_4 = \emptyset$

In this section let us show that, if $N_4 = \emptyset$, then one can check in polynomial time whether G has a d.i.m. M with $xy \in M$.

First let us introduce some assumptions, based on the fact that N_0 can be re-defined [by involving xy-forced M-edges] and on the Vertex Reduction [whose application will not disconnect the graph, by construction, and since $N_4 = \emptyset$], in order to simplify the scenario.

By Lemma 1 (iv) and the Vertex Reduction for the white vertices of S_3, we can assume:

(A1) $S_3 = \emptyset$, i.e., $N_3 = T_1 \cup \ldots \cup T_k$.

By Lemma 1 (v), we can assume:

(A2) For $i, j \in \{1, \ldots, k\}$, $i \neq j$, every vertex $t_i \in T_i$ has at most one neighbor in T_j.

In particular, if for some $i \in \{1, \ldots, k\}$, $T_i = \emptyset$, then there is no d.i.m. M of G with $xy \in M$, and if $|T_i| = 1$, say $T_i = \{t_i\}$, then $u_i t_i$ is an xy-forced M-edge. Thus, we can assume:

(A3) For every $i \in \{1, \ldots, k\}$, $|T_i| \geq 2$.

Let us say that a vertex $t \in T_i$, $1 \leq i \leq k$, is an *out-vertex* of T_i if it is adjacent to some vertex of T_j with $j \neq i$, and t is an *in-vertex* of T_i otherwise.

Recall that, by Lemma 1 (ii), T_i is the disjoint union of vertices and at most one edge say e_i. If $G[T_i]$ contains e_i, then at least one vertex of e_i is black, so that the isolated vertices of $G[T_i]$ are white and can be removed, i.e., one can apply the Vertex Reduction to such isolated vertices; it follows that, if vertices of e_i, say t' and t'', are in-vertices, then either t' or t'' is black (indifferently by symmetry); then one can re-define N_0 by involving $u_i t'$ (or indifferently, by symmetry, by involving $u_i t''$). If $G[T_i]$ does not contain e_i, then for finding a d.i.m. M with $xy \in M$ one can remove all but one in-vertices of T_i, i.e., one can apply the Vertex Reduction to all but one in-vertices in T_i.

Thus, let us assume:

(A4) For every $i \in \{1, \ldots, k\}$, T_i has at most one in-vertex.

Lemma 2. *Assume that G has a d.i.m. M with $xy \in M$. Then there are no three edges between T_i and T_j, $i \neq j$, and if there are two edges between T_i and T_j, say $t_i t_j \in E$ and $t'_i t'_j \in E$ for $t_i, t'_i \in T_i$ and $t_j, t'_j \in T_j$ then any other vertex in T_i or T_j is white.*

Proof. First, suppose to the contrary that there are three edges between T_1 and T_2, say $t_1 t_2 \in E$, $t'_1 t'_2 \in E$, and $t''_1 t''_2 \in E$ for $t_i, t'_i, t''_i \in T_i$, $i = 1, 2$. Then t_1 is black if and only if t_2 is white, t'_1 is black if and only if t'_2 is white, and t''_1 is black if and only if t''_2 is white. Without loss of generality, assume that t_1 is black, and t_2 is white. Then t'_1 is white, and t'_2 is black, but now, t''_1 and t''_2 are white, which is a contradiction.

Now, if there are exactly two such edges between T_1 and T_2, say $t_1 t_2 \in E$, $t'_1 t'_2 \in E$, then again, t_1 or t'_1 is black as well as t_2 or t'_2 is black, and thus, every other vertex in T_1 or T_2 is white.

Thus Lemma 2 is shown. □

By Lemma 2, we can assume:

(A5) For $i, j \in \{1, \dots, k\}$, $i \neq j$, there are at most two edges between T_i and T_j.

Let us point out that (A1) and (A3) hold under the assumptions of the case $N_4 = \emptyset$, which warrants that the Vertex Reduction does not disconnect the graph, while (A2), (A4), (A5) hold generally.

In the rest of this section let assume that (A1)-(A5) hold.

Let us write $T_{family} = \{T_1, \dots, T_k\}$. Let us assume that $G[\{u_1, \dots, u_k\} \cup T_1 \cup \dots \cup T_k]$ is connected, without loss of generality, else one can split the problem for the corresponding components. Then let us consider the following two exhaustive cases.

Case 1. There are no vertices $t_i \in T_i, t_j \in T_j, t_h \in T_h$, with $i, j, h \in \{1, \dots, k\}$ mutually distinct, which induce a P_3 in G.

Let us define a multi-graph $F = (T_{family}, E')$ as follows: for any $T_i, T_j \in T_{family}$ (with $i \neq j$), if in G there is an edge from vertices of T_i to vertices of T_j, then in F there is an edge from node T_i to node T_j; in particular [according to (A2)], if in G there are two edges from vertices of T_i to vertices of T_j, then in F there are two edges from node T_i to node T_j (in this case node T_i and node T_j form a cycle of F).

Let us recall that a *bridge* of a connected multi-graph is an edge of the multi-graph whose removal disconnects the multi-graph.

Let us recall that a multi-graph is *2-edge-connected* if it is connected and if it has no bridge [so that each edge of the multi-graph belongs to a cycle of the multi-graph].

Let us say that an induced subgraph of F is a *blue subgraph of F* if it is a maximal 2-edge-connected subgraph of G.

Then $V(F)$ can be partitioned into $\{V'(F), V''(F)\}$ where:

:: each node of $V'(F)$ belongs to some blue subgraph of F; in particular $V'(F)$ can be uniquely partitioned in order that each member of such a partition induces a blue subgraph of F, that is, the family of blue subgraphs of F is unique and its members have mutually no node of F in common; let say that every node of $V'(F)$ is a *blue node of F*;

:: each node of $V''(F)$ belongs to no blue subgraph of F; in particular $V'(F)$ induces a forest of F; let say that every node of $V''(F)$ is a *green node of F*.

P1. If all nodes of F are green, then G has a d.i.m. containing xy.

Proof. Note that in this case, as remarked above, each connected component of F is a tree. Then let us assume without loss of generality that F is a (rooted) tree, i.e., F is connected. Let node T_i be any leaf of F and let node T_j be the neighbor of node T_i in F. Then, by definition of green node, there is exactly one edge in G between T_i and T_j, say edge $t_i t_j$ with $t_i \in T_i$ and $t_j \in T_j$.

Claim 1. *Vertices of T_i are* ready for any coloring*, that is, for any choice of the color of t_j there is a feasible coloring of vertices of T_i.*

Proof. By construction, t_i is the only out-vertex of T_i; on the other hand, by (A4), T_i has at most one in-vertex, say $\bar{t}_i$; in particular, by (A3), vertex $\bar{t}_i$ does exist; then for any color of t_j, there is a feasible coloring of vertices of T_i, in details: t_i has a color different to that of t_j, while $\bar{t}_i$ has the same color as that of t_j. □

Then one can remove node T_i from F and iterate this argument for any leaf in the resulting tree. It follows that, for any feasible coloring of the vertices of the root of F [which does exist by Lemma 1 (ii)], there is a feasible coloring of the vertices of the nodes of F. This completes the proof of P1. □

P2. Let B be any blue subgraph of F. Then there are at most two feasible colorings of B and they can be computed in polynomial time.

Proof. Let C be any induced cycle of B.

