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A perturbation approach for the identification of uncertain structures

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Abstract This paper deals with the identification of a linear structural systems with random parameters. The stiffness matrix of a four-storey shear frame structure is assumed to be linearly dependent by a random parameter ruling the damage evolution of the columns. The evaluation of natural frequencies and the mode-shapes are in the context of random eigenvalue problems in structural dynamics. A perturbation technique is first applied to derive the asymptotic solution up to the second order and to identify the mass and stiffness matrices. Then, the evaluation of the statistic of the frequencies and mode-shapes are derived up to the second order. Finally a stochastic identification technique is proposed to characterize the statistics of the random parameter.

Keywords Stochastic structural system identification · Uncertain structures · Perturbation approach

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1 Introduction

The interest in the ability to monitor a structure and to detect its structural characteristics and damage at earliest possible stage is pervasive throughout the civil, mechanical and aerospace engineering communities. To this end, the number of identification techniques available in literature have increased in the last twenty years [5]. These can be divided in two large group of techniques, namely the time domain identification techniques and the frequency domain identification techniques [6, 10]. A further classification of identification techniques consider the question when the input of the structure is unknown, undetectable or unmeasurable, called *Output−Only* identification techniques [3, 12] as these are based on response measurement only [14, 13].

In the safety assessment of many engineering structures, randomness of various structural properties can be a crucial factor, especially when dynamic responses are of concern [4, 6, 2]. During structural dynamic analysis uncertainty is often present in structural parameters such as the material properties and dimensions. In the case of structural systems with random parameters one deals with stochastic structural system identification. However the probabilistic structure of these parameters is unknown in general and information need to be acquired via experimental setup. In literature many papers deal with the direct problem, admitting a known form of the probabilistic structure of the parameters and then evaluating the response statistics (static or dynamic) of a given structure to an assumed distribution of the parameters [11, 1]. However the inverse problem for structure with uncertain parameter has received less attention and only few contributions are available in literature [8, 9]. There is a need therefore to develop ad hoc identification techniques taking into account the uncertain nature of the parameters and to identify their statistical properties. Although the uncertain nature of structure parameters seems reasonable and confirmed by experimental evidence, taking into account the statistical nature of the parameters of a given structure increase the number of parameters to be identified as we will then be dealing with the statistics of these parameters of any order. The natural way to achieve information by experimental analysis is to proceed with the statistics of the acquired quantities. Under these circumstances it is necessary to incorporate probabilistic information into an overall model and to identify the statistics of the parameters, using both the model and available experimental data.

This papers focus on the estimation of the mean and variances of the parameters of a linear four-storey shear frame structure undergoing free vibrations. The stiffness matrix is assumed to be linearly dependent by a random Gaussian parameter ruling the damage evolution of the columns. The evaluation of natural frequencies and the mode-shapes are in the context of random eigenvalue problems in structural dynamics [1]. A perturbation technique [7] is first applied to derive the asymptotic solution up to the second order and to identify the mass and stiffness matrices. Then, the evaluation of the statistic of the eigenvalues and mode-shapes are derived up to the second order. Finally an identification technique is proposed to characterize the statistics of the random parameter.

2 Direct problem: perturbative approximation of eigensolution statistics

Let be ε a Gaussian distributed random parameter with mean μ_ε and variance σ_ε^2 and $\mathbf{M}(\varepsilon)$, $\mathbf{K}(\varepsilon)$ the mass and stiffness matrices of a linear structure with no damping. The free vibrations of the structure are ruled by the following random matrix equation

$$\mathbf{M}(\varepsilon)\ddot{\mathbf{u}}(t) + \mathbf{K}(\varepsilon)\mathbf{u}(t) = \mathbf{0} \tag{1}$$

where $\mathbf{u}(t)$ represent the displacement vector of the structure. In modal analysis it is well known that the solution of the equation of motion are given by

$$\mathbf{u}(t) = \sum_{i=1}^{n} \boldsymbol{\phi}_i \rho_i \sin(\omega_i t + \theta_i) = \boldsymbol{\Phi}\boldsymbol{q}(t) \tag{2}$$

where n is the number of degree of freedom of the structure, $\boldsymbol{q}(t)$ the modal coordinate vector and $\boldsymbol{\Phi} = [\boldsymbol{\phi}_1 \cdots \boldsymbol{\phi}_n]$ the modal matrix. Inserting the displacement vector $\mathbf{u}(t)$ in Eq. (1) requires the solution of the following eigenvalue problem

$$\begin{cases} (\mathbf{K}(\varepsilon) - \lambda_i \mathbf{M}(\varepsilon))\boldsymbol{\phi}_i = \mathbf{0} \\ \lambda_i = (2\pi f_i)^2 \end{cases} \qquad i = 1, \dots, n \tag{3}$$

where λ_i and $\boldsymbol{\phi}_i$ are the eigenvalues and eigenvectors of the dynamic system Eq. (1), and f_i the corresponding frequencies.

Pointing out that the related random characteristic equation

$$\det(\mathbf{K}(\varepsilon) - \lambda_i \mathbf{M}(\varepsilon)) = 0 \qquad i = 1, \dots, n \tag{4}$$

is nonlinear, in the following a perturbative approach will be considered [7]. For non defective systems, i.e., showing n distinct eigenvalues, it is possible to consider a Taylor expansion of the quantities of interest (fractional series expansions are needed for defective system).

