RIGIDITY OF OELJEKLAUS-TOMA MANIFOLDS

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ABSTRACT. We prove that Oeljeklaus-Toma manifolds are rigid, and that any line bundle on Oeljeklaus-Toma manifolds of simple type is flat.

INTRODUCTION

Oeljeklaus-Toma manifolds are complex non-Kähler manifolds. They have been introduced in [OT05] as counterexamples to a conjecture by Vaisman. Because of their construction using number fields techniques, many of their properties are encoded in the algebraic structure [OT05, Vul14, Dub14], and their class is well-behaved under such properties [Ver11, Ver13]. They generalize Inoue-Bombieri surfaces in class VII [Ino74, Tri82], and they are in fact solvmanifolds [Kas13].

For example, Oeljeklaus and Toma proved in [OT05, Proposition 2.5], among other results, that the line bundles $K_X^{\otimes k}$ varying $k \neq 0$ are flat. In this note, we use tools both from the number theoretic construction and from analytic geometry to prove more in general that:

Theorem 2.3. Any line bundle on an Oeljeklaus-Toma manifold of simple type is flat.

Here, by saying that the Oeljeklaus-Toma manifold X(K, U) associated to the algebraic number field K and to the admissible group U is of simple type, we understand that there exists no proper intermediate field extension $\mathbb{Q} \subset K' \subset K$ with $U \subseteq \mathcal{O}_{K'}^{*,+}$, that is, there exists no holomorphic foliation of X(K, U) with a leaf isomorphic to X(K', U) [OT05, Remark 1.7].

With similar techniques, we get a vanishing result:

Theorem 3.1. On Oeljeklaus-Toma manifolds X(K, U), for any non-trivial representation $\rho: U \to \mathbb{C}^*$, we have $H^1(X; L_{\rho}) = 0$.

As a corollary, we get rigidity, in the sense of the theory of deformations of complex structures of Kodaira-Spencer-Nirenberg-Kuranishi. Note that for the Inoue surface S_M , this is proven by Inoue in [Ino74, Proposition 2].

Corollary 3.2. Oeljeklaus-Toma manifolds are rigid.

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1. Oeljeklaus-Toma manifolds

Oeljeklaus-Toma manifolds [OT05] provide a beautiful family of examples of compact complex non-Kähler manifolds, generalizing Inoue-Bombieri surfaces [Ino74]. In this section, we briefly recall Oeljeklaus-Toma manifolds definition and main properties from [OT05]. See [OVu13] and [PV12, Section 6 of arXiv version for more details and algebraic number theory background.

Let K be an algebraic number field, namely, a finite extension of Q. Then $K \simeq \mathbb{Q}[X]/(f)$ as Qalgebras, where $f \in \mathbb{Z}[X]$ is a monic irreducible polynomial of degree $n = [K : \mathbb{Q}]$. By mapping X mod (f) to a root of f, the field K admits n = s + 2t embeddings in \mathbb{C} , more precisely, s real embeddings $\sigma_1, \ldots, \sigma_s \colon K \to \mathbb{R}$, and 2t complex embeddings $\sigma_{s+1}, \ldots, \sigma_{s+t}, \sigma_{s+t+1} = \overline{\sigma}_{s+1}, \ldots, \sigma_{s+2t} = \overline{\sigma}_{s+t} \colon K \to \mathbb{R}$ \mathbb{C} . Note that, for any choice of natural numbers s and t, there is an algebraic number field with s real embeddings and 2t complex embeddings, [OT05, Remark 1.1].

Denote by \mathcal{O}_K the ring of algebraic integers of K, namely, elements of K satisfying monic polynomial equations with integer coefficients. Note that, as a \mathbb{Z} -module, \mathcal{O}_K is free of rank n. Denote by \mathcal{O}_K^* the multiplicative group of units of \mathcal{O}_K , namely, invertible elements in \mathcal{O}_K . By the Dirichlet's unit theorem, \mathcal{O}_K^* is a finitely generated Abelian group of rank s + t - 1. Denote by $\mathcal{O}_K^{*,+}$ the subgroup of finite index of \mathcal{O}_K^* whose elements are totally positive units, namely, units being positive in any real embedding: $u \in \mathcal{O}_K^*$ such that $\sigma_j(u) > 0$ for any $j \in \{1, \ldots, s\}$. Let $\mathbb{H} := \{z \in \mathbb{C} : \text{Im } z > 0\}$ denote the upper half-plane. On $\mathbb{H}^s \times \mathbb{C}^t$, consider the following actions:

$$T: \mathcal{O}_K \odot \mathbb{H}^s \times \mathbb{C}^s,$$

$$T_a(w_1, \dots, w_s, z_{s+1}, \dots, z_{s+t}) := (w_1 + \sigma_1(a), \dots, z_{s+t} + \sigma_{s+t}(a)),$$

