Coherence and Fuzzy Reasoning

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Abstract

Upper and lower conditional previsions are defined by the Choquet integral with respect to the Hausdorff outer and inner measures when the conditioning events have positive and finite Hausdorff outer or inner measures in their dimension; otherwise, when conditioning events have infinite or zero Hausdorff outer or inner measures in their dimension, they are defined by a 0-1 valued finitely, but not countably additive probability. It is proven that, if we consider the restriction of the (outer) Haudorff measures to the Borel σ-field, these (upper) conditional and unconditional previsions satisfy the disintegration property in the sense of Dubins with respect to all countable partitions of Ω. This result is obtained as a consequence of the fact that non-disintegrability characterizes finitely as opposed to countably additive probability. Moreover upper and lower conditional previsions are proven to be coherent, in the sense of Walley, with the unconditional previsions.
Properties related to the coherence of upper conditional probabilities are extended to the case where information is represented by fuzzy sets. In particular, given an infinite set Ω, a conditioning rule for possibility distribution is proposed so that it is coherent and it is coherent with the unconditional possibility distribution.
Through this conditional possibility distribution, a conditional possibility measure with respect to the partition of all singletons of [0,1] is defined. It is proved it satisfies the conglomerative principle of de Finetti.

Keywords. Upper and lower conditional previsions, Hausdorff outer and inner measures, disintegration property, fuzzy reasoning, conditional possibility distribution.

1 Introduction

Fuzzy reasoning has been introduced as a tool to handle vague and ambiguous information about linguistic or numerical variables. In [2],[16],[17] probabilistic and fuzzy reasoning are compared. The common aim is to extend the conditions of coherence, which characterize upper and lower conditional previsions to uncertainty measures used to manage vague and ambiguous information, represented by fuzzy sets.
In this paper two different problems are considered; firstly we continue the research about the possibility to define coherent upper and lower conditional probabilities by a class of Hausdorff outer and inner measures. In particular, when the conditioning event has positive and finite Hausdorf outer (inner) measure in its dimension, upper (lower) conditional previsions are defined by the Choquet integral ([5]) with respect to the outer (inner) Hausdorff measures, which are particular examples of monotone set functions. Otherwise, when the conditioning event has Hausdorff outer (inner) measure in its dimension equal to zero or infinity, upper (lower) conditional previsions are defined by a 0-1 valued finitely but not countably additive probability.
Moreover when we consider the restriction of the (outer) Haudorff measures to the Borel σ-field (upper) conditional and unconditional previsions are proven to satisfy the disintegration property in the sense of Dubins with respect to all countable partitions of Ω and to be coherent in the sense of Walley.
The second problem analysed in this paper is a comparison between probabilistic and fuzzy reasoning.
If upper and lower conditional previsions are defined with respect to outer and inner Hausdorff measures some properties are assured. We focus the attention on the property (P1 Section 2), which assures that coherent conditional probability is an uncertainty measure able to manage precise information represented by the singletons and on the disintegration property.
If we represent information by fuzzy sets and partial knowledge by conditional possibility measures do we lose these properties assured by the coherence?
In Section 5 of this paper we define a conditional possibility distribution on an infinite set Ω that is coherent and coherent with respect to the unconditional

possibility distribution. Moreover, through this conditional possibility distribution we obtain a possibility conditional measure with respect to the partition of all singletons that is coherent and that satisfies the (weak) conglomerative principle of de Finetti.

2 Upper and Lower Conditional Previsions Separately Coherent and Coherent with respect to the Unconditional Prevision.

In the approach of Walley ([15]) coherent conditional previsions are required to be separately coherent and coherent with respect to a given unconditional lower prevision $\underline{P}$. Given a non empty set Ω, a gamble X is a bounded function from Ω to R (the set of real numbers) and let **L** be the set of all gambles on Ω. When **K** is a linear space of gambles a coherent lower prevision $\underline{P}$ is a real function defined on **K**, such that the following conditions hold for every X and Y in **K**:

1) $\underline{P}(X) \geq \inf(X)$

2) $\underline{P}(\lambda X) = \lambda \underline{P}(X)$ for each positive constant λ

3) $\underline{P}(X+Y) \geq \underline{P}(X) + \underline{P}(Y)$

Lower previsions have a behavioural interpretation. If the gambles X in **K** are regarded as uncertain rewards, the lower prevision $\underline{P}(X)$ can be regarded as a supremum buying price for the gamble X.

Suppose that $\underline{P}$ is a lower prevision defined on a linear space **K**, its conjugate upper prevision $\overline{P}$ is defined on the same domain **K** by $\overline{P}(X) = -\underline{P}(-X)$.

If **K** contains only gambles that are indicator functions of events then a coherent lower (upper) prevision $\underline{P}$ defined on **K** is a coherent lower (upper) probability. So in this note we use the same symbol for (conditional) probability measure and (conditional) prevision.

Let **B** denote a partition of Ω, which is a non-empty, pair wise-disjoint subsets whose union is Ω. For B in **B** let **H**(B) be the set of gambles defined on B which includes the gamble B (we denote with the same symbol the set that represents an event and the indicator function of the event). A lower conditional prevision $\underline{P}(X|B)$ is a real function defined on **H**(B). Lower coherent conditional previsions $\underline{P}(X|B)$, defined for B in **B** and X in **H**(B) are required to be *separately coherent,* that is for every conditioning event B $\underline{P}(\cdot|B)$ is a coherent lower prevision on the domain **H**(B) and $\underline{P}(B|B) = 1$.

$\underline{P}(X|B)$ can be interpreted as the supremum buying price for X after we make the observation of a set B, that is we learn that the true state ω is in B. (This interpretation amounts to one of several possible "conditionalization" principles).

A gamble X is **B**-*measurable* when it is constant on each set B in **B**. Given a σ-field **G** of subsets of Ω, a gamble X is **G**-measurable if for every Borel set C of R the sets $\{\omega \in \Omega : \ \omega \in X^{-1}(C)\}$ belong to **G**.