Claim 1. *If T_i, T_j, T_h are three nodes of C inducing a P_3 of C, say with center say T_j (without loss of generality by symmetry), then with reference to graph G one has that: (1) no vertex of T_j has a neighbor both in T_i and in T_h; (2) there are two distinct vertices, say $t_j, t'_j \in T_j$, such that vertex t_j has a neighbor in T_i (which is a nonneighbor of t'_j) and vertex t'_j has a neighbor in T_h (which is a nonneighbor of t_j).*

Proof. Statement (1) follows by assumption of Case 1 and since $G[N_3]$ is bipartite by Lemma 1 (iii). Statement (2) follows by assumption of Claim 1, by statement (1), and by (A2). □

Claim 2. *Let T_a, T_b be adjacent nodes in C and let $t_a t_b$ be any edge in G, between T_a and T_b, with $t_a \in T_a$ and $t_b \in T_b$. Then:*

— if the color of t_a is fixed black [and thus the color of all vertices of $T_a \setminus \{t_a\}$ is forced to be white by (R2)], then the color of all vertices of each node of B is forced by (R1)-(R3);

— if the color of t_a is fixed white [and thus the color of t_b is forced to be black by (R1) and the color of all vertices of $T_b \setminus \{t_b\}$ is forced to be white by (R2)], then the color of all vertices of each node of B is forced by (R1)-(R3).

Proof. It follows by construction, by definition of (R1)-(R3), and by Claim 1. □

Claim 3. *If all vertices of a node of C are colored [i.e. have a given feasible coloring], then the color of all vertices of each node of C is forced.*

Proof. It follows by construction, by Claim 1, and by Claim 2. □

Let us conclude the proof of P2. If B is an induced cycle, then by Claim 1 and by Claim 2 one has that B has at most two feasible colorings [and they can be computed in polynomial time]. If B is not an induced cycle, then one can proceed as follows: take any induced cycle, say Q, of B; let us write $Q = Q_0$; then let us define a procedure, with $|V(B)| - |V(Q)|$ steps, such that at each step $h = 1, \ldots, |V(B)| - |V(Q)|$, the procedure defines subgraph $Q_h = B[V(Q_{h-1}) \cup \{v_h\}]$, where v_h is any node of $B[V(B) \setminus V(Q_{h-1})]$ such that v_h is contained in at least one cycle of $B[V(Q_{h-1}) \cup \{v_h\}]$; let us observe that, since B is 2-edge-connected, such a procedure is well defined; summarizing, by Claim 1 and by Claim 2, one has that Q has at most two feasible colorings [and they can be computed in polynomial time], while by Claim 3 and by definition of the above procedure one has that, for each such two feasible colorings of Q, the color of all vertices of each node of B is forced [and that can be computed in polynomial time].

This completes the proof of P2. □

Then by P2, for any blue subgraph B of F such that there is at least one feasible coloring of B, the set of vertices of nodes of B can be partitioned into $\{Q1(B), Q2(B)\}$ where:

— $Q1(B)$ is formed by those vertices of nodes of B which have the same color for any feasible coloring of B, and

— $Q2(B)$ is formed by the other vertices of nodes of B; note that, by P2, $Q2(B)$ is nonempty only when there are exactly two feasible colorings of B.

Then let us introduce a method to check whether vertices of nodes of F admit a feasible coloring.

Preliminary Step.

:: Compute the family, say $\mathcal{B}$, of blue subgraphs of F.

:: Construct the *bridge-block tree* of F, say F^*, that is:

(i) each member of $\mathcal{B}$ is contracted into a respective node of F^* and is called a *big blue node* of F^*; each green node of F remains in F^* and is called a *green node* of F^*; then each node of F^* is either a big blue node of F^* or a green node of F^*;

(ii) in F^* two nodes are adjacent if and only if in G there is an edge [i.e. a bridge of G] between two respective vertices of such two nodes; in particular in F^*, between two adjacent nodes, there is exactly one edge;

(iii) F^* is a tree; then let us fix any vertex of F^* as the *root* of F^*;

(iv) for any node X of F^* different to the root of F^*, let us say that the vertex of node X which is adjacent (in G) to the vertex of the ancestor of X is the *up-vertex of* X, denoted as $u(X)$, and that the vertex of the ancestor of X which is adjacent (in G) to the up-vertex of X is the *ancestor-vertex for* X. □

The generic step of the method focuses on any leaf of F^*, i.e., it checks such a leaf (both concerning the forcing conditions which it has received possibly by previous steps and concerning the forcing conditions which it gives possibly for the next steps). After such a step/check, the leaf will be removed from F^*; then such a step/check is iterated for any leaf of the resulting tree; the method ends when it focuses on the root of F^*.

Generic Step.

Input: any leaf X of F^*.

:: if X is a green node of F^*, then:

:::: check if some vertex of X is colored (by previous steps); then, according to this possible partial coloring, check by Lemma 1 (ii) if there exist two feasible colorings of vertices of X in which the color of $u(X)$ is assumed respectively black and white; if none of such feasible colorings exist, then return "G has no d.i.m. M with $xy \in M$"; if there exists only a feasible coloring in which the color of $u(X)$ is assumed black (respectively, white), then fix the color of $u(X)$ as black (respectively, as white);

:::: if $u(X)$ has a fixed color (by the above), then color the ancestor-vertex for X by a color different to the color of $u(X)$;

:::: if $u(X)$ has not a fixed color, then do not color the ancestor-vertex for X; then $u(X)$, and generally X, is *ready for any color* of the ancestor-vertex for X;

:: if X is a big blue node of F^*, i.e. let B be the member of $\mathcal{B}$ which has been contracted into X, then:

:::: check if some vertex of X is colored (by previous steps); then, according to this possible partial coloring, compute the (at most two) possible feasible colorings of B by P2; if B does not admit any feasible coloring, then return "G has no d.i.m. M with $xy \in M$";

:::: if $u(X)$ belongs to $Q1(B)$, then color the ancestor-vertex for X by a color different to the color of $u(X)$;

:::: if $u(X)$ belongs to $Q2(B)$, then do not color the ancestor-vertex for X; then $u(X)$, and generally X, is *ready for any color* of the ancestor-vertex for X. □

Then let us formalize the main body of the method.

Main Body

1. Execute the Preliminary Step.
2. While F^* has a leaf say X do:
:: execute the Generic Step for X;
:: remove X from F^*, i.e., set $F^* := F^* - X$;
:: if X is the root of F^*, then: if there is a feasible coloring of X, then return "G has a d.i.m. M with $xy \in M$"; if there is no feasible coloring of X, then return "G has no d.i.m. M with $xy \in M$". □

Then the above method is correct and can be executed in polynomial time by the above.

This completes the proof for Case 1: let us point out that the assumption that G is P_{10}-free was not used for Case 1, i.e., the above result holds for the general case.

Case 2. There are vertices $t_i \in T_i, t_j \in T_j, t_h \in T_h$, with $i, j, h \in \{1, \ldots, k\}$ mutually distinct, which induce a P_3 in G.

P3. One can check if G has a d.i.m. containing xy in polynomial time in the following cases:

(a) there are vertices $t_i \in T_i, t_j \in T_j, t_h \in T_h, t_l \in T_l$, with $i, j, h, l \in \{1, \ldots, k\}$ mutually distinct, which induce a P_4 in G namely $t_i - t_j - t_h - t_l$;
(b) there are vertices $t_i \in T_i, t_j \in T_j, t_h, \bar{t_h} \in T_h$, with $i, j, h \in \{1, \ldots, k\}$ mutually distinct, which induce a P_4 in G namely $t_i - t_j - t_h - \bar{t_h}$;
(c) there are vertices $t_i \in T_i, t_j, \bar{t_j} \in T_j, t_h \in T_h$, with $i, j, h \in \{1, \ldots, k\}$ mutually distinct, which induce a P_4 in G namely $t_i - t_j - \bar{t_j} - t_h$;

Proof. First let us prove statement (a).