(1.1)

and

$$R: \mathcal{O}_K^{*,+} \circlearrowleft \mathbb{H}^s \times \mathbb{C}^t ,$$

$$R_u(w_1, \dots, w_s, z_{s+1}, \dots, z_{s+t}) := (w_1 \cdot \sigma_1(u), \dots, z_{s+t} \cdot \sigma_{s+t}(u)) .$$
(1.2)

For any subgroup $U \subset \mathcal{O}_K^{*,+}$, one has the fixed-point-free action $\mathcal{O}_K \rtimes U \circlearrowleft \mathbb{H}^s \times \mathbb{C}^t$. One can always choose an admissible subgroup [OT05, page 162], namely, a subgroup such that the above action is also properly discontinuous and cocompact. In particular, the rank of admissible subgroups is s. Conversely, when either s = 1 or t = 1, every subgroup U of $\mathcal{O}_K^{*,+}$ of rank s is admissible.

One defines the Oeljeklaus-Toma manifold associated to the algebraic number field K and to the admissible subgroup U of $\mathcal{O}_{K}^{*,+}$ as

$$X(K,U) := \mathbb{H}^s \times \mathbb{C}^t / \mathcal{O}_K \rtimes U$$

In particular, for K algebraic number field with s = 1 real embeddings and 2t = 2 complex embeddings, choosing $U = \mathcal{O}_K^{*,+}$ we obtain that X(K, U) is an Inoue-Bombieri surface of type S_M [Ino74].

The Oeljeklaus-Toma manifold X(K,U) is called of simple type when there exists no proper intermediate field extension $\mathbb{Q} \subset K' \subset K$ with $U \subseteq \mathcal{O}_{K'}^{*,+}$, that is, there exists no holomorphic foliation of X(K,U) with a leaf isomorphic to X(K',U) [OT05, Remark 1.7].

Oeljeklaus-Toma manifolds are non-Kähler solvmanifolds [Kas13, §6], with Kodaira dimension $\kappa(X) =$ $-\infty$ [OT05, Proposition 2.5]. Their first Betti number is $b_1 = s$, and their second Betti number in the case of simple type is $b_2 = {s \choose 2}$ [OT05, Proposition 2.3]. Their group of holomorphic automorphisms is discrete [OT05, Corollary 2.7]. The vector bundles Ω_X^1 , Θ_X , $K_X^{\otimes k}$ varying $k \neq 0$ are flat and admit no non-trivial global holomorphic sections [OT05, Proposition 2.5]. Other invariants are computed in [OT05, Proposition 2.5] and [TT15]. Oeljeklaus-Toma manifolds do not contain either any compact complex curve [Ver11, Theorem 3.9], or any compact complex surface except Inoue surfaces [Ver13, Theorem 3.5]. When t = 1, they admit a locally conformally Kähler structure [OT05, page 169], with locally conformally Kähler rank either $\frac{b_1}{2}$ or b_1 [PV12, Theorem 5.4]. This is the Tricerri metric [Tri82] in case s = 1 and t = 1.

In the case t > 2, no locally conformally Kähler metrics are known to exist, so far. The fact that such Oeljeklaus-Toma manifolds carry no locally conformally Kähler metric was proven for s = 1 already in the original paper [OT05, Proposition 2.9], later extended to the case s < t by [Vul14, Theorem 3.1], and eventually widely extended to almost all cases by [Dub14, Theorem 2]. Most likely, in the case $t \ge 2$, no Oeljeklaus-Toma manifold carries a locally conformally Kähler metric. However, note that Oeljeklaus-Toma manifolds admit no Vaisman metrics [Kas13, Corollary 6.2].

2. Flatness of line bundles on Oeljeklaus-Toma manifolds

Let X = X(K, U) be the Oeljeklaus-Toma manifold associated to the algebraic number field K and to the admissible subgroup $U \subseteq \mathcal{O}_K^{*,+}$. Let s be the number of real embeddings of K and 2t the number of complex embeddings of K. Recall that, given a group G acting on a manifold M, we denote by $H^*_{inv(G)}(M)$ the cohomology of the complex of invariant differential forms $(\wedge^*(M))^G$. Moreover, we will denote by \mathcal{O}_M the sheaf of holomorphic functions on a complex manifold M — not to be confused with the ring \mathcal{O}_K of algebraic integers of K.

For a better understanding of the cohomology of X, we start from its very definition, in the form of the following diagram of fibre-bundles:



Naively, since X factors through X^a , we would like to relate the cohomology of X with the cohomology of X^a and \tilde{X} . This is the reason for the following result, describing $H^1(X^a; \mathcal{O}_{X^a})$ in terms of invariant forms on $\mathbb{H}^s \times \mathbb{C}^t$. It will be part of the proof of Theorem 2.3, but we state it in a standalone form because it is useful for itself.