Measurability with respect to a partition is, in general, a stronger condition than the measurability with respect to a σ-field. In fact, given two σ-fields **F** and **G** with **G** properly contained in **F** and generated by the partition **B**, fix A in **F**-**G.** We have that the indicator function of A is **B**-measurable, but not **G**-measurable.

Let **G**(**B**) be the class of **B**-measurable gambles. We denote by $\underline{P}(X|\mathbf{B})$ the function from **H** into **G**(**B**) whose image is the collection of coherent lower previsions $\{\underline{P}(\cdot|B) : B \text{ in } \mathbf{B}\}$. $\underline{P}(X|\mathbf{B})$ is separately coherent if all the lower conditional previsions are separately coherent. Let $\overline{P}(\cdot \mid B)$ be the conjugate upper conditional prevision. If $\underline{P}(\cdot|\mathbf{B})$ are linear previsions, that is $\underline{P}(\cdot \mid B) = \overline{P}(\cdot \mid B)$ for every B in **B,** then a *linear conditional prevision* $P(X|\mathbf{B})$ is defined by $P(X \mid B) = \underline{P}(X \mid B) = \overline{P}(X \mid B)$ for every B in **B**.

Given a non-empty set Ω, a partition **B** of Ω and a **B**-measurable gamble X if upper and lower conditional previsions are separately coherent then we have that $\underline{P}(X \mid B) = \overline{P}(X \mid B) = X$ for every B in **B** (Walley [15] pag. 292).

In particular if **B** is the partition of Ω that consists of all singletons and $X = I_A(\omega)$ is the indicator function of an event A then the previous property implies that

(P1) $P(A|\{\omega\}) = I_A(\omega)$ for every $\omega \in \Omega$ and for every $A \subseteq \Omega$.

The intuitive meaning of property (P1) is that coherent conditional probability is an uncertainty measure able to manage "precise" information, which is represented by the singletons of Ω.

This basic property is not always satisfied, in the continuous case, by the axiomatic definition of conditional probability given by the Radon-Nykodim; in fact if the conditioning σ-field is not countably generated, we may have that the regular conditional distributions, given that σ-field, are maximally improper (Seidenfeld et al. [13]) and therefore it does not verify the property (P1). It implies that conditional probability defined by the Radon-Nikodym derivative cannot always be used to represent uncertainty (see Example 33.11 of Billingsley [1]).

Walley ([15], 6.3) discusses the conditions in which an unconditional lower prevision $\underline{P}$ is coherent with $\underline{P}(\cdot|\mathbf{B})$. Given a class **D** of gambles, we say that **D** is a class of desirable gambles if, for each X in **D** and positive δ, we are disposed to accept the gamble X+δ. X is almost-desirable if we are not necessarily disposed to accept X itself.

The link between unconditional and conditional previsions can be expressed in terms of desirability, by the conglomerative principle (Walley [15], 6.3.3):

If a gamble X is B-desirable, i.e we intend to accept X provided we observe only the event B, for every set B in the partition ***B****, then X is desirable.*

Definition 1. Let $\underline{P}$ be an unconditional lower prevision defined on **K** and $\underline{P}(\cdot|\mathbf{B})$ be a conditional lower prevision on the domain **H** separately coherent on **H**. Assume that **H** and **K** are linear spaces containing all constant gambles and denote by $G(X) = X - \underline{P}(X)$, $G(Y|\mathbf{B}) = Y - \underline{P}(Y|\mathbf{B})$ and $G(W|B) = B(W - \underline{P}(W|B))$. Say that $\underline{P}$ and $\underline{P}(\cdot|\mathbf{B})$ are coherent if

a) $\sup[G(X) + G(Y|\mathbf{B}) - G(Z)] \geq 0$
and
b) $\sup[G(X) + G(Y|\mathbf{B}) - G(W|B)] \geq 0$ if $X,Z\in$ **K**, $Y,W\in$ H and $B\in$ **B.**

The previous definition quantifies over infinitely many unconditional previsions and called-off previsions since the superior operation appears. It is an important difference with respect to de Finetti's criterion of coherence that permits only finitely many unconditional and called-off previsions to enter into an assessment of coherence. For this reason, in de Finetti's theory, coherence does not entail that $\underline{P}$ and $\underline{P}(\cdot|\mathbf{B})$ are coherent. Conditions a) and b) automatically hold when either domain **K** contains only constant gambles or **H** contains only **B**-measurable gambles.
In the first case we have that G(X) and G(Z) are equal to zero and by the separate coherence of $\underline{P}(\cdot|\mathbf{B})$ we have that conditions a) and b) are satisfied.
In the second case we have that $G(Y|B) = 0$ for every **B**-measurable gamble Y and by the coherence of $\underline{P}$ we have that conditions a) and b) are satisfied.
The gamble $G(X|\mathbf{B})$, in which we pay the uncertain price $\underline{P}(X|\mathbf{B})$ for X can be regarded as a two-stage gamble: firstly we observe B and pay price $\underline{P}(X|B)$, then we observe ω in B and receive X(ω).

A general characterization of coherence of the unconditional lower prevision with respect to the lower conditional prevision can be given by two axioms (Walley [15] 6.5.1).
Let $\overline{P}_B(Y|\mathbf{B})$ denote the gamble $B\,\underline{P}(Y|B) + B^c\,\overline{P}(Y|\mathbf{B})$.
Then $\underline{P}$ and $\underline{P}(\cdot|\mathbf{B})$ are coherent if and only if they satisfy the two axioms:
1) If $X\in$ **K**, $Y\in$ **H** and $X \geq Y$ then $\underline{P}(X) \geq \inf \underline{P}(\cdot|\mathbf{B})$
2) If $B\in$ **B**, $X\in$ **K**, $Y\in$ **H** and $X \leq Y$ then $\underline{P}(X) \leq \sup \overline{P}_B(Y|\mathbf{B})$.