Let us consider the Taylor series expansion of the mass and stiffness matrices around $\varepsilon = 0$

$$\begin{cases} \mathbf{M}(\varepsilon) = \mathbf{M}_0 + \varepsilon\mathbf{M}_1 + \varepsilon^2\mathbf{M}_2 + \dots \\ \mathbf{K}(\varepsilon) = \mathbf{K}_0 + \varepsilon\mathbf{K}_1 + \varepsilon^2\mathbf{K}_2 + \dots \end{cases} \qquad \|\varepsilon\| \ll 1 \tag{5}$$

with

$$\begin{cases} \mathbf{M}_j = \dfrac{1}{j!} \dfrac{\mathrm{d}^j \mathbf{M}(\varepsilon)}{\mathrm{d}\varepsilon^j}\Bigg|_{\varepsilon=0} \\ \mathbf{K}_j = \dfrac{1}{j!} \dfrac{\mathrm{d}^j \mathbf{K}(\varepsilon)}{\mathrm{d}\varepsilon^j}\Bigg|_{\varepsilon=0} \end{cases} \tag{6}$$

The Taylor series expansion around $\varepsilon = 0$ can be considered also for the eigenvalues and eigenvectors of the problem in Eqs. (3-4)

$$\begin{cases} \lambda_i = \lambda_{0i} + \varepsilon\lambda_{1i} + \varepsilon^2\lambda_{2i} + \dots \\ \boldsymbol{\phi}_i = \boldsymbol{\phi}_{0i} + \varepsilon\boldsymbol{\phi}_{1i} + \varepsilon^2\boldsymbol{\phi}_{2i} + \dots \end{cases} \tag{7}$$

with

$$\begin{cases} \lambda_{ji} = \dfrac{1}{j!} \dfrac{\mathrm{d}^j \lambda_i(\varepsilon)}{\mathrm{d}\varepsilon^j}\Bigg|_{\varepsilon=0} \\ \boldsymbol{\phi}_{ji} = \dfrac{1}{j!} \dfrac{\mathrm{d}^j \boldsymbol{\phi}_i(\varepsilon)}{\mathrm{d}\varepsilon^j}\Bigg|_{\varepsilon=0} \end{cases} \tag{8}$$

Substituting Eqs. (5-7) in Eq. (3) and equating to zero the coefficients with the same power in ε, we obtain the following set of relationships

$$\begin{cases} \varepsilon^0 : (\mathbf{K}_0 - \lambda_{0i}\mathbf{M}_0)\boldsymbol{\phi}_{0i} = \mathbf{0} \\ \varepsilon^1 : (\mathbf{K}_0 - \lambda_{0i}\mathbf{M}_0)\boldsymbol{\phi}_{1i} = \\ \qquad = (\lambda_{0i}\mathbf{M}_1 + \lambda_{1i}\mathbf{M}_0 - \mathbf{K}_1)\boldsymbol{\phi}_{0i} \\ \varepsilon^2 : (\mathbf{K}_0 - \lambda_{0i}\mathbf{M}_0)\boldsymbol{\phi}_{2i} = \\ \qquad = (\lambda_{0i}\mathbf{M}_1 + \lambda_{1i}\mathbf{M}_0 - \mathbf{K}_1)\boldsymbol{\phi}_{1i} + \\ \qquad\quad + (\lambda_{0i}\mathbf{M}_2 + \lambda_{1i}\mathbf{M}_1 + \lambda_{2i}\mathbf{M}_0 - \mathbf{K}_2)\boldsymbol{\phi}_{0i} \\ \dots \end{cases} \tag{9}$$

The perturbation procedure consider first the solution of the *generating equation* Eq. (9_a), an eigenvalue problem, to obtain the zero order solution $(\lambda_{0i}, \boldsymbol{\phi}_{0i})$. Then, one can solve Eq. (9_b) to get $(\lambda_{1i}, \boldsymbol{\phi}_{1i})$, Eq. ($9_c$) to get $(\lambda_{2i}, \boldsymbol{\phi}_{2i})$ and so on. It is worth noting that all these quantities are deterministic (as they do not depend on the random parameter ε) and that only one eigenvalue problem must be solved. It can be shown that the perturbative terms appearing in Eq.(9) are given by