Proposition 2.1. Let X = X(K, U) be an Oeljeklaus-Toma manifold associated to the algebraic number field K, with the notation as above. Consider the action $\mathcal{O}_K \oslash \mathbb{H}^s \times \mathbb{C}^t$ given by translations $a \mapsto T_{\sigma(a)}$. Extend it to the action $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R} \oslash \mathbb{H}^s \times \mathbb{C}^t$ by \mathbb{R} -linearity. Then

$$H^{1}(X^{a}; \mathcal{O}_{X^{a}}) \simeq H^{1}_{inv(\mathcal{O}_{K} \otimes_{\mathbb{Z}} \mathbb{R})}(\mathbb{H}^{s}; \mathcal{O}_{\mathbb{H}^{s}}) \oplus \mathbb{C} \left\langle d\bar{z}^{1}, \dots, d\bar{z}^{t} \right\rangle$$

where H_{inv}^1 denotes the cohomology of invariant forms, and (z^1, \ldots, z^t) are the coordinates on \mathbb{C}^t .

Proof. We need the following general fact about cohomology of invariant differential forms, which we extend to a more general context with respect to [FOT08, Theorem 1.28].

Lemma 2.2 (see, e.g. [FOT08]). Let X be a complex manifold. Let G be a Lie group acting holomorphically on X, and let H be a closed Lie subgroup of G. Suppose that G/H is a compact Lie group. Then the inclusion $(\wedge^{\bullet,\bullet}X)^G \hookrightarrow (\wedge^{\bullet,\bullet}X)^H$ induces isomorphisms in de Rham and Dolbeault cohomologies.

Proof of Lemma. For the sake of completeness, we recall the idea of the proof. Let $d\mu$ be a bi-invariant volume form on G/H with unitary volume [FOT08, Proposition 1.29]. Define the *average operator*

$$\mu \colon (\wedge^{\bullet,\bullet} X)^H \to (\wedge^{\bullet,\bullet} X)^G, \qquad \mu(\alpha) := \int_{G/H} r^* \alpha \, d\mu(r)$$

Clearly, $d \circ \mu = \mu \circ d$, and also $\overline{\partial} \circ \mu = \mu \circ \overline{\partial}$, since the action is holomorphic. Therefore it induces a morphism in de Rham and Dolbeault cohomologies. The statement follows as in [FOT08, Theorem 1.28] by reducing to a contractible neighbourhood of the unit.

Consider now the action $\mathcal{O}_K \mathfrak{O} \mathbb{H}^s \times \mathbb{C}^t$ given by translations $a \mapsto T_{\sigma(a)}$, and extend it to the action $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R} \mathfrak{O} \mathbb{H}^s \times \mathbb{C}^t$ by \mathbb{R} -linearity. It induces the compact Lie group holomorphic action

 $\mathbb{R}^{s+2t}/\mathbb{Z}^{s+2t} \simeq \mathcal{O}_K \otimes_\mathbb{Z} \mathbb{R}/\mathcal{O}_K \circ X^a$. Hence we can apply the Dolbeault Theorem and Lemma 2.2 with $X := X^a, G := \mathcal{O}_K \otimes_\mathbb{Z} \mathbb{R}/\mathcal{O}_K \circ X^a, H := \{1\}$ to obtain

$$H^{1}(X^{a}; \mathcal{O}_{X^{a}}) \simeq H^{0,1}(X^{a}) \simeq H^{0,1}_{\operatorname{inv}(\mathcal{O}_{K} \otimes_{\mathbb{Z}} \mathbb{R}/\mathcal{O}_{K})}(X^{a})$$

$$(2.2)$$

Looking at forms on the covering, we get

$$H^{0,1}_{\mathrm{inv}(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R}/\mathcal{O}_K)}(X^a) = H^1\left(\left(\wedge^{0,\bullet} X^a\right)^{\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R}/\mathcal{O}_K}, \overline{\partial}\right) = H^1\left(\left(\wedge^{0,\bullet} \tilde{X}\right)^{\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R}}, \overline{\partial}\right)$$
(2.3)

We are thus concerned with $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R}$ -invariant forms on $\tilde{X} = \mathbb{H}^s \times \mathbb{C}^t$. A crucial remark here is that, since the first *s* embeddings of *K* are real, $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R}$ spans $\mathbb{R}^s \times \mathbb{C}^t \subset \mathbb{H}^s \times \mathbb{C}^t$. Thus, any $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R}$ -invariant form on $\mathbb{H}^s \times \mathbb{C}^t$ has coefficients depending only on $(\operatorname{Im} w^1, \ldots, \operatorname{Im} w^s)$, where $(w^1, \ldots, w^s, z^1, \ldots, z^t)$ are the coordinates on $\mathbb{H}^s \times \mathbb{C}^t$:

$$H^{1}\left(\left(\wedge^{0,\bullet}\tilde{X}\right)^{\mathcal{O}_{K}\otimes_{\mathbb{Z}}\mathbb{R}},\overline{\partial}\right) = H^{1}\left(\mathcal{C}^{\infty}((\operatorname{Im}\mathbb{H})^{s};\mathbb{C})\otimes\wedge^{0,\bullet}\left\langle d\bar{w}^{1},\ldots,d\bar{w}^{s},d\bar{z}^{1},\ldots,d\bar{z}^{t}\right\rangle,\overline{\partial}\right).$$
(2.4)