The axioms simplify further when one of the domains contains the other.
In particular if **H** contains **K** $\underline{P}$ and $\underline{P}(\cdot|\mathbf{B})$ are coherent if and only if
3) $\underline{P}(X) \geq \inf \underline{P}(X|\mathbf{B})$ whenever $X\in$ **K**
4) $\underline{P}(X) \leq \sup \overline{P}_B(X|\mathbf{B})$ whenever $X\in$ **K** and $B\in$ **B.**

If $\underline{P}$ and $\underline{P}(\cdot|\mathbf{B})$ are respectively linear unconditional and conditional previsions their coherence can be characterized by simpler conditions. In particular in Walley ([15] Section 6.5.3 and section 6.5.7) the following result has been proven:

Proposition 1. *Given P defined on* **K** *and P(X|***B***) defined on* **H** *such that they are respectively linear unconditional and conditional previsions with* **H** *contained in* **K** *and P(X|***B***) separately coherent, then P and P(X|***B***) are coherent if and only if the following conglomerative property is satisfied*

$$P(X) = P(P(X|\mathbf{B})).$$

Given a partition **B** of Ω, the unconditional probability P(X) is **B**-conglomerable if it satisfies the conglomerative property in the partition **B**.
When the unconditional prevision P(X) is **B**-conglomerable for every partition **B** of Ω then it is called *fully conglomerable* (Walley [15] 6.8.1).
In the paper of Dubins ([7]) the following definitions are introduced.
Given a partition **B** of Ω, a linear prevision P(X) is *disintegrable* with respect to linear conditional previsions P(X|**B**) if the equality P(X) = P(P(X|B)) is satisfied for every bounded variable X on Ω and for every B in **B**.
A linear prevision P(X) is defined to be *conglomerative* with respect to a partition **B** of Ω if the following condition is satisfied: for every bounded variable X and for every B in **B** we have that $P(X|B) \geq 0$ implies $P(X) \geq 0$.
It has been proven (Theorem 1 of [7]) that a prevision is disintegrable with respect to a partition if and only if it is conglomerative with respect to the same partition.
When **H** and **K** are equal to the set **L** of all bounded gambles on Ω then the conglomerative property of Walley is equivalent to the notion of *disintegrability* of a prevision P(X) with respect to a partition of Ω, introduced by Dubins ([7]). The author calls *strategies* linear conditional previsions that are separately coherent and defined on the set of all bounded gambles on Ω.
So if linear conditional previsions P(X|**B**) and linear unconditional prevision P(X), defined on the class of all bounded gambles, are such that they satisfy the disintegration property with respect to a given partition **B** of Ω, then they are coherent.
The notion of conglomerability given by Dubins can be seen as a generalization to the class of all bounded variables of the *conglomerative principle*, introduced by de Finetti ([4] pp.99) for probabilities:

Given a partition **B** *of Ω we say that the probability P is conglomerable with respect to the partition* **B** *if for every* *event A and for every* B *in* **B** *we have that* $a \leq P(A|B) \leq b$ *implies* $a \leq P(A) \leq b$.

Generally the disintegrability in the sense of Dubins is stronger than the conglomerability in the sense of de Finetti.
In fact if the conglomerative principle is satisfied it does not imply that the disintegration property is satisfied; but when the domain of the conditional and unconditional linear previsions is a linear space then the notion of conglomerability in the sense of de Finetti is equivalent to the notion of disintegrability in the sense of Dubins.

An important aspect, analysed in literature is the relationship between conglomerability and countable additivity.
In Schervish et. al. [11] it has been proven that when a probability P is defined on a σ-field, it takes infinitely many values and it is countably additive then it is disintegrable (conglomerable) in the sense of Dubins in every countable partition of Ω.
In particular if P is defined on the class of all subsets of Ω and it takes infinitely many different values then it is fully conglomerable if and only if it is countably additive on every partition of Ω.
We have that for non-countable partitions countable additivity of the unconditional prevision is not a sufficient condition to assure that it is coherent with the conditional previsions (Kadane, Schervish, Seidenfeld [9] Example 6.1).
The previous results imply that there is no fully conglomerable linear prevision P defined on the set of all bounded gambles **L** that takes many different values on events and satisfies $P(\{\omega\}) = 0$ for all $\omega \in \Omega$. For example there is no fully conglomerable linear extension of Lebesgue measure to all bounded gambles on the unit interval. Otherwise the Lebesgue lower prevision on **L**, which is the natural extension of the Lebesgue (inner) measure to all bounded gambles is fully conglomerable (Walley [15], 6.9.6), that is there is a lower conditional prevision with respect to every partition **B** coherent with the Lebesgue lower previsions.

3 Hausdorff Outer and Inner Measures

In this section we recall some preliminaries about Hausdorff measures, that we use to define conditional previsions P(X|B) when the conditioning events B have finite and positive Hausdorff measure in their dimension. For more details about Hausdorff measures see for example Falconer ([8]).
Let (Ω,d) be the Euclidean metric space with $\Omega = [0,1]$. The diameter of a nonempty set U of Ω is defined as $|U| = \sup\{|x-y|: x,y \in U\}$ and if a subset A of Ω is such that $A \subset \bigcup_i U_i$ and $0 < |U_i| < \delta$ for each i, the class $\{U_i\}$ is called a δ-cover of A. Let s be a non-negative number. For $\delta > 0$ we define $h^s_\delta(A) = \inf \sum_{i=1}^{\infty} |U_i|^s$, where the infimum is over all δ-covers $\{U_i\}$. The Hausdorff s-dimensional outer measure of A, denoted by $h^s(A)$, is defined as $h^s(A) = \lim_{\delta \to 0} h^s_\delta(A)$. This limit exists, but may be infinite, since $h^s_\delta(A)$ increases as δ decreases.
The *Hausdorff dimension* of a set A, $\dim_H(A)$, is defined as the unique value, such that

$$h^s(A) = \begin{cases} \infty & \text{if} \quad 0 \leq s \leq \dim_H(A) \\ \\ 0 & \text{if} \quad \dim_H(A) < s < \infty \end{cases}$$