$$\begin{cases}\lambda_{1i} = \dfrac{\boldsymbol{\phi}_{0i}^{\top}(\mathbf{K}_1 - \lambda_{0i}\mathbf{M}_1)\boldsymbol{\phi}_{0i}}{\boldsymbol{\phi}_{0i}^{\top}\mathbf{M}_0\boldsymbol{\phi}_{0i}} \\ \lambda_{2i} = \dfrac{\boldsymbol{\phi}_{0i}^{\top}(\mathbf{K}_1 - \lambda_{0i}\mathbf{M}_1 - \lambda_{1i}\mathbf{M}_0)\boldsymbol{\phi}_{1i}}{\boldsymbol{\phi}_{0i}^{\top}\mathbf{M}_0\boldsymbol{\phi}_{0i}} + \\ \quad + \dfrac{\boldsymbol{\phi}_{0i}^{\top}(\mathbf{K}_2 - \lambda_{0i}\mathbf{M}_2 - \lambda_{1i}\mathbf{M}_1)\boldsymbol{\phi}_{0i}}{\boldsymbol{\phi}_{0i}^{\top}\mathbf{M}_0\boldsymbol{\phi}_{0i}} \\ \boldsymbol{\phi}_{1i} = \alpha_{ii}\boldsymbol{\phi}_{0i} + \displaystyle\sum_{\substack{j=1\\j\neq i}}^{n} \dfrac{\boldsymbol{\phi}_{0j}^{\top}(\mathbf{K}_1 - \lambda_{0i}\mathbf{M}_1)\boldsymbol{\phi}_{0i}}{(\lambda_{0i} - \lambda_{0j})\boldsymbol{\phi}_{0j}^{\top}\mathbf{M}_0\boldsymbol{\phi}_{0j}}\boldsymbol{\phi}_{0j} \\ \boldsymbol{\phi}_{2i} = \beta_{ii}\boldsymbol{\phi}_{0i} + \\ \quad + \displaystyle\sum_{\substack{j=1\\j\neq i}}^{n} \Bigg(\dfrac{\boldsymbol{\phi}_{0j}^{\top}(\mathbf{K}_1 - \lambda_{0i}\mathbf{M}_1 - \lambda_{1i}\mathbf{M}_0)\boldsymbol{\phi}_{1i}}{(\lambda_{0i} - \lambda_{0j})\boldsymbol{\phi}_{0j}^{\top}\mathbf{M}_0\boldsymbol{\phi}_{0j}} + \\ \quad\quad + \dfrac{\boldsymbol{\phi}_{0j}^{\top}(\mathbf{K}_2 - \lambda_{0i}\mathbf{M}_2 - \lambda_{1i}\mathbf{M}_1)\boldsymbol{\phi}_{0i}}{(\lambda_{0i} - \lambda_{0j})\boldsymbol{\phi}_{0j}^{\top}\mathbf{M}_0\boldsymbol{\phi}_{0j}} \Bigg)\boldsymbol{\phi}_{0j} \end{cases} \tag{10}$$

where α_{ii} and β_{ii} are real coefficients that can be fixed imposing a normalization conditions.

Since in real life applications, the estimation of frequencies is more easy and reliable than the evaluation of mode-shapes, only the eigenvalue statistics are considered in the following relationships. In detail, for the identification purpose, we start from Eq. (7_a) and use the stochastic average operator $E[\bullet]$ in order to find the first and second order statistics of eigenvalues

$$\begin{cases} E[\lambda_i] = \lambda_{0i} + E[\varepsilon]\lambda_{1i} + E[\varepsilon^2]\lambda_{2i} \\ E[\lambda_i\lambda_j] = \lambda_{0i}\left(\lambda_{0j} + \lambda_{1j}E[\varepsilon] + \lambda_{2j}E[\varepsilon^2]\right) + \\ \quad +\lambda_{1i}\left(\lambda_{0j}E[\varepsilon] + \lambda_{1j}E[\varepsilon^2] + \lambda_{2j}E[\varepsilon^3]\right) + \\ \quad +\lambda_{2i}\left(\lambda_{0j}E[\varepsilon^2] + \lambda_{1j}E[\varepsilon^3] + \lambda_{2j}E[\varepsilon^4]\right) \end{cases} \tag{11}$$

Manipulating and simplifying Eqs. (11), the following relations for mean and covariance of λ_i are obtained

$$\begin{cases} \mu_{\lambda_i} = \lambda_{0i} + \mu_\varepsilon\lambda_{1i} + \left(\mu_\varepsilon^2 + \sigma_\varepsilon^2\right)\lambda_{2i} + o(\varepsilon^2) \\ \sigma_{\lambda_i\lambda_j} = \sigma_\varepsilon^2\lambda_{1i}\lambda_{1j} + 2\sigma_\varepsilon^2\mu_\varepsilon(\lambda_{1i}\lambda_{2j} + \lambda_{1j}\lambda_{2i}) + \\ \quad\quad +(2\sigma_\varepsilon^4 + 4\sigma_\varepsilon^2\mu_\varepsilon^2)\lambda_{2i}\lambda_{2j} + o(\varepsilon^2) \end{cases} \tag{12}$$

obviously, if the $\lambda_{2i}, \lambda_{2j}$ terms are neglected, a first order solution is employed; in this case, because of the linear relationship between the eigensolution and the parameter, also the eigenvalues appear as Gaussian variables.

3 Inverse problem: identification of uncertain parameters

Different well posed techniques are available in literature for the identification of the modal model of a structure. They can be divided in time domain (e.g., Ibrahim Time Domain), frequency domain (e.g., Peak Picking) and mixed time-frequency (e.g., Wavelet Based) identification techniques. The main goal of these techniques consists in the identification of the main dynamic properties of the structure, i.e. frequency and mode shapes [6].