Assume that there are such vertices, say without loss of generality $t_1 \in T_1, t_2 \in T_2, t_3 \in T_3, t_4 \in T_4$, which induce a P_4 in G. Then let us prove two claims.

Claim 1. *Assume that there are vertices, say without loss of generality $t_5 \in T_5, t_6 \in T_6, t_7 \in T_7$, which induce a P_3 in G. Then $\{t_1, \ldots, t_4\}$ contacts $\{t_5, t_6, t_7\}$.*

Proof. By contradiction assume that $\{t_1, \ldots, t_4\}$ does not contact $\{t_5, t_6, t_7\}$. Then let P be any induced path in G from u_1 and u_5 through $N_0 \cup N_1$ (let us recall that, by construction, $G[N_0 \cup N_1]$ is connected). Then the subgraph of G induced by $t_4, t_3, t_2, t_1, u_1, P, u_5, t_5, t_6, t_7$ contains an induced P_{10}, a contradiction. ⋄

Claim 2. *One can check if G has a d.i.m. containing xy in polynomial time.*

Proof. The proposed method is based on Claim 1.

Assume that all vertices of $T_1, \ldots, T_4$ have an assigned color, and assume to repeatedly apply forcing rules (R1)-(R3), in order to possibly color vertices of members of T_{family}. Then T_{family} can be partitioned into: T'_{family}, formed by those members whose vertices are all colored, and T''_{family}, formed by those members whose vertices are not all colored. Clearly $T_1, \ldots, T_4 \in T'_{family}$. Concerning the other members of T_{family}: note that for any triple $t_a \in T_a, t_b \in T_b, t_c \in T_c$, with $\{t_a, t_b, t_c\}$ inducing a P_3, with $a, b, c \in \{1, \ldots, k\}$ and $a, b, c \geq 5$, one has that by Claim 1 and by (R1) at least one vertex in $\{t_a, t_b, t_c\}$, say t_d with $d \in \{a, b, c\}$, is forced to be black, so that the color of all vertices of T_d is forced by (R2), so that $T_d \in T'_{family}$. Then T''_{family} enjoys Case 1.

Summarizing, to check if G has a d.i.m. containing xy, one can proceed as follows: For each $(t'_1, \ldots, t'_4) \in T_1 \times \ldots \times T_4$ assign color black to $t'_1, \ldots, t'_4$; then repeatedly apply forcing rules (R1)-(R3) in order to possibly color vertices of members of T_{family}; let T'_{family} and T''_{family} be defined as above; if no contradiction arose, then a feasible coloring of vertices of T'_{family} is directly obtained, while a feasible coloring of vertices of T'_{family} (if one exists) can be obtained since T''_{family} enjoys Case 1 [in details: one can check if T''_{family} admits a feasible coloring, which is consistent with the (possible) forced consequences of the above forcing rules (R1)-(R3), by referring to Case 1]. That is correct by the above and can be executed in polynomial time since the procedure of Case 1 can be executed in polynomial time. ⋄

Then statement (a) follows by Claim 2.

Then let us prove statement (b). The proof is very similar to that of statement (a): that is based on the fact if there are such vertices, say without loss of generality $t_1 \in T_1, t_2 \in T_2$, and $t_3, \bar{t_3} \in T_3$, which induce a P_4 in G, and if there are vertices, say without loss of generality $t_4 \in T_4, t_5 \in T_5, t_6 \in T_6$, which induce a P_3 in G, then $\{t_1, t_2, t_3, \bar{t_3}\}$ contacts $\{t_4, t_5, t_6\}$ (cf. Claim 1).

Then let us prove statement (c). The proof is very similar to that of statement (b). □

Remark 2. Let us assume that statements (a)-(b)-(c) of P3 do not occur [else one can apply P3].

Let us recall that, by (A1), $N_3 = T_1 \cup \ldots \cup T_k$.

Let K be any component of $G[N_3]$ containing vertices $t_i \in T_i, t_j \in T_j, t_h \in T_h$, with $i, j, h \in \{1, \ldots, k\}$ mutually distinct, which induce a P_3 in G. Note that, once the color of any vertex of K is fixed, then the color of all vertices of K is forced by (R1). Furthermore, by (A2), every induced P_3 of K is such that its three vertices belong to respective different members of T_{family}.

Let us introduce some preliminary definition.

— Let us say that a member T_i of T_{family} is *critical* for K if $|T_i \cap K| = 1$.

— For any induced P_3 of K, say of vertices $t_i \in T_i, t_j \in T_j, t_h \in T_h$, with $i, j, h \in \{1, \ldots, k\}$, let us say that the P_3 *involves* T_i, T_j, T_h.

— Let us say that an induced P_3 of K is *max-critical* for K if the P_3 involves a maximum number of members of T_{family} which are critical for K. Note that such a maximum number is at most 3.

Now let us assume, without loss of generality, that a *max-critical* P_3 of K is induced by vertices $t_1 \in T_1, t_2 \in T_2, t_3 \in T_3$, with edges $t_1 t_2$ and $t_2 t_3$.

P4. Assume that all vertices of T_1, T_2, T_3 have an assigned color, and assume to repeatedly apply forcing rules (R1)-(R3), in order to possibly color vertices of members of T_{family}. If there are vertices $t_i \in T_i, t_j \in T_j, t_h \in T_h$, with $i, j, h \in \{4, \ldots, k\}$ mutually distinct, which induce a P_3 in G, then all vertices of at least one set in $\{T_i, T_j, T_h\}$ are colored.

Proof. Without loss of generality let us assume that $i = 4,\ j = 5,\ h = 6$.

Let us write $A = \{u_1, u_2, u_3\} \cup \{t_1, t_2, t_3\}$ and $B = \{u_4, u_5, u_6\} \cup \{t_4, t_5, t_6\}$.

As a preliminary let us observe that P4 follows as soon as some vertex in $\{t_4, t_5, t_6\}$ is colored: in fact, in this case, by (R1) at least one vertex in $\{t_4, t_5, t_6\}$ is black, say t_h with $h \in \{4, 5, 6\}$, and then by (R2) all vertices of T_h are colored.

Claim 1. *If there is a path through $G[N_3]$ from a vertex of $T_1 \cup T_2 \cup T_3$ to a vertex of $\{t_4, t_5, t_6\}$, then P4 follows.*

Proof. In fact, if such a path should exist, then P4 would follow by (R1) and since all vertices of T_1, T_2, T_3 are colored. □

Assumption 3. Let us assume that there is no path through $G[N_3]$ from a vertex of $T_1 \cup T_2 \cup T_3$ to a vertex of $\{t_4, t_5, t_6\}$, else P4 follows, by Claim 1.

Then let us consider a shortest path, say P, in $G[\{u_1, \ldots, u_k\} \cup T_1 \cup \ldots \cup T_k]$ from A to B; then, let a be the vertex of $P \setminus (A \cup B)$ which is adjacent to some vertex of A, and let b be the vertex of $P \setminus (A \cup B)$ which is adjacent to some vertex of B; note that vertex a and vertex b may coincide.