We can observe that if $0 < h^s(A) < \infty$ then $\dim_H(A) = s$, but the converse is not true. We assume that the Hausdorff dimension of the empty set is equal to –1 so no event has Hausdorff dimension equal to the empty set. If an event A is such that $\dim_H(A) = s < 1$, then the Hausdorff dimension of the complementary set A^c is equal to 1 since the following relation holds:

$$\dim_H(A \cup B) = \max\{\dim_H(A); \dim_H(B)\}.$$

A subset A of Ω is called measurable with respect to the outer measure h^s if it decomposes every subset of Ω additively, that is if $h^s(E) = h^s(AE) + h^s(E-A)$ for all sets $E \subset \Omega$.

The restriction of h^s to the σ-field of h^s-measurable sets, containing the σ-field of the Borel sets, is called *Hausdorff s-dimensional measure*. In particular the Hausdorff 0-dimensional measure is the counting measure and the Hausdorff 1-dimensional measure is the Lebesgue measure.

4 Upper and Lower Conditional Previsions defined by the Hausdorff Outer and Inner Measures

In [6] upper and lower conditional probabilities are obtained as *natural extensions* (Theorem 3.1.5 [15]) of a finitely additive conditional probability in the sense of Dubins, assigned by a class of Hausdorff measures. They are proven to be separately coherent and so they satisfy the necessary condition for the coherence (P1).
In this Section upper and lower conditional previsions are defined as extensions of the previous upper and lower conditional probabilities. In particular, when the conditioning event has positive and finite Hausdorff outer (inner) measure in its dimension, they are defined

by the Choquet integral ([5]) with respect to outer (inner) Hausdorff measures, which are particular examples of monotone set functions. Otherwise, when the conditioning event has Hausdorff outer (inner) measure in its dimension equal to zero or infinity, they are defined by a 0-1 valued finitely but not countably additive probability.

In Theorem 2 and 3 of this Section we prove that when Hausdorff measures are defined on the Borel σ-field and the class of all Borel-measurable gambles is considered, then linear conditional and unconditional previsions defined with respect to Hausdorff measures satisfy the Dubin's disintegration property with respect to every countable partition of Ω.

In Theorem 2 we consider Ω equal to [0,1] and in Theorem 3 we consider the general case in which Ω is an infinite set with Hausdorff measure equal to 1 in its dimension.

Moreover linear conditional previsions are coherent with the unconditional previsions in the sense of Walley, since in this case coherence in the sense of Walley is equivalent to the disintegration property of Dubins (see Proposition 1 of Section 2).

The role of Hausdorff measures in the previous results is crucial.

In fact it is important to observe that if we define conditional and unconditional previsions with respect to a coherent finitely but not countably additive probability we cannot obtain the same results.

In fact from Theorem 3.1 of ([11]) we have that for each finitely but not countable additive probability P defined on a σ-field there is a partition (in that σ-field) where P is not disintegrable in the sense of Dubins.

This implies that linear conditional and unconditional previsions defined with respect to a merely finitely additive probability cannot be disintegrable on every countable partition of Ω.

We recall some results given in ([6]).

Let Ω be a non empty set and let **F** and **G** be two fields of subsets of Ω, with $\mathbf{G}\subseteq\mathbf{F}$ or with **G** an additive subclass of **F**, P* is a *finitely additive conditional probability* ([7]) defined on **(F,G)** if it is a real function defined on $\mathbf{F}\times\mathbf{G}^0$, where $\mathbf{G}^0=\mathbf{G}-\varnothing$, such that the following conditions hold:

I) given any $H\in\mathbf{G}^0$ and $A_1,...,A_n\in\mathbf{F}$ with $A_iA_j=\varnothing$ for $i\neq j$, the function $P^*(\cdot|H)$ defined on **F** is such that

I)$P^*(A|H)\ \geq\ 0$, $\quad P^*(\bigcup_{k=1}^{n}A_k\ |H)\ =\ \sum_{k=1}^{n}P^*(A_k\ |H)$,

$P^*(\Omega|H) = 1$

II) $P^*(H|H) = 1$ if $H\in\mathbf{FG}^0$

III) given $E\in\mathbf{F}$, $H\in\mathbf{F}$ $EH\in\mathbf{F}$ with $A\in\mathbf{G}^0$ and $EA\in\mathbf{G}^0$ then $P^*(EH|A) = P^*(E|A)P^*(H|EA)$.

From conditions I) and II) we have

II') $P^*(A|H) = 1$ if $A\in\mathbf{F}$, $H\in\mathbf{G}^0$ and $H\subset A$.

Such approach to conditional probability allows to give probability assessments on arbitrary finite family of conditional events through the notion of *coherence* as proposed by de Finetti ([3], [4]). In fact, if **F** and **G** are arbitrary finite families of subsets of Ω, then the real function P, defined on $\mathbf{F}\times\mathbf{G}^{\mathbf{0}}$ is *coherent* if and only if it is the restriction of a finitely additive conditional probability defined on $\mathbf{D}\times\mathbf{D}^0$, where **D** is the field generated by the sets of **F** and **G**.

In [6] a finitely additive conditional probability in the sense of Dubins is defined by a class of Hausdorff dimensional measures. Moreover, upper (lower) conditional probability is given by Hausdorff s-dimensional outer (inner) measures if the conditioning event has positive and finite Hausdorff s-dimensional outer (inner) measure in its dimension; otherwise upper conditional probability is defined by a 0-1 finitely additive (but not countable additive) probability so that condition III) of a finitely additive conditional probability in the sense of Dubins is satisfied. They are proven to be separately coherent in the sense of Walley. The unconditional probability is obtained as particular case when the conditioning event is Ω.