In the framework of stochastic perturbation identification techniques, the equations previously derived may be used to stochastically quantify the stiffness and mass matrix of a given structure. In particular, we assume that $(\mathbf{M}_0, \mathbf{K}_0)$ are known matrices, obtained from the project analysis or from a preliminary experimental campaign; it is clear that in this work we are assuming these spatial quantities of the structure in its reference configuration as deterministic. Thus we proceed with the evaluation of the statistics of the uncertain parameter from the knowledge of the measured and analytical eigenvalues statistics; since the random variables ε is assumed to be gaussian, its mean and standard deviation allow a complete statistical description of its PDF.

In our approach we also admit that the perturbative quantities $(\lambda_{1i}, \lambda_{2i}, \dots)$ are known, that is, we assume to known where the uncertain parameter acts; this assumption can be justified starting from the physical meaning of the parameters. Since the formulation is designed for a dynamic system without damping, the range of possibilities is reduced to the analysis of alterations of mass or stiffness

- change in mass: it mainly occurs when one or more parts of the structure undergo a change of use or when a structural reinforcement is performed (indeed, the variations for mass degradation and/or damage may be considered negligible);
- increase of stiffness: it occurs in the presence of a structural reinforcement;
- reduction of stiffness: it occurs when the structure is affected by a damage, visible or not.

In the first two cases, change in the mass and/or increase of the stiffness, the dependence of the model from the physical parameters representative of the structural changes can certainly be considered known *a priori*. For a stiffness reduction, i.e., a damage uncertain parameter, undoubtedly the most interesting case from an engineering point of view, in general the damage is not visible (because the damage is not macroscopic or because the area is not accessible); in this case the dependence of the model from the parameter can be gathered employing at upstream an *ad hoc* technique able to locate the damage, i.e., one of the so called level two damage identification technique [5].

Summarizing, in the applications under consideration the number of unknown is, regardless the perturbation order, equal to two, the couple $(\mu_\varepsilon, \sigma_\varepsilon)$; regarding the number of equations, since in practical applications only a small number of eigenvalues can be measured, in what follows we will refer to an amount of measured eigenvalues n_λ less than n and, accordingly, Eqs. (12) furnish $2n_\lambda$ governing relationships. As it is well known, a problem of this kind may be approached computing an objective function: to this aim we define the following functional

$$\mathcal{E}(\mu_\varepsilon, \sigma_\varepsilon) = \sum_{i=1}^{n_\lambda} \left(1 - \frac{{}^K\lambda_i}{{}^K\tilde{\lambda}_i}\right)^2 \tag{13}$$

having indicated with ^{K} a fractile measure and where the amount ${}^K\lambda_i$ are analytic quantities, depending only by $(\mu_\varepsilon, \sigma_\varepsilon)$, while the terms ${}^K\tilde{\lambda}_i$ represent the relevant measured counterparts. In other words, the objective function in Eq. (13) is a sum of squared errors defined as normalized eigenvalues discrepancies. Therefore, the proposed identification procedure belongs to the class of least square methods, for which the solving equation can be written in the form

$$(\tilde{\mu}_\varepsilon, \tilde{\sigma}_\varepsilon) = \arg\min \mathcal{E}(\mu_\varepsilon, \sigma_\varepsilon) \qquad \sigma_\varepsilon \geq 0 \tag{14}$$

that is, we look for the couple $(\tilde{\mu}_\varepsilon, \tilde{\sigma}_\varepsilon)$ that minimize the objective function with the constrain $\sigma_\varepsilon \geq 0$, plus relevant other constraints dictated by the physics of the problem (the mass and the stiffness of an element are strictly positive quantities).

The advantage of the proposed approach consists in the possibility of knowing the quantities $(\lambda_{1i}, \lambda_{2i}, \dots)$, that is to keep the advantages of the computational techniques based on perturbative approaches, regardless of the chosen order of approximation. The following section discusses the choice of the fractile, while the section 4 shows an application of the proposed technique.

3.1 Choice of fractile

In the formulation of the objective function we preserved the stochastic character of the problem expressing the eigenvalues with reference to fractile measures. It's a simple and intuitive method to account for uncertainties, and, moreover, it's commonly used in the direct problem.

As regards the choice of the fractile, the calculation may not be so straightforward: even if the eigenvalue problem is a function of a Gaussian variable, usually the eigensolution is not (it is so only for first order solution). A simple solution to the issue, however, can be found through an alternative definition of fractile; if it's regarded as an upper limit of the response [2], then we can simply put

$${}^K\lambda_i = \mu_{\lambda_i} + 3\,\sigma_{\lambda_i} \tag{15}$$

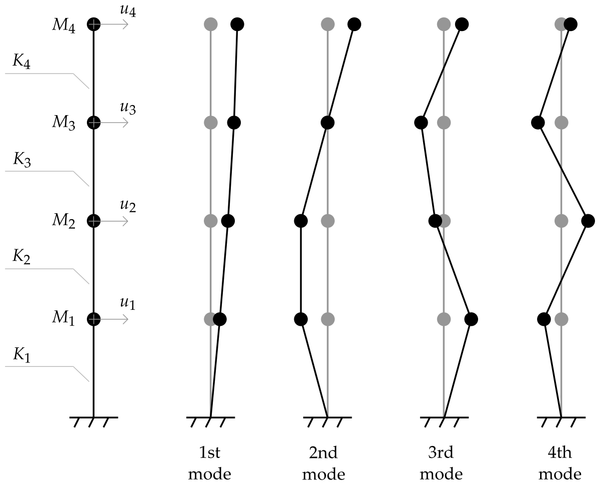

Fig. 1 Structural model and its unperturbed mode-shapes

(and similar for the measured quantities) where the choice of the value 3 comes from a similarity condition with the value of the coefficient c such that if x is a Gaussian variable with mean μ_x and standard deviation σ_x, then

$${}^K x = \mu_x + c\,\sigma_x \tag{16}$$

has a probability of not exceeding equal to 99.9 %, approximately.