Take $[\omega]$ a 1-class in the cohomology of this complex:

$$\omega = \sum_{h=1}^{s} a_h d\bar{w}^h + \sum_{k=1}^{t} b_k d\bar{z}^k \, ,$$

where $a_1, \ldots, a_s, b_1, \ldots, b_t$ are $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R}$ -invariant functions over $\mathbb{H}^s \times \mathbb{C}^t$, namely, they depend only on $(\operatorname{\mathsf{Im}} w^1, \ldots, \operatorname{\mathsf{Im}} w^s)$. From $\overline{\partial} \omega = 0$ and (2.4) we get

$$0 = \frac{\partial a_h}{\partial \bar{z}^k} - \frac{\partial b_k}{\partial \bar{w}^h} = -\frac{\partial b_k}{\partial \bar{w}^h}, \quad \text{for } h \in \{1, \dots, s\}, \quad k \in \{1, \dots, t\},$$

whence it follows that b_k is a holomorphic function in (w^1, \ldots, w^s) depending only on $(\operatorname{Im} w^1, \ldots, \operatorname{Im} w^s)$, hence constant. We have proved that

$$H^{1}\left(\mathcal{C}^{\infty}((\operatorname{Im}\mathbb{H})^{s};\mathbb{C})\otimes\wedge^{0,\bullet}\left\langle d\bar{w}^{1},\ldots,d\bar{w}^{s},d\bar{z}^{1},\ldots,d\bar{z}^{t}\right\rangle,\overline{\partial}\right)$$

$$= H^{1}\left(\mathcal{C}^{\infty}((\operatorname{Im}\mathbb{H})^{s};\mathbb{C})\otimes\wedge^{0,\bullet}\left\langle d\bar{w}^{1},\ldots,d\bar{w}^{s}\right\rangle,\overline{\partial}\right)\oplus\mathbb{C}\left\langle d\bar{z}^{1},\ldots,d\bar{z}^{t}\right\rangle.$$
(2.5)

The statement follows by noting that

$$H^{1}\left(\mathcal{C}^{\infty}((\operatorname{\mathsf{Im}}\mathbb{H})^{s};\mathbb{C})\otimes\wedge^{0,\bullet}\left\langle d\bar{w}^{1},\ldots,d\bar{w}^{s}\right\rangle,\overline{\partial}\right)=H^{1}_{\operatorname{inv}((\mathcal{O}_{K}\otimes_{\mathbb{Z}}\mathbb{R})}(\mathbb{H}^{s};\mathcal{O}_{\mathbb{H}^{s}}),\qquad(2.6)$$

and by assembling equivalences (2.2), (2.3), (2.4), (2.5), (2.6).

Theorem 2.3. Any line bundle on an Oeljeklaus-Toma manifold of simple type is flat.

Proof. Recall that (equivalence classes of) line bundles on X are given by $H^1(X; \mathcal{O}_X^*)$, and that the flat ones are given by the image of the map $n: H^1(X; \mathbb{C}_X^*) \to H^1(X; \mathcal{O}_X^*)$ induced by $\mathbb{C}_X \hookrightarrow \mathcal{O}_X$. The statement is then equivalent to prove that the map

$$n: H^1(X; \mathbb{C}^*_X) \to H^1(X; \mathcal{O}^*_X)$$

is an isomorphism.

The map n appears naturally from the following morphism of short exact sequences of sheaves:



and the corresponding induced morphism of long exact sequences in cohomology:

By the Five Lemma, it suffices to prove that, in diagram (2.7), m is an isomorphism and q is injective. To this aim, consider the following exact sequence of sheaves:

 $0 \longrightarrow \mathbb{C}_X \longrightarrow \mathcal{O}_X \longrightarrow d\mathcal{O}_X \longrightarrow 0$

and the induced exact sequence in cohomology:

$$H^0(X; d\mathcal{O}_X) \longrightarrow H^1(X; \mathbb{C}_X) \xrightarrow{m} H^1(X; \mathcal{O}_X)$$
.

Note that $H^0(X; d\mathcal{O}_X) = 0$, since $H^0(X; \Omega^1_X) = 0$ by [OT05, Proposition 2.5]. Therefore *m* is injective. Using the fact that $\dim_{\mathbb{C}} H^1(X; \mathbb{C}_X) = s$ [OT05, Proposition 2.3], we have reduced the proof of Theorem 2.3 to the following two claims:

Claim H1. dim_{\mathbb{C}} $H^1(X; \mathcal{O}_X) = s$.