Claim 2. *The following statements hold:*

(i) *if a contacts $\{u_1, u_2, u_3\}$, then a is the endpoint of an induced P_5 together with four vertices of A; if a contacts $\{t_1, t_2, t_3\}$ and does not contact $\{u_1, u_2, u_3\}$, then a is the endpoint of an induced P_4 together with three vertices of A and belongs to N_3.*

(ii) *if b contacts $\{u_4, u_5, u_6\}$, then b is the endpoint of an induced P_5 together with four vertices of B; if b contacts $\{t_4, t_5, t_6\}$ and does not contact $\{u_4, u_5, u_6\}$, then b is the endpoint of an induced P_4 together with three vertices of B and belongs to N_3.*

Proof. Let us just prove statement (i), since statement (ii) can be proved similarly, by symmetry.

First assume that a is adjacent to u_1; then $a \in T_1$; then a is nonadjacent to t_2 by (A2). If a is nonadjacent to t_1, then a, u_1, t_1, t_2, u_2 induce a P_5. If a is adjacent to t_1, then by Remark 2 one has that a is adjacent to t_3; by Assumption 3 and by construction, there is a vertex say a' in $P \setminus (A \cup B)$ adjacent to a, in particular a' belongs to some member T_i of T_{family} with $i \notin \{1, 2, 3\}$; then, by Remark 2 and since $G[N_3]$ is bipartite, a' is adjacent to t_2; but this contradicts the definition of a, i.e., it is not possible that a is adjacent to t_1.

Then assume that a is adjacent to u_2; then $a \in T_2$; then a is nonadjacent to t_1 and to t_3 by (A2). If a is nonadjacent to t_2, then a, u_2, t_2, t_3, u_3 induce a P_5. If a is adjacent to t_2, then let us consider the following argument; by Assumption 3 and by construction, there is a vertex say a' in $P \setminus (A \cup B)$ adjacent to a, in particular a' belongs to some member T_i of T_{family} with $i \notin \{1, 2, 3\}$; then, by Remark 2 and since $G[N_3]$ is bipartite, a' is adjacent to t_1 or to t_3; but this contradicts the definition of a, i.e., it is not possible that a is adjacent to t_2.

Finally assume that a is adjacent to u_3; then, by symmetry, this occurrence can be treated similarly to that in which a is adjacent to u_1.

Now let us assume that a contacts $\{t_1, t_2, t_3\}$ and does not contact $\{u_1, u_2, u_3\}$. Then, by construction, a belongs to N_3 [i.e. to some member T_h of T_{family} with $h \notin \{1, 2, 3\}$]: if a is adjacent to t_1, then a, t_1, t_2, u_2 induce a P_4; if a is adjacent to t_2; then a, t_2, t_3, u_3 induce a P_4; if a is adjacent to t_3, then a, t_3, t_2, u_2 induce a P_4. □

Claim 3. *$P \setminus (A \cup B)$ has at most 3 vertices.*

Proof. It follows by construction, by Claim 2, and since G is P_{10}-free. □

Then, by Claim 3, let us consider the following exhaustive occurrences.

OCCURRENCE 1: $P \setminus (A \cup B)$ has exactly 3 vertices.

Then $P \setminus (A \cup B)$ is an induced P_3, of vertices a, b, and say z, and of edges az and zb.

Since G is P_{10}-free, by Claim 2 one has that: a contacts $\{t_1, t_2, t_3\}$ and does not contact $\{u_1, u_2, u_3\}$; b contacts $\{t_4, t_5, t_6\}$ and does not contact $\{u_4, u_5, u_6\}$; furthermore $a, b \in N_3$; then by Assumption 3, one has $z \in N_2$, say $z = u_7$; it follows that $a, b \in T_7$.

Then let us consider the following exhaustive cases.

As a preliminary (recalling that $G[N_3]$ is bipartite) let us observe that one can assume that either a is adjacent to t_2 or a is adjacent to t_1 and to t_3: in fact by Remark 2, if a is adjacent to t_1 (or to t_3), then a is adjacent to t_3 (to t_1) as well.

:: Assume that T_7 is not critical for K. Then, since by construction $a \in K$ [that is $|K \cap T_7| \geq 1$], one has $|K \cap T_7| \geq 2$. Note that $b \notin K$, by Assumption 3 and by construction. Then there is $t \in T_7$, different to a and to b, such that $t \in K$: in particular let us choose vertex t in order that, over vertices of $(K \cap T_7) \setminus \{a\}$, t has a minimum distance in K from $\{t_1, t_2, t_3\}$. Clearly t does not contact $\{u_4, u_5, u_6\}$ since $t \in T_7$; furthermore, by Assumption 3, t [and more generally any vertex of K] does not contact $\{t_4, t_5, t_6, b\}$; furthermore one can assume that t is adjacent to no vertex of $K \cap T_7$, else either t or its neighbor in $K \cap T_7$ would be black, so that by (R2) vertex b would be white, and then by (R1) vertices of $\{t_4, t_5, t_6\}$ are colored, i.e., P4 follows; in particular t is nonadjacent to a.

Then let us consider a shortest path say P^* in K from t to $\{t_1, t_2, t_3\}$.

If t contacts $\{t_1, t_2, t_3\}$, then by (A2) t is adjacent to a vertex of $\{t_1, t_2, t_3\}$ which is nonadjacent to a, so that (by the above preliminary) a and t have different colors, so that by (R2) vertex b is white, and then by (R1) vertices of $\{t_4, t_5, t_6\}$ are colored, i.e., P4 follows.

If t does not contact $\{t_1, t_2, t_3\}$, then let $\bar{t}$ be the first vertex in P^* (going from t to $\{t_1, t_2, t_3\}$) which contacts A;

if $\bar{t}$ contacts $\{u_1, u_2, u_3\}$ [so that $\bar{t} \neq a$], then $\bar{t}$ is the endpoint of an induced P_5 together with four vertices in A, so that t is the endpoint of an induced P_6 together with (at least) five vertices in $P^* \cup A$, so that [by the choice of t, by the choice of $\bar{t}$, and since a does not belong to P^* by construction] such an induced P_6 together with vertices u_7, b, and two vertices of $\{t_4, t_5, t_6\}$ (depending on the neighbors of b) induce a P_{10}, i.e., this occurrence is not possible;

if $\bar{t}$ contacts $\{t_1, t_2, t_3\}$ and does not contact $\{u_1, u_2, u_3\}$, then let us consider the vertex say $\bar{t}'$ in P^* which precedes $\bar{t}$ (going from t to $\{t_1, t_2, t_3\}$), let us observe that $\bar{t}$ and $\bar{t}'$ do not contact $\{u_1, u_2, u_3\}$ (by the above), and let us conclude that $\bar{t}', \bar{t}$, and other two vertices of $\{t_1, t_2, t_3\}$ (depending on the neighbors of $\bar{t}$) induce a P_4 which contradicts Remark 2, i.e., this occurrence is not possible.

:: Assume that T_7 is critical for K. Then let us show that a contradiction arises, i.e., this occurrence is not possible. In details, let us show that a is the endpoint of an induced P_5 together with four vertices of A, which is not possible since G is P_{10}-free [in fact by definition of b and by Claim 2, b is the endpoint of an induced P_4 together with three vertices of B, and in turn u_7 is the endpoint of an induced P_5].

Then let us consider the following exhaustive cases according to the above preliminary.

If a is adjacent to t_1 and to t_3, then, since $\{t_1, t_2, t_3\}$ induces a *max-critical* P_3 of K, T_2 is critical for K (else one would have considered the P_3 induced by t_1, a, t_3); it follows that, since $t_2 \in K$ and since by (A3) $|T_2| \geq 2$, there is a vertex $t_2' \in T_2 \setminus K$; but then a P_5 is induced by a, t_1, t_2, u_2, t_2'.