4 Numerical example

Let us consider a four degrees of freedom planar structural model as in Fig. 1, sketch on the left.

The equation of motion Eq.(1) can be written as follows (cfr. [4])

$$\begin{aligned} &\begin{bmatrix} M_1 & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 \\ 0 & 0 & M_3 & 0 \\ 0 & 0 & 0 & M_4 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \\ \ddot{u}_4 \end{Bmatrix} + \\ &+ \begin{bmatrix} K_1+K_2 & -K_2 & 0 & 0 \\ -K_2 & K_2+K_3 & -K_3 & 0 \\ 0 & -K_3 & K_3+K_4 & -K_4 \\ 0 & 0 & -K_4 & K_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \mathbf{0} \end{aligned} \tag{17}$$

The following values of mass and stiffness are considered

$$\begin{cases} M_i = 1 \text{ kg} & i = 1, 2, 3, 4 \\ K_i = 1800 \text{ N/m} & i = 1, 2, 4 \\ K_3 = 1800\,(1-\varepsilon) \text{ N/m} & \varepsilon : p(\varepsilon) = \mathcal{N}(\mu_\varepsilon,\, \sigma_\varepsilon^2) \end{cases} \tag{18}$$

where $p(\varepsilon)$ represents the probability density function (PDF) of the variable ε.

Table 1 Modal characteristics of the unperturbed system

Mode	λ_{0i}, (rad/s)^2	ω_{0i}, rad/s	f_{0i}, Hz	T_{0i}, s
1	217.1	14.735	2.345	0.426
2	1800.0	42.426	6.752	0.148
3	4225.1	65.001	10.345	0.097
4	6357.8	79.736	12.690	0.079

DoF	ϕ_{01}	ϕ_{02}	ϕ_{03}	ϕ_{04}
1	0.347	1.000	1.000	-0.653
2	0.653	1.000	-0.347	1.000
3	0.879	0.000	-0.879	-0.879
4	1.000	-1.000	0.653	0.347

Following the perturbative approach, Eqs.(6) return

$$\begin{cases} \mathbf{M}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ \mathbf{K}_0 = \begin{bmatrix} 3600 & -1800 & 0 & 0 \\ -1800 & 3600 & -1800 & 0 \\ 0 & -1800 & 3600 & -1800 \\ 0 & 0 & -1800 & 1800 \end{bmatrix} \end{cases} \tag{19}$$

for the 0th-order terms,

$$\mathbf{M}_1 = \mathbf{0} \qquad \mathbf{K}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1800 & 1800 & 0 \\ 0 & 1800 & -1800 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{20}$$

for the 1st-order terms,

$$\mathbf{M}_2 = \mathbf{0} \qquad \mathbf{K}_2 = \mathbf{0} \tag{21}$$

for the 2nd-order terms.

For the unperturbed system $(\mathbf{M}_0, \mathbf{K}_0)$, Table 1 shows the eigenvalues λ_{0i} and the corresponding angular frequencies ω_{0i}, cyclic frequencies f_{0i} and periods T_{0i}; in the same table are also listed the components of the eigenvectors $\boldsymbol{\phi}_{0i}$, that are sketched in Fig. 1.

Two cases are taken under consideration

- case 1 - deterministic: $\mu_\theta \in (0, 0.5]$, $\sigma_\theta^2 = 0$;
- case 2 - stochastic : $\mu_\theta \in (0, 0.5]$, $\sigma_\theta^2 = (0.05\,\mu_\theta)^2$;

that is, a deterministic/stochastic damage localized at the third level of the structure. The mean value is increased up to 0.5 in order to explore a wide neighbourhood of the given configuration (50 % reduction of the third stiffness) and a constant coefficient of variation of the 5 % is considered to describe an increasing of dispersion with damage increasing.