Claim H2. The map $q: H^2(X; \mathbb{C}_X) \to H^2(X; \mathcal{O}_X)$ is injective.

In order to describe the cohomology of X and prove the above claims, we use again diagram (2.1): we would like to relate the cohomology of X with the U-invariant cohomology of X^a . In what follows, we use group cohomology and the Lyndon-Hochschild-Serre spectral sequence to accomplish this task.

In general, whenever one has a map $\pi: \tilde{X} \to X = \tilde{X}/G$, for a free and properly discontinuous action of a group G on \tilde{X} , and a sheaf \mathcal{F} on X, there is an induced map

$$H^p(G, H^0(\tilde{X}; \pi^*\mathcal{F})) \to H^p(X; \mathcal{F}) ,$$
 (2.8)

where the first is the group cohomology of G with coefficients in the G-module $H^0(X; \pi^* \mathcal{F})$, see for instance [Mum74, Appendix at page 22]. If, moreover, $\pi^* \mathcal{F}$ is acyclic over \tilde{X} , then the map (2.8) is an isomorphism.

Using the previous argument on the $\mathcal{O}_K \rtimes U$ and the \mathcal{O}_K maps in diagram (2.1), with $\mathcal{F} = \mathcal{O}_X$ and $\mathcal{F} = \mathcal{O}_{X^a}$ respectively, and noting that $\mathcal{O}_{\tilde{X}}$ is acyclic over $\tilde{X} = \mathbb{H}^s \times \mathbb{C}^t$, we obtain the isomorphisms

$$H^p(\mathcal{O}_K \rtimes U; H^0(\tilde{X}; \mathcal{O}_{\tilde{X}})) \simeq H^p(X; \mathcal{O}_X) \text{ and } H^p(\mathcal{O}_K; H^0(\tilde{X}, \mathcal{O}_{\tilde{X}})) \simeq H^p(X^a; \mathcal{O}_{X^a}).$$

Hereafter, for the sake of notation, we denote by R the $\mathcal{O}_K \rtimes U$ -module $H^0(\tilde{X}; \mathcal{O}_{\tilde{X}})$. The previous isomorphisms are then written as

$$H^p(\mathcal{O}_K \rtimes U; R) \simeq H^p(X; \mathcal{O}_X) \quad \text{and} \quad H^p(\mathcal{O}_K; R) \simeq H^p(X^a; \mathcal{O}_{X^a}).$$
 (2.9)

The extension $\mathcal{O}_K \hookrightarrow \mathcal{O}_K \rtimes U \twoheadrightarrow U$ gives the associated Lyndon-Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(U; H^q(\mathcal{O}_K; R)) \Rightarrow H^{p+q}(\mathcal{O}_K \rtimes U; R) ,$$

and the cohomology five-term exact sequence yields



From (2.9), we get $H^0(\mathcal{O}_K; R) \simeq H^0(X^a; \mathcal{O}_{X^a}) = \mathbb{C}$, see [OT05, Lemma 2.4], whence $H^1(U; H^0(\mathcal{O}_K; R)) = \mathbb{C}^{\mathrm{rk}(U)} = \mathbb{C}^s$. Applying again (2.9), the cohomology five-term exact sequence becomes

$$0 \longrightarrow \mathbb{C}^{s} \longrightarrow H^{1}(X; \mathcal{O}_{X}) \longrightarrow H^{1}(X^{a}; \mathcal{O}_{X^{a}})^{U}$$

$$H^{2}(U; \mathbb{C}_{U}) \xrightarrow{} H^{2}(X; \mathcal{O}_{X}) .$$

$$(2.10)$$

Claim H1 follows then from the following.

Claim H1 α . The map $H^1(X; \mathcal{O}_X) \to H^1(X^a; \mathcal{O}_{X^a})^U$ in diagram (2.10) is the zero map.

Proof of Claim H1 α . We have to show that any class in $H^1(X; \mathcal{O}_X)$ yields a zero class in $H^1(X^a; \mathcal{O}_{X^a})^U$. By Proposition 2.1, we have

$$H^{1}(X^{a};\mathcal{O}_{X^{a}})^{U} \simeq \left(H^{1}_{\mathrm{inv}(\mathcal{O}_{K}\otimes_{\mathbb{Z}}\mathbb{R})}(\mathbb{H}^{s};\mathcal{O}_{\mathbb{H}^{s}}) \oplus \mathbb{C}\left\langle d\bar{z}^{1},\ldots,d\bar{z}^{t}\right\rangle\right)^{U} \simeq H^{1}_{\mathrm{inv}(\mathcal{O}_{K}\otimes_{\mathbb{Z}}\mathbb{R})}(\mathbb{H}^{s};\mathcal{O}_{\mathbb{H}^{s}})^{U}$$

where the last equivalence is due to the fact that U acts by multiplication, thus $\mathbb{C} \langle d\bar{z}^1, \ldots, d\bar{z}^t \rangle^U = 0.$

Since X is compact we can apply Hodge theory, and choose an harmonic representative for any cohomology class in $H^1(X; \mathcal{O}_X)$. Therefore we are reduced to show that any class in $H^1(X^a; \mathcal{O}_{X^a})^U \simeq H^1_{\text{inv}(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R})}(\mathbb{H}^s; \mathcal{O}_{\mathbb{H}^s})^U$ represented by a harmonic representative on X is the zero class. Please note that the following argument is inspired by [TT15, Lemma 3.1], where explicit computations are performed for the (0, 1)-Hodge number in the case s = 2 and t = 1.