Theorem 1. *Let Ω=[0,1], let **F** be the σ-field of all subsets of Ω and let **G** be an additive sub-class of **F**. Let us denote by h^s the Hausdorff s-dimensional outer measure and let us define on $\mathbf{C}=\mathbf{F}\times\mathbf{G}^{\mathbf{0}}$ the function $\overline{\mathrm{P}}$ by*

$$\overline{\mathrm{P}}(A|H)=\begin{cases}\dfrac{h^s(AH)}{h^s(H)} & \text{if}\quad 0<h^s(H)<\infty\\[2ex] m(AH) & \text{if}\quad h^s(H)=0,\infty\end{cases}$$

where m is a 0-1 valued finitely additive (but not countably additive) probability measure. Then the function $\overline{\mathrm{P}}$ is an upper conditional probability.

The existence of the measure m is a consequence of the prime ideal theorem.

The conjugate lower conditional probability $\underline{\mathrm{P}}$ can be defined as in Theorem 1 if h^s denotes the Hausdorff s-dimensional inner measure.

When the family of the conditioning events is a partition of Ω the conditional probabilities can be defined in a similar way.

Definition 2. *Let Ω = [0,1], let **F** be the σ-field of all subsets of Ω and let **B** be a partition of Ω. Let us denote* *by s the Hausdorff dimension of the conditioning event B belonging to* **B** *and by* h^s *the outer Hausdorff s-dimensional measure. Let us define an upper conditional probability on* **F**$\times$**B** *by the function*

$$\overline{P}(A|B) = \begin{cases} \dfrac{h^s(AB)}{h^s(B)} & \text{if} \quad 0 < h^s(B) < \infty \\ m(AB) & \text{if} \quad h^s(B) = 0, \infty \end{cases}$$

where m is a 0-1 valued finitely additive (but not countably additive) probability measure.

The two definitions of upper conditional probabilities can be compared when **G** is the σ-field generated by the partition **B**. In particular, given a probability space (Ω,**F**,P), let **G** be equal to or contained in the σ-field generated by a countable class **C** of subsets of **F** and let **B** be the partition generated by the class the **C.** Denote by Ω' = **B** and $\psi_{\mathbf{B}}$ the function from Ω to Ω' that associates to every $\omega \in \Omega$ the atom B of the partition **B** that contains ω; then we have that $\overline{P}(\cdot|\mathbf{G}) = \overline{P}(\cdot|\mathbf{B}) \circ \psi_{\mathbf{B}}$ (See Koch [10] p. 262).

Upper (lower) conditional prevision is obtained as extension of upper (lower) conditional probability assigned by a class of outer Hausdorff measures.
It is defined by the Choquet integral ([5]) with respect to outer (inner) Hausdorff measures, which are particular examples of monotone set functions.

Definition 3. Let *Ω = [0,1], let* **L** *be the class of all bounded gambles on Ω and let* **B** *be a partition of Ω . Let us denote by s the Hausdorff dimension of the conditioning event B belonging to* **B** *and by* h^s *the Hausdorff s-dimensional outer measure. Let us define an upper conditional prevision on* **L**$\times$**B** *by the function*

$$\overline{P}(X|B) = \begin{cases} \dfrac{1}{h^s(B)} \displaystyle\int_B X dh^s & \text{if} \quad 0 < h^s(B) < \infty \\ m(XB) & \text{if} \quad h^s(B) = 0, \infty \end{cases}$$

where m is a 0-1 valued finitely additive (but not countably additive) probability measure.

From the definition it follows that upper conditional previsions are separately coherent for every partition **B** of Ω.
We prove that when the (outer) Hausdorff measures are defined on the Borel σ-field and **L** is the class of all Borel-measurable gambles, then linear conditional and unconditional previsions defined with respect to Hausdorff measures satisfy the Dubin's disintegration property with respect to every countable partition of Ω and they are coherent in the sense of Walley.
The following results can be obtained as a consequence of the fact that non-disintegrability characterizes finitely as opposed to countably additive probability as proven in [11]. Each arbitrary finitely additive probability P can be decomposed uniquely into a convex combination of a countably additive probability P_c and a purely finitely additive probability P_D , that is

$$P = \alpha P_c + \beta P_D \qquad \text{with } \alpha + \beta = 1, \ \alpha, \beta \geq 0.$$

In [11] the coefficient β has been proven to be an upper bound for failures of conglomerability in all denumerable partitions.
In Theorem 3.1 of [11] it has been proven that if $\beta \neq 0$, if the range of P is not limited to finitely many distinct values and if P is defined on a σ-field of event, then the upper bound on the failure of conglomerability, β, must be approached.

Theorem 2. Let *Ω = [0,1], let* ***F*** *be the Borel σ-field of subsets of Ω and let* **L** *be the class of all Borel-measurable gambles on Ω. If* **B** *is a countable partition of Ω, consisting of sets belonging to* ***F****, then the linear conditional prevision defined on* ***L***$\times$**B** *by Definition 3, is coherent with the unconditional prevision P(·|Ω).*

Proof. Since Ω is equal to [0,1] then the linear unconditional prevision P(·|Ω) is defined with respect to the Hausdorff measure of order 1, h^1, that is the Lebesgue measure. It is defined on the Borel σ-field, it takes infinitely many different values and it is countably additive. As shown in [11] this is equivalent to the disintegrability of h^1 in the sense of Dubins with respect to all countable partitions of Ω.
Since for every s, the σ-field of h^s-measurable sets contains the Borel σ-field and **L** is the class of all Borel–measurable gambles, we also have that the conditional previsions are linear.
So the unconditional and conditional previsions are coherent in the sense of Walley; in fact from Proposition 1 of Section 2, disintegrability in the sense of Dubins is equivalent to the coherence of linear conditional previsions with respect to the linear unconditional prevision.□

The previous result can be generalized to the case where Ω is an infinite set with Hausdorff measure in its dimension equal to 1.