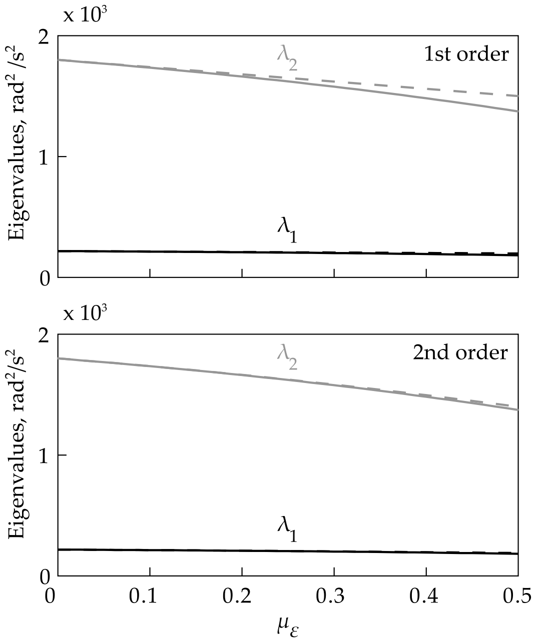

Fig. 2 Eigenvalues sensitivity, case 1

Hereafter we suppose that the first two frequencies has been detected during the tests (here numerically simulated). Before analyzing the inverse problem, in order to better interpret the results, we study the sensitivity of the eigensolution. The solution obtained with the perturbative approach (PA) using relationships (12) is compared with the solution of the eigenvalue problem (1): for the deterministic case this can be easily done substituting the current value of μ_ε (ES); for the stochastic case a Monte Carlo (MC) analysis was performed with up to 5000 simulations for each couple of $(\mu_\varepsilon, \sigma_\varepsilon^2)$. The results are shown in Fig. 2, for case 1, and in Fig. 3, for case 2:

- caso 1 - deterministic: continuous line ES, dashed line PA;
- caso 2 - stochastic: continuous line MC, dashed line PA; thick and thin lines refer to mean and mean plus/minus three standard deviations values, respectively (although for the first eigenvalue λ_1 the related curves are practically indistinguishable).

These figures show that, as suggested by the physics of the problem, the error committed by the perturbative approach increases by increasing the intensity of the sensitivity parameter ε, both in the deterministic (case 1) and stochastic (case 2) problem; it is also clear the contribution of the second order terms. To quantify the discrepancy when using the perturbative approach, the following comparisons are developed

- case 1 - deterministic: comparison between ES and PA;

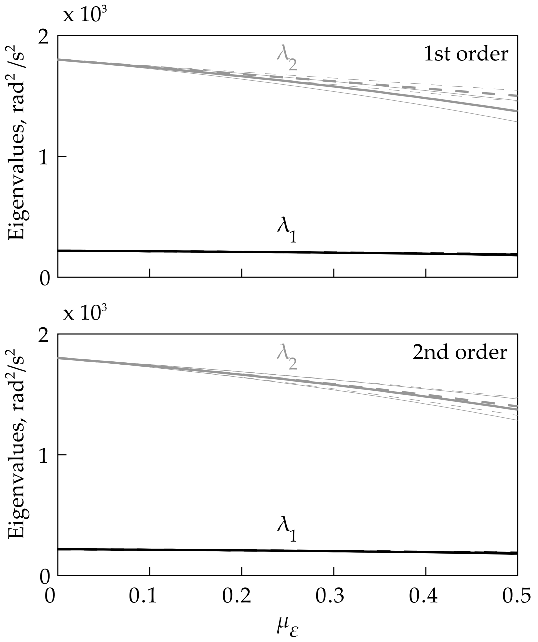

Fig. 3 Eigenvalues sensitivity, case 2

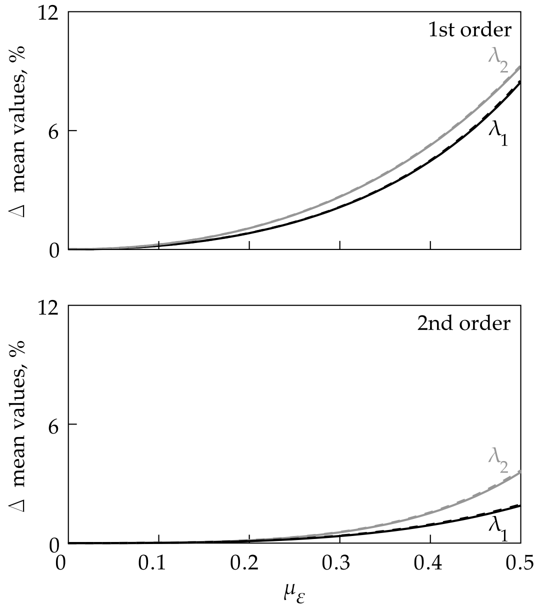

Fig. 4 Percentage error for the mean values, cases 1 and 2

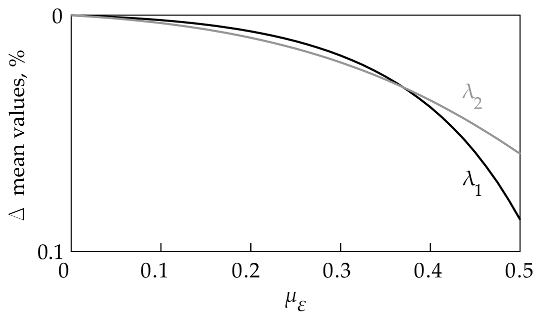

Fig. 5 Percentage error for the mean values, ES vs MC

- case 1 vs case 2: comparison among the mean values obtained by the deterministic eigensolution (ES) and the ones obtained in the stochastic case by Monte Carlo simulations (MC);
- case 2 - stochastic: comparison between the mean values and the standard deviations given by Monte Carlo simultation (MC) with that obtained by the perturbative approach (PA).