If a is adjacent to t_2, then, since $\{t_1, t_2, t_3\}$ induces a *max-critical* P_3 of K, T_1 is critical for K (else one would have considered the P_3 induced either by t_3, t_2, a); it follows that, since $t_1 \in K$ and since by (A3) $|T_1| \geq 2$, there is a vertex $t_1' \in T_1 \setminus K$; but then a P_5 is induced by a, t_2, t_1, u_1, t_1'.

OCCURRENCE 2: $P \setminus (A \cup B)$ has exactly 2 vertices.

Then $P \setminus (A \cup B)$ is an induced P_2 of vertices a, b.

First let us assume that a contacts $\{u_1, u_2, u_3\}$, i.e., $a \in T_1 \cup T_2 \cup T_3$; it follows that $b \in N_3$; then, by Claim 2 and since G is P_{10}-free, b contacts $\{t_4, t_5, t_6\}$; but this contradicts Assumption 3, i.e., this occurrence is not possible.

Then let us assume that a contacts $\{t_1, t_2, t_3\}$ and does not contact $\{u_1, u_2, u_3\}$, i.e., $a \in N_3$, say $a \in T_7$, without loss of generality; it follows that $b \in N_3$, else $b = u_7$, so that b could not contact B; then, by Assumption 3, b does not contact $\{t_4, t_5, t_6\}$; then by construction, b contacts $\{u_4, u_5, u_6\}$, say $b \in T_h$ for some $h \in \{4, 5, 6\}$; then b, a, and two vertices of $\{t_1, t_2, t_3\}$ (depending on the neighbors of a) induce a P_4 which contradicts Remark 2, i.e., this occurrence is not possible.

OCCURRENCE 3: $P \setminus (A \cup B)$ has exactly 1 vertex.

Then $P \setminus (A \cup B)$ is a singleton namely $a = b$. Then, by construction, $a \in N_3$.

First let us assume that a contacts $\{u_1, u_2, u_3\}$, i.e., $a \in T_1 \cup T_2 \cup T_3$; it follows that a does not contact $\{u_4, u_5, u_6\}$ by construction, i.e., a contacts $\{t_4, t_5, t_6\}$. This contradicts Assumption 3, i.e., this occurrence is not possible.

Then let us assume that a contacts $\{t_1, t_2, t_3\}$ and does not contact $\{u_1, u_2, u_3\}$. Note that, by Assumption 3, a does not contact $\{t_4, t_5, t_6\}$. Then a contacts $\{u_4, u_5, u_6\}$, say a is adjacent to u_h for some $h \in \{4, 5, 6\}$, and in particular a is the endpoint of an induced P_5 together with four vertices of B by Claim 2.

Then let us consider the following exhaustive cases.

As a preliminary (recalling that $G[N_3]$ is bipartite) let us observe that one can assume that either a is adjacent to t_2 or a is adjacent to t_1 and to t_3: in fact by Remark 2, if a is adjacent to t_1 (or to t_3), then a is adjacent to t_3 (to t_1) as well.

As a preliminary let us observe that one can assume that the color of a (let us recall that, by assumptions of P4, the color of a is forced since $a \in K$) is white: in fact otherwise, by (R2) the color of t_h is forced to be black and then by (R1) vertices of $\{t_4, t_5, t_6\}$ are colored, i.e., P4 follows.

:: Assume that T_h is not critical for K. Then, since by construction $a \in K$ [that is $|K \cap T_h| \geq 1$], one has $|K \cap T_h| \geq 2$. Let us recall that $t_h \notin K$ by Assumption 3. Then there is $t \in T_h$, different to a and to t_h, such that $t \in K$: in particular let us choose vertex t in order that, over vertices of $(K \cap T_h) \setminus \{a\}$, t has a minimum distance in K from $\{t_1, t_2, t_3\}$. Let us observe that t does not contact $\{t_4, t_5, t_6, \}$ by Assumption 3; furthermore one can assume that t is adjacent to no vertex of $K \cap T_h$, else either t or its neighbor in $K \cap T_h$ would be black, so that by (R2) vertex t_h would be white, and then by (R1) vertices of $\{t_4, t_5, t_6\}$ are colored, i.e., P4 follows; in particular t is nonadjacent to a.

Then let us consider a shortest path say P^* in K from t to $\{t_1, t_2, t_3\}$. By Assumption 3 one can assume that no vertex of P^* contacts $\{t_4, t_5, t_6\}$. Furthermore, by the choice of t, one has that t is the endpoint of an induced P_4 say Z together with three vertices of $\{u_h\} \cup \{t_4, t_5, t_6\}$ and that no vertex of Z contacts $P^* \setminus \{t\}$. It follows that, since G is P_{10}-free, $P^* \setminus \{t, t_1, t_2, t_3\}$ has *at most* three vertices, say z_1, z_2, z_3, inducing a path $z_1 - z_2 - z_3$.

If t contacts $\{t_1, t_2, t_3\}$, then by (A2) t is adjacent to a vertex of $\{t_1, t_2, t_3\}$ which is nonadjacent to a, so that (by the above preliminary) a and t have different colors, so that by (R2) vertex t_h is white, and then by (R1) vertices of $\{t_4, t_5, t_6\}$ are colored, i.e., P4 follows.

If t does not contact $\{t_1, t_2, t_3\}$, then let us consider the following exhaustive cases; in particular, let us assume that z_1 contacts $\{t_1, t_2, t_3\}$, without loss of generality.

:::: If $P^* \setminus \{t, t_1, t_2, t_3\} = \{z_1\}$, then: if z_1 is adjacent to t_1, then z_1 is nonadjacent to u_2 by (A2), and then (independently to the fact that z_1 is adjacent or nonadjacent to u_1) one has that t, z_1, t_1, t_2 induce a P_4 which contradicts Remark 2; if z_1 is adjacent to t_2, then z_1 is nonadjacent to u_1 by (A2), and then (independently to the fact that z_1 is adjacent or nonadjacent to u_2) one has that t, z_1, t_2, t_1 induce a P_4 which contradicts Remark 2; if z_1 is adjacent to t_3, then one can proceed similarly to the case in which if z_1 is adjacent to t_1, by symmetry.

:::: If $P^* \setminus \{t, t_1, t_2, t_3\} = \{z_1, z_2\}$, then: if z_1 is adjacent to t_1, then z_1 is nonadjacent to u_2 by (A2), so that, *either* z_1 is adjacent to u_1 and in this case [since z_2 would be nonadjacent to u_1 by Lemma 1 (*ii*)] t, z_2, z_1, t_1 induce a P_4 which contradicts Remark 2, *or* z_1 is nonadjacent to u_1 and in this case [since z_2 would be nonadjacent to u_1 by (A2)] t, z_2, z_1, t_1 induce a P_4 which contradicts Remark 2; if z_1 is adjacent to t_2, then z_1 is nonadjacent to u_1 and to u_3 by (A2), while z_2 is nonadjacent to u_2 [either by (A2), if z_1 is nonadjacent to u_2, or by Lemma 1 (*ii*), if z_1 is adjacent to u_2] and is nonadjacent to at least one vertex of $\{u_1, u_3\}$ by construction, say z_2 is nonadjacent to u_1 (without loss of generality by symmetry), and then z_2, z_1, t_2, t_1 induce a P_4 which contradicts Remark 2; if z_1 is adjacent to t_3, then one can proceed similarly to the case in which if z_1 is adjacent to t_1, by symmetry.