Theorem 3. Let *Ω be an infinite set with Hausdorff measure equal to 1 in its dimension, let* ***F*** *be the Borel σ-field of subsets of Ω and let* **L** *be the class of all Borel-measurable gambles on Ω. If* **B** *is a countable partition* *of Ω, consisting of sets belonging to **F**, then the conditional prevision defined on **L**×**B** as Theorem 2, is coherent with the unconditional prevision P(·|Ω).*

Proof. Denoted by s the Hausdorff dimension of Ω, then the unconditional prevision P(·|Ω) is defined with respect to the s-dimensional Hausdorff measure h^s, which is a probability since $h^s(\Omega)$ = 1, it is defined on the Borel σ-field, it takes infinitely many different values and it is countably additive since for every s, the σ-field of h^s-measurable sets contains the Borel σ-field.
Then the result can be obtained in a similar way of Theorem 2.□

Remark 1. It is important to note the crucial role of the Hausdorff measures in the previous theorems. In fact if the unconditional prevision is defined with respect to the s-dimensional Hausdorff measure, where s is the Hausdorff dimension of Ω and **F** is the Borel σ-field, then in Theorem 2 and in Theorem 3 the unconditional prevision is defined with respect to a countably additive probability. This implies ([11]) that the disintegration property in the sense of Dubins is satisfied on every countable partition of Ω.
Otherwise if we define the unconditional prevision with respect to a coherent finitely but not countably additive probabilility P, defined on a σ-field then there is (Theorem 3.1 of [11]) a countably partition where P fails disintegrability in the sense of Dubins.

Example 1. We recall the definition of the Cantor set, which is the most familiar set of real numbers of non-integer Hausdorff dimension.

Let E_0 = [0,1], E_1 = [0,1/3] ∪ [2/3,1], E_2 = [0,1/9] ∪ [2/9,1/3] ∪ [2/3,7/9] ∪ [8/9,1], etc., where E_{j+1} is obtained by removing the open middle third of each interval in E_j. The Cantor's set is the perfect set E $= \bigcap_{j=0}^{\infty} E_j$ .The Hausdorff dimension of the Cantor set is s = log2/log3 and $h^s(E)$ = 1 (see [8] Theorem 1.14).

Let Ω be equal to the Cantor set, let **F** be the Borel σ-field of subsets of Ω and let **L** be the class of all Borel-measurable gambles on Ω. If **B** is a countable partition of Ω, consisting of sets belonging to **F,** then the conditional prevision defined on **L**× **B** as in Theorem 2, is coherent with the unconditional prevision P(·|Ω).

5 Coherence of Conditional Possibility Distribution

A first criterion to decide if an uncertainty measure is a good tool to handle imprecise and vague information about a linguistic or numerical variable is to verify if it is, first of all, able to manage "precise" information, which is represented by the singletons of Ω.
In the theory of imprecise probabilities this property is formalised by property (P1) as recalled in Section 2.
(P1) for every x belonging to Ω P(A|{x}) is equal to 1 if x belongs to A and it is equal to 0 if x does not belong to A.
We analyze the possibility to extend the properties assured by the coherence to uncertainty measures used when information is represented by fuzzy sets.
In this Section a conditional possibility distribution on an infinite set Ω that is coherent and coherent with respect to the unconditional possibility distribution.
The conditional possibility distribution satisfies the property (P1). Moreover, through this conditional possibility distribution we obtain a possibility conditional measure with respect to the partition of all singletons that is coherent and such that the (weak) conglomerative principle of de Finetti is verified.

In ([2], [16]) possibility measures are proven to be an important special class of upper probabilities; moreover in [17] a necessary and sufficient condition for the coherence of rules for defining conditional possibility distributions is given when the possibility space Ω is finite. In the quoted paper conditioning on variables rather then events is considered. Given two variables X and Y whose sets of possible values are finite the problem of examining whether a conditioning rule produces conditional distribution π(x|y) that is coherent with the joint possibility distribution π(x,y) has been investigated; moreover it has been investigated when they generate possibility measures (or equivalently upper probability measures) Π and Π(·|y) that are coherent.
Given an infinite set Ω, in this section we consider conditioning on the class of fuzzy sets of Ω, and we investigate the problem to define conditional possibility distribution π(x|y) coherent with the unconditional possibility distribution π. Moreover the coherence of the conditional possibility measures Π(·|y) with the unconditional possibility measure Π is analyzed.
Several conditioning rules are proposed in literature for defining conditional possibility distributions or measures from unconditional ones.
The approach followed in this section is quite different: firstly we define the conditional possibility distribution π(x|y) and the conditional possibility measure Π(·|y) such that they satisfy the condition (P1) for the coherence, then we consider their coherence with the unconditional possibility distribution and the unconditional possibility measure.

Given a non-empty set Ω a *fuzzy set* A is defined by a membership function that associates to each element x of Ω a real number A(x) between 0 and 1, which represents the degree to which x belongs to A.