The analysis of the numerical results allows for the following conclusions

- the continuous curves, for the case 1 (ES vs PA), and the dashed lines, for the case 2 (MC vs PA), of Fig. 4 show the percentage error in the mean values evaluation. The error increase by increasing the mean value of the damage under consideration. The maximum error (in absolute value) is approximately equal to 10 and 4 %, for respectively first and second order approach;
- the overlaps among the previous curves can be regarded as a low influence of the parameter variance σ_ε^2 on the mean of the eigenvalues. This circumstance is confirmed from an analytical point of view by the results in Fig. 5: the percentage discrepancy between the mean eigenvalues obtained by the deterministic eigensolution (ES) and the ones obtained in the stochastic case by Monte Carlo simulations (MC) is less than 0.1 %;
- in the stochastic case, the variations of the standard deviation (MC vs PA), Fig.s 6, highlight a maximum error (in absolute value) of about 70 and 40 %, for respectively first and second order approach.

These results suggest that, yet retaining the second order terms, the neighborhood properly identifiable in the inverse problem will shrink strongly going from the deterministic to the stochastic case.

Going to the inverse problem, the experimental quantities ${}^K\tilde{\lambda}_i$ are here obtained numerically, starting from the solution of the eigenvalue problem (1): for the deterministic case by replacing the value of the parameter μ_ε, for the uncertain case performing a Monte Carlo simulation of 5000 samples drawn from a normal distribution $\mathcal{N}(\mu_\varepsilon, \sigma_\varepsilon^2)$. Afterwards we try to identify the uncertain parameter applying the relationships (13-14).

For the case 1, deterministic, the results obtained are those of Fig. 7, which contains both the the first and second order solutions for the identification of mean value of ε. Each of the graphs shows a dashed line, the bisector of the first quadrant of the effective-identified

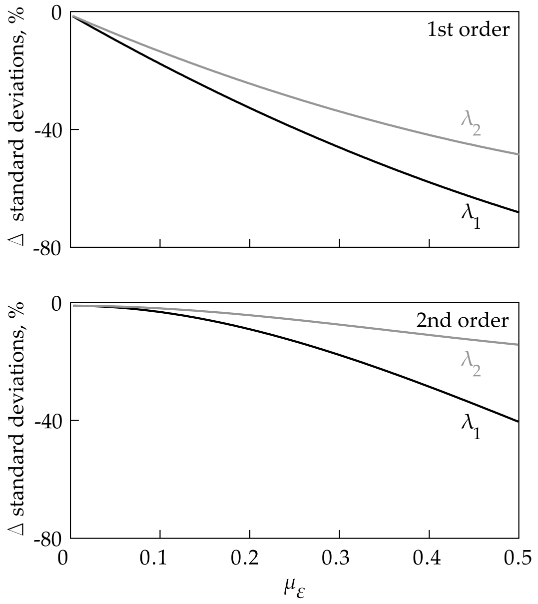

Fig. 6 Percentage error for the standard deviations, case 2

plane of the parameter, representing the path of ideal minimum of the objective function (the identified parameter equals the actual parameter). In this manner, it's easy to gather a qualitative measure of the goodness of the technique as the deviation of the obtained path, represented in the plots with a solid line, from the ideal (dashed) one. A similar convention is adopted for the case 2, stochastic, see Fig. 8, for the mean value, and Fig. 9, for the standard deviation.

Analyzing these results, it turns out that the error made in measuring the damage in our four degrees of freedom planar structural model can be void (in numerical limits) if

- deterministic damage: the stiffness reduction is smaller than 10 %, in the case of first order solution, and than 25 %, in the case of second order solution;
- stochastic damage: the average stiffness reduction is smaller than 10 % (corresponding to a standard deviation of 0.05 for the damage parameter, since a coefficient of variation of 5 % has been assumed) and at least a second order solution is adopted.

In other words, in the stochastic case not only the neighborhood properly identifiable is smaller than the one obtained for the deterministic case (as we expected from the sensitivity analysis), but also the second order terms become essential for a suitable implementation of the technique; indeed, if the first order solution suddenly tends to diverge from the effective statistics of parameter, the second order approximations is able to explore the neighborhood of the unperturbed solution.