:::: If $P^* \setminus \{t, t_1, t_2, t_3\} = \{z_1, z_2, z_3\}$, then:

assume that z_1 is adjacent to t_1; then z_1 is nonadjacent to u_2 by (A2); if z_1 is adjacent to u_1, then z_2 is nonadjacent to u_1 by Lemma 1 (*ii*), and then, *either* z_2 is nonadjacent to u_2, and then z_2, z_1, t_1, t_2 induce a P_4 which contradicts Remark 2, *or* z_2 is adjacent to u_2, and then [since z_3 is nonadjacent to u_1 by (A2)] t, z_3, z_2, z_1 induce a P_4 which contradicts Remark 2;

assume that z_1 is adjacent to t_2; then z_1 is nonadjacent to u_1 and to u_3 by (A2), while z_2 is nonadjacent to u_2 [either by (A2) if z_1 is nonadjacent to u_2, or by Lemma 1 (*ii*) if z_1 is adjacent to u_2] and is nonadjacent to at least one vertex of $\{u_1, u_3\}$ by construction, say that z_2 is nonadjacent to u_1 (without loss of generality by symmetry), and then z_2, z_1, t_2, t_1 induce a P_4 which contradicts Remark 2;

assume that z_1 is adjacent to t_3; then one can proceed similarly to the case in which z_1 is adjacent to t_1, by symmetry.

:: Assume that T_h is critical for K. Then let us show that a contradiction arises, i.e., this case is not possible.

Then let us consider the following exhaustive cases according to the above preliminary.

If a is adjacent to t_1 and to t_3, then, since $\{t_1, t_2, t_3\}$ induces a *max-critical* P_3 of K, T_2 is critical for K (else one would have considered the P_3 induced by t_1, a, t_3); it follows that, since $t_2 \in K$ and since by (A3) $|T_2| \geq 2$, there is a vertex $t'_2 \in T_2 \setminus K$; then a is the endpoint of an induced P_5 say $\overline{P}$ together with vertices t_3, t_2, u_2, t'_2; in particular, by Assumption 3, t'_2 does not contact $\{t_4, t_5, t_6\}$; then a is nonadjacent to u_4, else an induced P_{10} arises, involving u_4, t_4, t_5, t_6, u_6, and $\overline{P}$; furthermore, a is nonadjacent to u_6, by symmetry; then a is adjacent to u_5; now, by (A3), let t'_4 be *any* vertex of $T_4 \setminus \{t_4\}$; let us observe that t'_4 is nonadjacent to t_4 [else t'_4 would be adjacent to t_6 by Remark 2; then, since by Assumption 3 t'_4 does not contact $\{a\} \cup T_1 \cup T_2 \cup T_3$, vertices u_4, t'_4, t_6, t_5, u_5, and $\overline{P}$ would induce a P_{10}]; furthermore one can assume that t'_4 is nonadjacent to a and to t_2 [in fact, since by the above preliminary the color of a is white, and the color of t_2 is white too (since a is adjacent to t_1), the color of t'_4 would be black, so that the color of t_4 would be white by (R2), and P4 would follow]; furthermore one can assume that t'_4 is nonadjacent to t'_2, in case t'_2 should be white, by an argument similar to that of the previous sentence; summarizing, in order to avoid a P_{10} induced by t'_4, u_4, t_4, t_5, u_5, and $\overline{P}$, one has that t'_4 is adjacent either to t_1 [which is black since a is white], or to t_3 [which is black since a is white], or to t'_2 [in case t'_2 should be black]; but then one can conclude that *any* vertex of $T_4 \setminus \{t_4\}$ is white, so that by (R3) the color of t_4 is black, and P4 follows.

If a is adjacent to t_2, then one applies a similar argument [not reported for brevity, in particular, T_3 is critical for K etc.], in order to get to a similar conclusion. □

P5. One can check if G has a d.i.m. containing xy in polynomial time.

Proof. The proposed method is based on Remark 2 and P4.

Let K be a component of $G[N_3]$ containing vertices $t_i \in T_i, t_j \in T_j, t_h \in T_h$, with $i, j, h \in \{1, \dots, k\}$ mutually distinct, which induce a P_3 in G. Now let us assume, without loss of generality, that a *max-critical* P_3 of K is induced by vertices $t_1 \in T_1, t_2 \in T_2, t_3 \in T_3$, with edges t_1, t_2 and t_2, t_3.

Assume that all vertices of $T_1, \dots, T_3$ have an assigned color, and assume to repeatedly apply forcing rules (R1)-(R3), in order to possibly color vertices of members of T_{family}. Then T_{family} is partitioned into: T'_{family}, formed by those members whose vertices are all colored, and T''_{family}, formed by those members whose vertices are not all colored. Clearly $T_1, \dots, T_3 \in T'_{family}$. Concerning the other members of T_{family}: note that for any triple $t_a \in T_a, t_b \in T_b, t_c \in T_c$, with $\{t_a, t_b, t_c\}$ inducing a P_3, with $a, b, c \in \{1, \dots, k\}$ and $a, b, c \geq 4$, one has that by P4 at least one vertex in $\{t_a, t_b, t_c\}$, say t_d with $d \in \{a, b, c\}$, is forced to be black, so that the color of all vertices of T_d is forced by (R2), so that $T_d \in T'_{family}$. Then T''_{family} enjoys Case 1.

Summarizing, to check if G has a d.i.m. containing xy, one can proceed as follows: For each $(t'_1, \dots, t'_3) \in T_1 \times \dots \times T_3$ assign color black to $t'_1, \dots, t'_3$; then repeatedly apply forcing rules (R1)-(R3) in order to possibly color vertices of members of T_{family}; let T'_{family} and T''_{family} be defined as above; then, if no contradiction arose, then a feasible coloring of vertices of T'_{family} is directly obtained, while a feasible coloring of vertices of T'_{family} (if one exists) since T''_{family} enjoys Case 1 [in details: one can check if T''_{family} admits a feasible coloring, which is consistent with the (possible) forced consequences of the above forcing rule (R1)-(R3), by referring to Case 1]. That is correct by the above and can be executed in polynomial time since the procedure of Case 1 can be executed in polynomial time. □

This completes the proof for Case 2.

4. The case $N_4 \neq \emptyset$

In this section let us show that, if $N_4 \neq \emptyset$ [i.e., in the general case, with possibly $N_4 \neq \emptyset$], then one can check in polynomial time whether G has a d.i.m. M with $xy \in M$.

Recall that $N_k = \emptyset$ for $k \geq 6$ according to Algorithm DIM(G).

Then let us assume that $G[N_3 \cup N_4 \cup N_5]$ is connected, without loss of generality, else one can split the problem for each component of $G[N_3 \cup N_4 \cup N_5]$.

Observation 5. *If $v \in N_i$ for $i \geq 4$, then v is an endpoint of an induced P_6, say with vertices $v, v_1, v_2, v_3, v_4, v_5$ such that $v_1, v_2, v_3, v_4, v_5 \in N_0 \cup N_1 \cup \dots \cup N_{i-1}$ and with edges $vv_1 \in E$, $v_1v_2 \in E$, $v_2v_3 \in E$, $v_3v_4 \in E$, $v_4v_5 \in E$. Analogously, if $v \in N_3$, then v is an endpoint of a corresponding induced P_5.*

Proof. If $i \geq 5$, then clearly there is such a P_6. Thus, assume that $v \in N_4$. Then $v_1 \in N_3$ and $v_2 \in N_2$. Recall that y, x, r induce a P_3. If $v_2r \in E$ then v, v_1, v_2, r, x, y induce a P_6. Thus assume that $v_2r \notin E$. Let $v_3 \in N_1$ be a neighbor of v_2. Now, if $v_3x \in E$ then v, v_1, v_2, v_3, x, r induce a P_6, and if $v_3x \notin E$ but $v_3y \in E$, then v, v_1, v_2, v_3, y, x induce a P_6. Analogously, if $v \in N_3$ then v is an endpoint of an induced P_5 (which could be part of the P_6 above). Thus, Observation 5 is shown. □

For any component of $G[N_4 \cup N_5]$, say K, let us say that:
K is *pure* if K contacts exactly one member of T_{family},
K is *impure* if K contacts at least two members of T_{family}.