If the membership function is equal to the indicator function then A is a *crisp* set.
The *support* of a fuzzy set is the crisp set where the membership function of the fuzzy set is greater than zero; the *core* of a fuzzy set is the crisp set where the membership function is equal to one. A fuzzy singleton is a fuzzy set whose core is a singleton.
Given two fuzzy sets A(x) and B(x) their union is defined by $A(x) \cup B(x) = \max\{A(x), B(x)\}$ for every x in Ω.
Given an infinite set Ω we denote by $P(\Omega)$ the class of the fuzzy sets of Ω; a fuzzy measure over $P(\Omega)$ is a function m: $P(\Omega) \to [0,1]$ such that $m(\emptyset) = 0$ and $m(\Omega) = 1$, $E \subset F \Rightarrow m(E) \leq m(F)$.
A measure of possibility is a fuzzy measure Π such that $\Pi\left(\bigcup_{j \in J} A_j\right) = \sup_{j \in J} \Pi(A_j)$.
A possibility distribution over Ω is a function π: $\Omega \to [0,1]$ such that $\pi(\omega) = \Pi(\{x\})$
Using a possibility distribution π over Ω, it is possible to construct a possibility measure Π over $P(\Omega)$ by the formula
$\Pi(A) = \sup_{\omega \in \Omega} \{\min(\pi(\omega), A(\omega))\}$;
A possibility distribution and a possibility measure are normalized if $\Pi(\Omega) = \sup\{\pi(\omega) : \omega \in \Omega\} = 1$.
In this paper we assume they are normalized.
As recalled in Walley ([16] p. 35) the information represented by a fuzzy set, for example "Mary is young" can be modeled by a possibility distribution defined on the set of possible ages. The number $\pi(\omega)$ lies between zero and one and it represents " the degree to which it is possible that Mary has a precise age ω, given she is young"
In the same way we can interpret a conditional possibility distribution $\pi(x|y)$ as "the degree to which it is possible that Mary has a precise age x, given she is y years old"
So if we want the conditional distribution to satisfy the condition (P1) necessary for the coherence of an upper conditional probability we have to define

$$\pi(x|y) = \begin{cases} 1 & \text{if} \quad x = y \\ 0 & \text{if} \quad x \neq y \end{cases} \qquad (1)$$

that is the conditional distribution $\pi(x|y)$ is equivalent to the indicator function of the (fuzzy) singleton; so for every y in Ω $\pi(x|y)$ is concentrated on the singleton y.

In order to find conditions that assure the coherence of the conditional distribution $\pi(x|y)$ with the unconditional distribution π it is important to determine relations between them.

Given two fuzzy sets A and B we introduce a joint possibility distribution $\pi(x,y)$ for all $x \in A$ and $y \in B$.
According to Hisdial a conditional possibility distribution $\pi(x|y)$ is implicitly defined as

$$\pi(x,y) = \min(\pi(y), \pi(x|y)) \qquad (2)$$

If we define conditional possibility distribution by (1) and we require that also (2) is satisfied we obtain

$$\pi(x, y) = \begin{cases} \pi(y) & \text{if} \quad x = y \\ 0 & \text{if} \quad x \neq y \end{cases}$$

At this point the question is:" how do we have to choose $\pi(y)$ so that the conditional possibility distribution $\pi(x|y)$ and the unconditional possibility distribution are coherent?"

We observe that the definition of conditional possibility distribution proposed in this section is similar to the one proposed by Ramer [14]. This conditioning rule consists in picking one x_0 such that $\pi(x_0,y) = \pi(y)$, letting $\pi(x_0,y) = 1$ and $\pi(x|y) = \pi(x,y)$. It produces normal $\pi(\cdot|y)$, but it has the disadvantage of requiring an arbitrary choice whenever there is more than one x that maximizes $\pi(\cdot|y)$. Moreover, Ramer's rule produces conditional possibility distributions which are incoherent with joint distribution if $0 < \pi(x,y) < \pi(y) < 1$ as pointed out in [17].
The definition of conditional possibility distribution given in this section by (1) avoids this problem since the only value, which maximizes $\pi(\cdot|y)$ is $x_0 = y$
Given $y \in \Omega$ and for every $x \in \Omega$ we define $\pi(x,y)$ equal to

$$\pi(x, y) = \begin{cases} 1 & \text{if} \quad x = y \\ 0 & \text{if} \quad x \neq y \end{cases}$$

So that $\pi(\cdot)$ is the 0-1 valued finitely additive measure concentrated on the singleton $\{y\}$.

When Ω is a finite set a necessary and sufficient condition for the coherence of $\pi(\cdot,\cdot)$ and $\{\pi(\cdot \mid y) : y \in \Omega\}$ has been proposed in [17]; it claims that $\pi(\cdot,\cdot)$ and $\{\pi(\cdot \mid y) : y \in \Omega\}$ are coherent if and only if the conditional possibility distribution $\pi(x|y)$ is greater or equal to the conditional possibility distribution defined by the Dempster's rule ($\pi_{DE}(x, y)$) and it is less or equal to the conditional possibility distribution defined by the natural extension ($\pi_{NE}(x, y)$) if $\pi(y) > 0$.

The definition of conditional possibility distribution $\pi(x|y)$ given by (1) satisfies the previous condition in fact we have that
$\pi(x|y) = \pi(x,y) = \pi(y)= 1= \pi_{DE}(x, y) = \pi_{NE}(x, y)$.
If Ω is a countable set the unconditional possibility distribution $\pi(\cdot)$ and the conditional possibility distribution $\pi(\cdot|y)$ defined by (1) are equal to the measure concentrated on the singleton $\{y\}$.
They are coherent as proven in Seidenfeld et al. ([12] Lemma 1).

Definition 4. Let us define a conditional possibility measure by

$$\Pi(A \mid y) = \sup_{x \in A}\{\min(\pi(x \mid y), A(y))\}$$

then we obtain

$$\Pi(A \mid y) = A(y)$$

Remark 2. If A is a crisp set, so that its membership function A(x) is equal to the indicator function of A, then $\Pi(A \mid y)$ is the indicator function of A and so property (P1) necessary for the coherence is satisfied; moreover from the definition of possibility measure we have that $\Pi(A \cup B \mid y) = \max\{\Pi(A \mid y), \Pi(B \mid y)\}$.
So, if A is a crisp set, the conditional possibility measure $\Pi(A \mid y)$ is the 0-1 finitely additive measure concentrated on the singleton $\{y\}$, which is a particular kind of (upper) conditional probability

Example 2. Let Ω be the set of natural numbers N, then the conditional possibility measure $\Pi(A \mid y)$ is the 0-1 finitely additive measure concentrated on the singleton $\{y\}$ and we have that

$$\Pi(N \mid y) = \sup_{x \in N}\{\min(\pi(x \mid y), N(y))\} = 1$$

We have defined the conditional possibility distribution $\pi(\cdot)$ equal to the 0-1 valued finitely additive measure concentrated on the singleton $\{y\}$. So we obtain that the conditional possibility measure is equal to

$$\Pi(A) = \sup_{y \in \Omega}\{\min(\pi(y), A(y))\} = \sup_{y \in \Omega}\{A(y)\}$$

In particular if A is a fuzzy singleton $A = \{x\}$ we have that $\Pi(\{x\}) = \pi(x) = 1$.