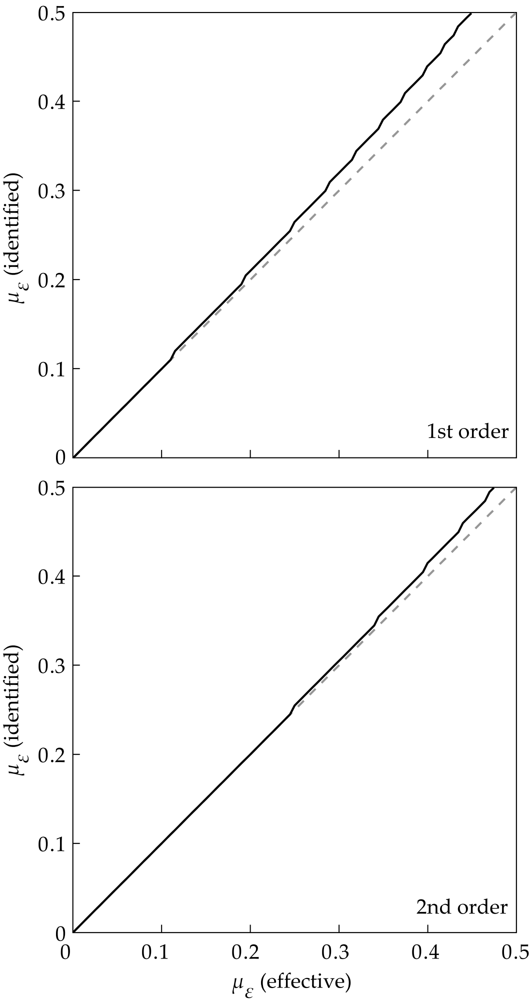

Fig. 7 Identification of the mean value μ_ε, case 1

5 Final remarks and further developments

In this paper the identification of a linear structural systems with random parameters is performed. The structural system under consideration is a four-storey shear frame structure with a stiffness matrix linearly dependent by a random parameter ruling the damage evolution of the columns. Using a perturbative approach the natural frequencies and mode-shapes are pursued in the context of random eigenvalue problems in structural dynamics. The perturbation technique is first applied to derive the asymptotic solution up to the second order to identify the mass and stiffness matrices. Then, the evaluation of the statistic of the eigenvalues and mode-shapes are derived up to the second order. A stochastic identification technique is proposed to characterize the statistics of the quantities of interest and of the random parameter. Particular attention has been devoted in this paper with the identification of the first two frequencies of the structural system and to the mean

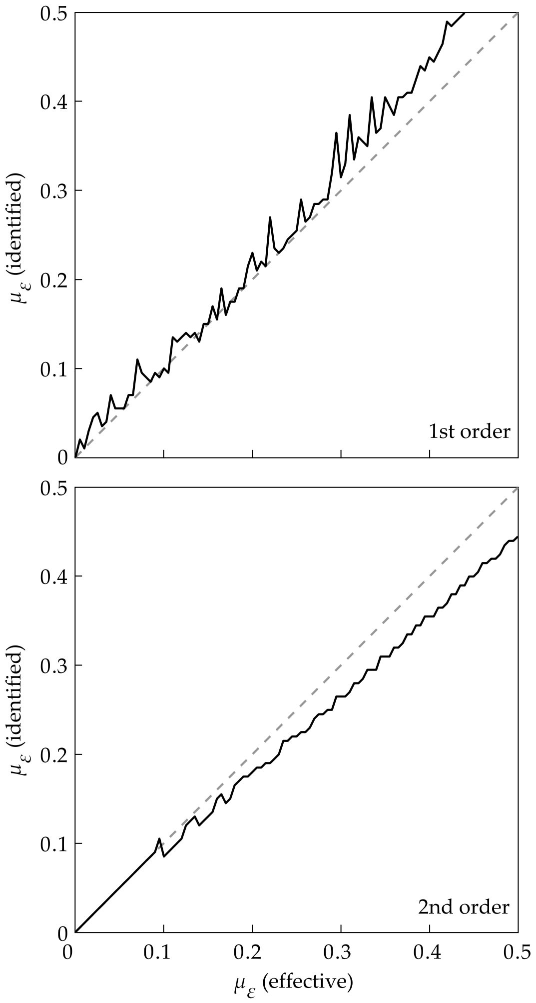

Fig. 8 Identification of the mean value μ_ε, case 2

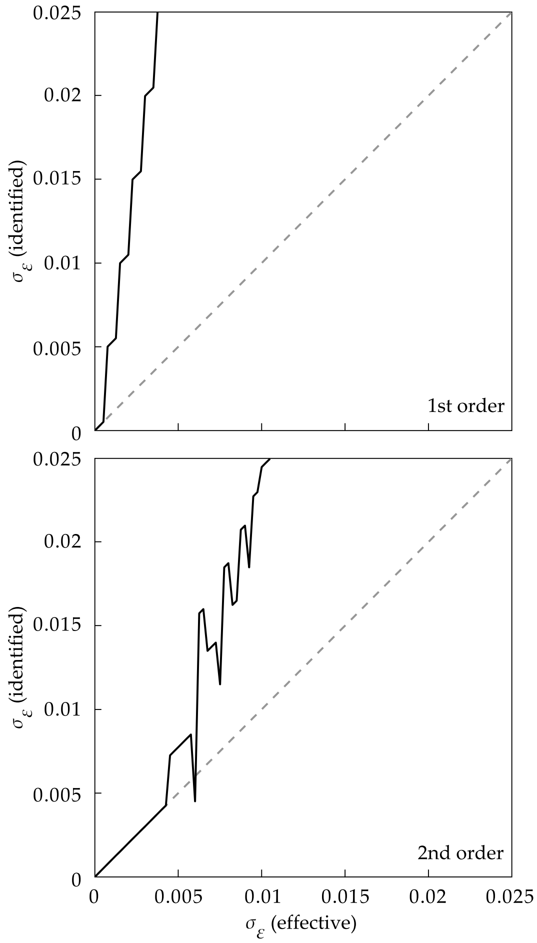

Fig. 9 Identification of the standard deviation σ_ε, case 2

and variance of the random parameter assumed Gaussian without loss of generality. The numerical analysis show that the proposed identification technique is capable of identifying with very limited error the frequencies and the statistics of the random parameter. As further developments we mention here the question of non Gaussianity of the frequencies and mode-shape, the dependance of the parameters by a random vector (multiparametric case), the noise effect and finally the model updating and continuous model updating. All these aspects are under examination and will be tackle in the near future by the authors.

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