For convenience, consider holomorphic coordinates $\{w^j := x^j + \sqrt{-1}y^j\}_{j \in \{1,...,s\}}$ on \mathbb{H}^s . For $j \in \{1,\ldots,s\}$, the (1,0)-form

$$\varphi^j := \frac{1}{y^j} dw^j$$

is $\mathcal{O}_K \rtimes U$ -invariant, whence globally defined on X. We can extend it to a global $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R}$ -invariant co-frame $\{\varphi^j\}_{j \in \{1,...,s,s+1,...,s+t\}}$ of (1,0)-forms on X, see [Kas13, §6]. In fact, notice that this co-frame is associated to a presentation of X as a solvmanifold with left-invariant complex structure. That is to say, its associated structure equations are given by constants.

Consider the Hermitian metric g on X such that $\{\varphi^j, \bar{\varphi}^j\}_{j \in \{1,\dots,s+t\}}$ is orthonormal. Let

$$\alpha = \sum_{j=1}^{s+t} \alpha_j \bar{\varphi}^j$$

be a harmonic (0, 1)-form on X, with respect to the Hodge Laplacian associated to g, and consider its class in $H^1(X^a; \mathcal{O}_{X^a})^U$. Since the complex structure and the metric are compatible with the isomorphisms in Proposition 2.1, we can argue in the same way that

$$\alpha_j = \alpha_j(y_1, \dots, y_s) \quad \text{for } j \in \{1, \dots, s\}, \alpha_j = 0 \quad \text{for } j \in \{s+1, \dots, s+t\}.$$

$$(2.11)$$

We use the following notations:

$$\alpha_{k,j} := y^j \cdot \frac{\partial}{\partial y^j} \alpha_k , \qquad \alpha_{k,jj} := (y^j)^2 \cdot \frac{\partial^2}{(\partial y^j)^2} \alpha_k .$$
(2.12)

We now use the fact that α is harmonic. The condition $\overline{\partial}\alpha = 0$ yields the equations

$$\alpha_{k,j} = \alpha_{j,k} \quad \text{for } j,k \in \{1,\ldots,s\}, \ j \neq k .$$

The condition $\overline{\partial}^* \alpha = 0$ yields the equation

$$\sum_{j=1}^{s} (\alpha_{j,j} + T_j(\alpha_j)) = 0 ,$$

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where T_j is a differential operator of order zero with constant coefficients. From the condition $\partial \overline{\partial}^* \alpha = 0$ we get the equations

$$L(\alpha_k) := \sum_{j=1}^s \left(\alpha_{k,jj} + (1+T_j)(\alpha_{k,j}) \right) = 0 \quad \text{for } k \in \{1, \dots, s\} .$$

Note that L is a second order linear elliptic differential operator, compare (2.12). The function α_k being defined on X compact, by the Hopf maximum principle, see *e.g.* [GT01], we get that α_k is constant on X, for any $k \in \{1, \ldots, s\}$. But constant and U-invariant implies zero, and by using also (2.11), we get

$$0 = \left[\sum_{j=1}^{s} \alpha_{j} \bar{\varphi}^{j}\right] = [\alpha] \in H^{1}(X^{a}; \mathcal{O}_{X^{a}})^{U},$$

concluding the proof of Claim H1 α and hence of Claim H1.

We now prove the second claim.

Claim H2. The map $q: H^2(X; \mathbb{C}_X) \to H^2(X; \mathcal{O}_X)$ is injective.

Proof of Claim H2. First of all, we argue as we did for diagram (2.10), the only difference being that this time we forgot the holomorphic structure. Namely, we use $\mathcal{F} = \mathbb{C}_X$ instead of $\mathcal{F} = \mathcal{O}_X$. Everything works the same way, thanks to $H^j(\tilde{X}; \mathbb{C}_{\tilde{X}}) = 0$ for any $j \geq 1$. Denoting by $S := H^0(\tilde{X}; \mathbb{C}_{\tilde{X}})$, the Lyndon-Hochschild-Serre spectral sequence reads

$$E_2^{p,q} = H^p(U; H^q(\mathcal{O}_K; S)) \Rightarrow H^{p+q}(\pi_1(X); S) ,$$

and the associated cohomology five-term exact sequence yields

$$0 \longrightarrow \mathbb{C}^{s} \longrightarrow H^{1}(X; \mathbb{C}_{X}) \longrightarrow H^{1}(X^{a}; \mathbb{C}_{X^{a}})^{U}$$
$$H^{2}(U; \mathbb{C}_{U}) \xrightarrow{\checkmark} H^{2}(X; \mathbb{C}_{X}) .$$