The next result shows that for every fuzzy set A the normalized possibility measure Π and the normalized conditional possibility measure $\Pi(A \mid y)$ satisfy the conglomerative principle of de Finetti with respect to the partition of all singletons of an infinite set Ω.

Theorem 4. *Let Ω be an infinite set and let Π be a normalized possibility measure over $P(\Omega)$ the class of all fuzzy sets of Ω defined by* $\Pi(A) = \sup_{y \in \Omega}\{\min(\pi(y), A(y))\} = \sup_{y \in \Omega}\{A(y)\}$. *Moreover let* $\Pi(A \mid y)$ *be the conditional possibility measure defined by* $\Pi(A \mid y) = A(y)$. *Then for every y belonging to Ω, we have that*
$a \le \Pi(A|y) \le b$ implies $a \le \Pi(A) \le b$.

Proof. Since we have that

$$\Pi(A \mid y) = \sup_{x \in A}\{\min(\pi(x \mid y), A(y))\}$$

from the definition of conditional possibility distribution $\pi(x|y)$ given by (1) we obtain that $\Pi(A \mid y) = A(y)$.
So, if for every y in Ω we have that

$$a \le \Pi(A \mid y) = A(y) \le b$$

it implies that

$$a \le \sup_{y \in \Omega}\{A(y)\} \le b$$

that is $a \le \Pi(A) \le b$.□

6 Summary and Conclusions

A new model of upper and lower conditional previsions is proposed in this paper.
When the conditioning event has positive and finite Hausdorf outer (inner) measure in its dimension, upper (lower) conditional previsions are defined by the Choquet integral ([5]) with respect to the outer (inner) Hausdorff measures, which are particular examples of monotone set functions. Otherwise, when the conditioning event has Hausdorff outer (inner) measure in its dimension equal to zero or infinity, upper (lower) conditional previsions are defined by a 0-1 valued finitely but not countably additive probability.
These upper and lower conditional previsions are proven to be separately coherent for every partition **B** of Ω.
Moreover when we consider the restriction of the (outer) Haudorff measures to the Borel σ-field (upper) conditional and unconditional previsions are proven to satisfy the disintegration property in the sense of Dubins with respect to all countable partitions of Ω and to be coherent in the sense of Walley.
Another problem analyzed in this paper is the extension of upper conditional probability properties assigned by a class of Hausdorff outer measures when information is represented by fuzzy sets.
A conditional possibility distribution on an infinite set Ω that is coherent and coherent with respect to the unconditional possibility distribution is defined.

Moreover through this conditional possibility distribution we obtain a possibility conditional measure with respect to the partition of all singletons that is coherent and that satisfies the conglomerative principle of de Finetti.

Acknowledgements

I wish to thank the reviewers for pointing out trivial aspects in the first version of the paper and for their valuable remarks.

References

[1] P.Billingsley, *Probability and measure*, John Wiley, New York, 1985.

[2] G. de Cooman, A behavioural model for vague probability assessments, *Fuzzy sets and Systems*, 154, 305-358, 2005.

[3] B. de Finetti, *Teoria della Probabilità*, Einaudi Editore, Torino, 1970.

[4] B. de Finetti, *Probability, Induction and Statistics*. New York, Wiley, 1972.

[5] D. Denneberg, *Non-additive measure and integral*, Kluwer Academic Publishers, 1994.

[6] S. Doria, Probabilistic independence with respect to upper and lower conditional probabilities assigned by Hausdorff outer and inner measures, accepted for publication in *International Journal of Approximate Reasoning*, 2007.

[7] L. Dubins Finitely additive conditional probabilities, conglomerability and disintegrations, *The Annals of Probability*, Vol.3, No1, 89-99, 1975.

[8] K.J. Falconer, *The geometry of fractal sets*, Cambridge University Press, 1986.

[9] J.B. Kadane, M. Schervish, T. Seidenfeld, Statistical implications of finitely additive probability. In *Bayesian Inference and Decision Techniques With Applications* (P.Goel and A. Zellner, eds.) 59-76. North-Holland, Amsterdam, 1986.

[10] G. Koch, *La matematica del probabile*, Aracne Editrice, 1997.

[11] M. Schervish, T. Seidenfeld, J.B. Kadane, The extent of non-conglomerability of finitely additive probabilities. *Z. Warsch.Verw.Gebiete* 66, 205-226, 1984.

[12] T. Seidenfeld, M. Schervish, J.B. Kadane, Non-conglomerability for finite-valued, finitely additive probability, *The Indian Journal of Statistics*, Special issue on Bayesian Analysis, Vol.60, Series A, 476-491, 1998.

[13] T. Seidenfeld, M. Schervish, J.B. Kadane, Improper regular conditional distributions, *The Annals of Probability*, Vol.29, No 4, 1612-1624, 2001

[14] A.Ramer Conditional possibility measures. *Cybernetics and Systems*, 20:233-247, 1989.

[15] P.Walley *Statistical Reasoning with Imprecise Probabilities*, Chapman and Hall, 1991.

[16] P.Walley. Measures of uncertainty in expert systems. *Artificial Intelligence*, 83: 1.-58, 1996.

[17] P.Walley, G.de Cooman, Coherence of rules for defining conditional possibility, *International Journal of Approximate Reasoning*, vol. 21, 63-107, 1999.