P6. One can assume that every component of $G[N_4 \cup N_5]$ is non-trivial, i.e., that every vertex of N_4 is not isolated in $G[N_4 \cup N_5]$.

Proof. In fact if a vertex $v \in N_4$ is isolated in $G[N_4 \cup N_5]$, then by (7) and by definition of d.i.m. the color of v is forced white, so that one can apply the Vertex Reduction to v [without disconnecting the graph]. □

P7. One can assume that every component of $G[N_4 \cup N_5]$ is impure.

Proof. Let us assume that there are pure components of $G[N_4 \cup N_5]$. Then, for every member $T_i \in T_{family}$, let Q_i denote the union of vertex-sets of those pure components of $G[N_4 \cup N_5]$ which contact T_i.

Let us observe that T_i is a cut-set for G separating Q_i: in particular, if G has a d.i.m. M with $xy \in M$, then $G[\{u_i\} \cup T_i \cup Q_i]$ has a d.i.m. M^* with $u_it \in M^*$ for some $t \in T_i$.

Note that since $dist_G(u_i, q) \leq 3$ for any $q \in \{u_i\} \cup T_i \cup Q_i$ (by construction), one can check *for each* $t \in T_i$ if $G[\{u_i\} \cup T_i \cup Q_i]$ has a d.i.m. M^* with $u_it \in M^*$ in polynomial time, by referring to the case $N_4 = \emptyset$ of the previous section with u_it instead of xy.

Then, for every member $T_i \in T_{family}$ such that $Q_i \neq \emptyset$, one can proceed as follows:

:: For each $t \in T_i$, check if there is a d.i.m. M^* of $G[\{u_i\} \cup T_i \cup Q_i]$, with $u_it \in M^*$: if the answer is *no*, then color t by white.

:: Remove Q_i from V.

Then, by the above, G has a d.i.m. M with $xy \in M$ if and only if the resulting graph [together with the above possible forcing conditions for vertices of T_i] has a d.i.m. M' with $xy \in M'$. □

P8. For any feasible partial coloring of $N_0 \cup N_1 \cup N_2 \cup N_3 \cup N_4$ [with vertices x and y black] such that all vertices of N_4 are colored, one can check in polynomial time if there is a feasible coloring of N_5 which is consistent with it.

Proof. Let us fix any feasible partial coloring of $N_0 \cup N_1 \cup N_2 \cup N_3 \cup N_4$ [with vertices x and y black] such that all vertices of N_4 are colored. By definition of d.i.m., vertices of N_4 can be partitioned into $\{W(N_4), B'(N_4), B''(N_4)\}$, where: vertices of $W(N_4)$ are white, vertices of $B'(N_4)$ are black and have no black neighbor in N_4 [and more generally in $N_3 \cup N_4$ by (7)], vertices of $B''(N_4)$ are black and have one black neighbor in N_4.

Claim 1. *Vertices of $W(N_4)$ force their neighbors in N_5 to be black. Vertices of $B''(N_4)$ force their neighbors in N_5 to be white. Vertices of $B'(N_4)$ should have one black neighbor in N_5: however, if a vertex of N_5 has two neighbors in $B'(N_4)$, then it is forced to be white.*

Proof. The first two statements follow by definition of d.i.m. The third statement follows by definition of d.i.m. and by (7). □

Let N_5^* be the set of vertices of N_5 which enjoy none of the forcing conditions of Claim 1, i.e., each vertex of N_5^* has exactly one neighbor in N_4, namely, a vertex of $B'(N_4)$.

Then − once checked if the forcing conditions of Claim 1 lead to a contradiction − the problem is to check in polynomial time if there is a feasible coloring of $B'(N_4) \cup N_5^*$ which is consistent with the current feasible partial coloring.

Let us assume that $G[B'(N_4) \cup N_5^*]$ is connected, without loss of generality, else one can split the problem for the corresponding components.

Claim 2. *There exists a vertex $u_i \in S_2$ such that $dist_G(u_i, z) = 3$ for any $z \in N_5^*$.*

Proof. First let us observe that $B'(N_4)$ is an independent set (by construction).

Assume by contradiction that such a vertex does not exist. Then let $u_i \in S_2$ be such that the number of vertices $z \in N_5^*$ such that $dist_G(u_i, z) = 3$ for any $z \in N_5$ is maximum over all vertices of S_2. Then, by assumption of contradiction and by construction, there are vertices $u_j \in S_2, t_j \in T_j, b \in B'(N_4), z \in N_5^*$, with $j \neq i$, and vertices $t_i \in T_i, b' \in B'(N_4), z' \in N_5^*$, such that: $\{u_j, t_j, b, z\}$ and $\{u_i, t_i, b', z'\}$ respectively induce a P_4, and between such P_4's there is at most one edge possibly between z and z' [note in particular that t_j and t_i are white].

Now, since $G[B'(N_4) \cup N_5^*]$ is connected, consider any shortest path in $G[B'(N_4) \cup N_5^*]$ say P from $\{b, z\}$ to $\{b', z'\}$. Note that P has at most one interior vertex [where interior vertices are those not in $\{b, z\} \cup \{b', z'\}$], else an induced P_{10} arises in the subgraph induced by P, by the above vertices, and by $N_0 \cup N_1$. And if P has exactly one interior vertex, then a similar induced P_{10} arises, recalling that by Claim 1 we assumed that no vertex of N_5^* is adjacent to two vertices of $B'(N_4)$. This leads to a contradiction. □

Then let $u_i \in S_2$ be according to Claim 2, i.e., such that $dist_G(u_i, v) \leq 3$ for any $v \in B'(N_4) \cup N_5^*$. Let us focus on the subgraph of G, say G', induced by $N'_0 = \{u_i, u\}$ where $u \in N_1$ is any neighbor of u_i, $N'_1 = N(u_i) \cap N(B'(N_4))$, $N'_2 = B'(N_4)$, and $N'_3 = N_5^*$. Then, by referring to the case $N_4 = \emptyset$ of the previous section with $u_i u$ instead of xy, one can check in polynomial time if there is a feasible coloring of $B'(N_4) \cup N_5^*$ which is consistent with the current feasible partial coloring.

This completes the proof of P8. □

P9. For any $i \in \{1, \dots, k\}$ and for any $t \in T_i$, with $N(t) \cap N_4 \neq \emptyset$, if the color of t is fixed black, then there are (at most) polynomially many feasible partial colorings of $N_0 \cup N_1 \cup N_2 \cup N_3 \cup N_4$ [with vertices x and y black] such that all vertices of N_4 are colored.

Proof. For any $i \in \{1, \dots, k\}$ and for any $t \in T_i$, with $N(t) \cap N_4 \neq \emptyset$, let us fix black the color of t. Then the color of vertices in $T_i \setminus \{t\}$ is forced (white): then, by (R1), the color of vertices in $N(T_i) \cap N_4$ is forced. Then let us consider any vertex $z \in N(T_j) \cap N_4$, for any $j \in \{1, \dots, k\}$, with $j \neq i$: then let $t_j \in T_j$ be adjacent to z.