The map $\mathbb{C}_{\tilde{X}} \to \mathcal{O}_{\tilde{X}}$ induces a map $R \to S$, and hence a morphism of exact sequences

In fact, as in the proof of Claim H1 α , the maps $H^1(X; \mathbb{C}_X) \to H^1(X^a; \mathbb{C}_{X^a})^U$ and $H^1(X; \mathcal{O}_X) \to H^1(X^a; \mathcal{O}_{X^a})^U$ are the zero maps. Moreover, by Proposition 2.1, since the coefficients of forms representing classes in $H^1(X^a; \mathcal{O}_{X^a})^U$ depends only on the imaginary part of variables in \mathbb{H}^s , we have that the map $H^1(X^a; \mathbb{C}_{X^a})^U \to H^1(X^a; \mathcal{O}_{X^a})^U$ is surjective. Finally, the map $H^2(U; \mathbb{C}_U) \to H^2(X; \mathbb{C}_X)$ is surjective: indeed, the map $H^2(U; \mathbb{C}_U) \to E_{\infty}^{2,0}$ is surjective, and $E_2^{0,2} = 0 = E_2^{1,1}$, see [OT05, pages 166–167]. Here we use the hypothesis that X is of simple type.

At the end, the diagram reduces to

from which we get that q is injective by diagram chasing.

Claim H1 and Claim H2 imply Theorem 2.3. Thus, we have proved that any line bundle on an Oeljeklaus-Toma manifold of simple type is flat. \Box

Remark 2.4. A well-known result by Ornea and Verbitsky [OVe11] and, in full generality, by Battisti and Oeljeklaus [BO15], states that Oeljeklaus-Toma manifolds of simple type have no divisors. Under the additional hypothesis that $H_1(X)$ has no torsion, this result is a consequence of Theorem 2.3.

Proof. Take any line bundle on X, which is then flat, and let ρ be the associated representation. Under the hypothesis, any representation $\rho: \pi_1(X) \to U$ induces the identity on \mathcal{O}_K [Bra15, Proposition 6]. Therefore the pull-back of L_{ρ} to X^a is trivial, and its sections are constants. Therefore L_{ρ} has no trivial sections on X.

Remark 2.5. The same argument works without the hypothesis on $H_1(X)$ being torsion-free, if Theorem 2.3 is extended to a larger class of generalised OT-manifolds in the sense of [MT15], namely, finite unramified covers of Oeljeklaus-Toma manifolds.

3. RIGIDITY OF OELJEKLAUS-TOMA MANIFOLDS

In this section we extensively apply techniques similar to the ones used in Section 2, to prove the following vanishing result.

Theorem 3.1. Let X = X(K, U) be an Oeljeklaus-Toma manifold. Take any non-trivial representation $\rho: U \to \mathbb{C}^*$, and let L_{ρ} be its associated flat line bundle on X. Then $H^1(X; L_{\rho}) = 0$.

Proof. We use group cohomology, with the action of $U \ni u$ given by $u \mapsto \rho(u) \cdot R_u$, where R_u is the rotation given by equation (1.2). Consider the $\mathcal{O}_K \rtimes U$ and the \mathcal{O}_K maps in diagram (2.1). Since the pull-back of L_ρ to \tilde{X} is trivial, we get

$$H^1(\mathcal{O}_K \rtimes U; R) \simeq H^1(X; L_{\rho}),$$

where $R = H^0(\tilde{X}; \mathcal{O}_{\tilde{X}})$ as in Section 2. From the Lyndon-Hochschild-Serre spectral sequence and the cohomology five-term exact sequence we obtain, as in diagram (2.10), the exact sequence

$$H^1(U; H^0(\mathcal{O}_K; R)) \longrightarrow H^1(X; L_\rho) \longrightarrow H^1(\mathcal{O}_K; R)^U$$

We first show that $H^1(U; H^0(\mathcal{O}_K; R)) = 0$. Indeed, $H^0(\mathcal{O}_K; R) = \mathbb{C}$. Moreover, recall that

$$H^1(U;\mathbb{C}) = \mathbb{C}/\left\{\rho(u)z - z : z \in \mathbb{C}, u \in U\right\} .$$

If ρ is non-trivial, then $\{\rho(u)z - z : z \in \mathbb{C}, u \in U\}$ is non-trivial, whence $H^1(U; H^0(\mathcal{O}_K; R)) = 0$.

We next show that the map $H^1(X; L_{\rho}) \to H^1(\mathcal{O}_K; R)^U$ is the zero map, arguing by Hodge theory as in the proof of Claim H1 α at page 6. Fix a Hermitian metric g on X, and a Hermitian metric h on the line bundle L_{ρ} . Recall that $H^1(\mathcal{O}_K; R) = H^1(X^a; \mathcal{O}_{X^a}) = H^1_{inv(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R})}(\mathbb{H}^s; \mathcal{O}_{\mathbb{H}^s}) \oplus \mathbb{C} \langle d\bar{z}^1, \ldots, d\bar{z}^t \rangle$.

Since ρ is non-trivial, then it suffices to prove that any class in $H^1(X^a; \mathcal{O}_{X^a})$ represented by a harmonic representative on X with values in L_ρ with respect to the metric $g \otimes h$ is the zero class. We interpret harmonicity as follows. Let ϑ be the closed 1-form determined by ρ as $\rho(\gamma) = \exp \int_{\gamma} \vartheta$. Then the (de Rham) cohomology of X with values in the complex line bundle L_ρ corresponds to the cohomology of the trivial bundle $X \times \mathbb{C}$ with respect to the flat connection $d_\vartheta := d + \vartheta \wedge$ -. We split $d_\vartheta = \overline{\partial}_\vartheta + \partial_\vartheta$ where $\overline{\partial}_\vartheta :=$ $\overline{\partial} - \vartheta^{0,1} \wedge$ -. Here, $\vartheta^{0,1}$ is the (0, 1)-component of ϑ . The (Dolbeault) cohomology of X with value in the holomorphic line bundle L_ρ corresponds to the cohomology of the trivial bundle with respect to the flat connection $\overline{\partial}_\vartheta$. Moreover, we can choose metrics compatible with the isomorphisms in Proposition 2.1. Indeed, up to gauge transformations, ϑ depends just in its class in $H^1(X; \mathbb{C})$, which is $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R}/\mathcal{O}_K$ invariant. Then Hodge theory applies with the operator $[\overline{\partial}_\vartheta, \overline{\partial}_\vartheta^*]$. Note indeed that the operator is elliptic, since the second-order part of it is equal to the second-order part of $[\overline{\partial}, \overline{\partial}^*]$. We claim that the zeroth-order part of $[\overline{\partial}_\vartheta, \overline{\partial}^*_\vartheta]$ is positive (with respect to the L^2 -pairing). Indeed, note that $\overline{\partial}^*_\vartheta = -*\overline{\partial}_{-\vartheta}*$. Therefore the zeroth-order term is given by $\vartheta^{0,1} \wedge *(\vartheta^{0,1} \wedge *-) + *(\vartheta^{0,1} \wedge *(\vartheta^{0,1} \wedge -))$. Note that, on 1-forms γ , it holds

 $\langle \vartheta^{0,1} \wedge *(\vartheta^{0,1} \wedge *\gamma) | \gamma \rangle = \| \vartheta^{0,1} \wedge *\gamma \|^2 \ge 0$, and, similarly, $\langle *(\vartheta^{0,1} \wedge *(\vartheta^{0,1} \wedge \gamma)) | \gamma \rangle = \| \vartheta^{0,1} \wedge \gamma \|^2 \ge 0$. It follows that the Hopf maximum principle applies, and the argument proceeds as in the proof of Claim H1 α at page 6.

As a corollary, we get rigidity in the sense of the theory of deformations of complex structures of Kodaira-Spencer-Nirenberg-Kuranishi. See [Ino74, Proposition 2] for rigidity in the case s = t = 1 of Inoue-Bombieri surfaces.

Corollary 3.2. Oeljeklaus-Toma manifolds are rigid.

Proof. Note that $\Theta_{\mathbb{H}^s \times \mathbb{C}^t} = \left\langle \frac{\partial}{\partial w^1}, \ldots, \frac{\partial}{\partial w^s}, \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^t} \right\rangle$, and $\mathcal{O}_K \rtimes U \ni (a, u)$ acts on $\frac{\partial}{\partial w^h}$, respectively $\frac{\partial}{\partial z^k}$, as multiplication by $\sigma_h(u)$, respectively $\sigma_{s+k}(u)$. Whence the holomorphic tangent bundle of an Oeljeklaus-Toma manifold splits as

$$\Theta_X = \bigoplus_{j=1}^{s+t} L_{\sigma_j} \,.$$

where L_{σ_j} are the line bundle associated to the embeddings σ_j . By Theorem 3.1, we get $H^1(X; \Theta_X) = 0$, proving the claim.

Remark 3.3. For the case t = 1, a stronger result was obtained by Braunling. He proves in [Bra15, Proposition 1] that, if two Oeljeklaus-Toma manifolds $X' = X(K', \mathcal{O}_{K'}^{*,+})$ and $X'' = X(K'', \mathcal{O}_{K''}^{*,+})$, both having t = 1, are homotopy equivalent, then they are isomorphic.

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