Let us recall some preliminary:

by Observation 5, for any $\bar{t} \in N_3$, let $Q(\bar{t})$ denote the induced P_5 of G whose vertices except for $\bar{t}$ are in $N_0 \cup N_1 \cup N_2$;

by (9), there is no triangle in G with one vertex in N_3 and two vertices in N_4;

by P6 and by P7, every component of $G[N_4 \cup N_5]$ is non-trivial and impure.

Let $G[D]$, with vertex set D, be any component of $G[N_4 \cup N_5]$ such that $N(t) \cap D \neq \emptyset$ [i.e. $N(t) \cap D \cap N_4 \neq \emptyset$] according to the assumption.

By Observation 5 and since G is P_{10}-free, D can be partitioned into $\{D_1, D_2, D_3, D_4\}$, where D_j for $j = 1, 2, 3, 4$, denotes the set of vertices of D at distance j from t in $G[\{t\} \cup D]$.

Note that: vertices of D_1 are forced to be white; vertices of D_2 are forced to be black; $D_2 \neq \emptyset$ [by (9) and by P6]. Then let us distinguish the following two exhaustive cases with the aim of determining the color of z.

Case A. No vertex of D_3 is fixed black.

Assume that $z \in D$. Then, since by assumption of Case A either $D_3 = \emptyset$ or all vertices of D_3 are fixed white, the color of z is forced (in particular, if $v \in D_4$, then v is forced to be black).

Assume that $z \notin D$. Recall that [by P7] the component of $G[N_4 \cup N_5]$ containing z, say D', is impure. Then D' contacts T_h, for some $h \in \{1, \ldots, k\}$, with $h \neq j$: let $t_h \in T_h$ contact D'. Then there is a shortest path, say P, through D' from t_h to z, say of consecutive vertices $t_h, z_1, \ldots, z_l, z$: note that $l \geq 1$ (since z can not be adjacent to t_h by construction), and that t_j is nonadjacent to z_1 (by construction) and to z_l (by (9)). On the other hand, recall that $D_2 \neq \emptyset$, then let $d_1 \in D_1$, $d_2 \in D_2$ induce a P_2.

If $h \neq i$, then: if t_h contacts $\{d_1, d_2\}$, then the subgraph induced by $\{t, d_1, d_2, t_h\} \cup P \cup Q(t_j)$, contains an induced P_{10}; else, the subgraph induced by $\{d_2, d_1, t, u_i\} \cup N_1 \cup \{x, y\} \cup \{u_h, t_h\} \cup P \cup \{t_j\}$, contains an induced P_{10}; then this occurrence is not possible.

If $h = i$, then: if $t = t_h$, then the subgraph induced by $\{d_2, d_1, t\} \cup P \cup Q(t_j)$, contains an induced P_{10}; if $t \neq t_h$, then: if t_h contacts $\{d_1, d_2\}$, then the subgraph induced by $\{t, d_1, d_2, t_h\} \cup P \cup Q(t_j)$, contains an induced P_{10}; else, the subgraph induced by $\{d_2, d_1, t, u_i, t_h\} \cup P \cup \{t_j, u_j\} \cup N_1$, contains an induced P_{10}; then this occurrence is not possible.

Case B. A vertex of D_3 is fixed black.

Then let $d_1 \in D_1$, $d_2 \in D_2$, $d_3 \in D_3$ induce a P_3 (recall that d_1 is white and d_2 is black) and let us assume that d_3 is fixed black.

Assume that $z \in D$. If $z \in D_1 \cup D_2$, then by the above the color of z is forced. If $z \in D_3$, then there is a induced path say $z - z_2 - z_1$ with $z_2 \in D_2$ and $z_1 \in D_1$; if $\{z, z_2\}$ contacts $\{d_2, d_3\}$ (or if such sets should have a nonempty intersection), then the color of z is forced; else, if z_1 contacts $\{d_2, d_3\}$ [that is z_1 is adjacent to d_2], then d_3, d_2, z_1, z_2, z, and $Q(t_j)$ induce a P_{10}, which is not possible; else, if d_1 contacts $\{z_1, z_3\}$, then one similarly would get an induced P_{10}, which is not possible; else, one similarly would get an induced P_{10} [involving vertex t], which is not possible.

Assume that $z \notin D$. Then one can refer to the corresponding proof for Case A.

Summarizing, to obtain all possible feasible colorings of N_4, one can proceed as follows: (a) if $D_3 = \emptyset$, then derive the color of all vertices of N_4, by Case A; else: (b) fix white the color of all vertices of D_3 and then derive the color of all vertices of N_4, by Case A, and (c) for each vertex $d_3 \in D_3$, fix black the color of d_3 and then derive the color of all vertices of N_4, by Case B.

This completes the proof of P9. □

P10. For any $i \in \{1, \ldots, k\}$ and for any $t \in T_i$, with $N(t) \cap N_4 \neq \emptyset$, one can check if G has a d.i.m. M with $u_i t \in M$ and $xy \in M$ in polynomial time.

Proof. For any $i \in \{1, \ldots, k\}$ and for any $t \in T_i$, with $N(t) \cap N_4 \neq \emptyset$, let us fix black the color of t. Then, by P9, there are (at most) polynomially many feasible partial colorings of $N_0 \cup N_1 \cup N_2 \cup N_3 \cup N_4$ [with vertices x and y black] such that all vertices of N_4 are colored: then, let us fix one of such feasible partial colorings, say γ.

Let us consider $N_0 \cup N_1 \cup N_2 \cup N_3 \cup N_4$. Then, by referring to the case $N_4 = \emptyset$ of the previous section, one can check in polynomial time if there is a feasible coloring of $N_0 \cup N_1 \cup N_2 \cup N_3 \cup N_4$ which is consistent with γ.

Let us consider N_5. Then, by construction and by P8, one can check in polynomial time if there is a feasible coloring of vertices of N_5 which is consistent with γ. □

Finally let us show that one can check if G has a d.i.m. M with $xy \in M$ in polynomial time, by the following algorithm, which is based on P10.

:: Let $T = \{t \in T_i\colon i \in \{1, \ldots, k\}$ and $N(t) \cap N_4 \neq \emptyset\}$.
:: For each $t \in T$ do
:::: check if G has a d.i.m. M with $u_i t \in M$ and $xy \in M$ [by P10]
:::: if *yes*, then return the corresponding d.i.m. and STOP;
:: Assign color white to all vertices in T and repeatedly apply (R1), so to obtain a partial coloring of $N_3 \cup N_4$ such that all vertices of N_4 are colored, by definition of T.
:: Check if G has a d.i.m. M with $xy \in M$ which is consistent with the above partial coloring of $N_3 \cup N_4$, by referring to the case $N_4 = \emptyset$ of the previous section and by P8 [that is in details: concerning $N_0 \cup N_1 \cup N_2 \cup N_3 \cup N_4$, by referring to the case $N_4 = \emptyset$ of the previous section, one can check in polynomial time if there is a feasible coloring of $N_0 \cup N_1 \cup N_2 \cup N_3 \cup N_4$ which is consistent with the above partial coloring of $N_3 \cup N_4$; concerning N_5, by construction and by P8, one can check in polynomial time if there is a feasible coloring of vertices of N_5 which is consistent with the above partial coloring of $N_3 \cup N_4$]; if *yes*, then return the corresponding d.i.m. and STOP; if *no*, then return "G has no d.i.m. M with $xy \in M$" and STOP.

This completes the proof for the case $N_4 \neq \emptyset$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

The authors would like to thank the reviewers for their helpful comments and suggestions which improved the paper under different aspects. The second author would like to witness that he just tries to pray a lot and is not able to do anything without that - ad laudem Domini.

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