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Examples of non-trivial rank in locally conformal Kähler geometry

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Abstract We consider locally conformal Kähler geometry as an equivariant, homothetic Kähler geometry (K, Γ). We show that the de Rham class of the Lee form can be naturally identified with the homomorphism projecting Γ to its dilation factors, thus completing the description of locally conformal Kähler geometry in this equivariant setting. The rank r_M of a locally conformal Kähler manifold is the rank of the image of this homomorphism. Using algebraic number theory, we show that r_M is non-trivial, providing explicit examples of locally conformal Kähler manifolds with $1 \nless \mathrm{r}_M \nless b_1$. As far as we know, these are the first examples of this kind. Moreover, we prove that locally conformal Kähler Oeljeklaus-Toma manifolds have either $\mathrm{r}_M = b_1$ or $\mathrm{r}_M = b_1/2$.

1 Introduction

For many reasons, Kähler manifolds are considered the most interesting objects of complex geometry. However, strong topological properties -like formality- even Betti numbers of odd index and others, obstruct the existence of Kähler metrics on many compact manifolds, some of them very simple ones, like the Hopf or Kodaira surfaces. From the Riemannian viewpoint, the natural place to look for metrics with a given property is a conformal class. When this is not possible, then local metrics with the said property can be searched for, subject to some condition on the overlaps.

This is exactly the way Izu Vaisman arrived to the notion of *locally conformal Kähler* (briefly, LCK) metric [9]. The original definition puts the accent on a fixed metric which is locally conformal with local Kähler ones. Equivalently, it requires the existence of a closed

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one-form (the Lee form) which, together with the fundamental two-form, generates a differential ideal. On the other hand, any metric globally conformal with a LCK metric is again LCK. This allows talking about a LCK structure, in which no metric is fixed and only the cohomology class of the Lee form is given. This understanding of LCK geometry is consistent with the fact that any Kähler cover of a LCK manifold bears a Kähler metric with respect to which the covering group acts by holomorphic homotheties. LCK geometry can thus be seen as the pair (K, Γ) of a Kähler manifold and a group of holomorphic homotheties. This viewpoint has been suggested in [2, Remark 5.9], and then developed in [3], where two key notions were introduced: the *presentation* (in this paper called LCK-presentation), which is the pair described above, and the *rank* of the subgroup of $\mathbb{R}^+$ given by dilation factors of Γ, which measures the "true" homothety part of the group.

In the present paper we go a bit further, showing that the Lee form can also be read in these terms. This completes the description of LCK geometry in terms of presentations. Moreover, we show that the examples of LCK manifolds constructed in [6] have highly non-trivial rank: their rank is either equal to the first Betti number or to half of it. In particular, this provides a first example of LCK manifold of rank $\neq 1$ and strictly less than b_1.

2 LCK-presentations for complex manifolds

For convenience of the reader, we here briefly review notation established in [3].

Let M be a complex manifold. A *locally conformal Kähler* metric is a conformal class $[g]$ of Hermitian metrics on M such that $[g]$ is given locally by Kähler metrics. The conformal class $[g]$ gives raise to a de Rham cohomology class $[\omega_g] \in H^1(M)$, whose representative ω_g is defined as the unique closed 1-form satisfying $d\Omega_g = \omega_g \wedge \Omega_g$, where Ω_g denotes the fundamental form of g. The 1-form ω_g is called the *Lee form* of g.

One can immediately see that a complex manifold M of complex dimension at least two admits a locally conformal Kähler metric if and only if there is a complex covering space K of Kähler type such that $\pi_1(M)$ acts on K by holomorphic homotheties with respect to the Kähler metric.

The above discussion motivates the following definitions, first given in [3]. For the notion of minimal cover in the more general setting of conformal geometry, see also [1].

Definition 2.1 Let K be a homothetic Kähler manifold (that is, a complex manifold K on which we fixed a class of homothetical Kähler metrics) and Γ a discrete Lie group of biholomorphic homotheties acting freely and properly discontinuously on K.

- The pair (K, Γ) is called a LCK-*presentation*.
- If M is a complex manifold and $M = K/\Gamma$ as complex manifolds, (K, Γ) is called a LCK-*presentation for* M.
- If Γ does not contain isometries other than the identity, then (K, Γ) is called *minimal*, and if K is simply connected then (K, Γ) is called *maximal*.

Remark 2.2 Given a complex manifold M, the statement "(K, Γ) is a LCK-presentation for M" is just a shortcut for "K is a complex covering space of M, and Γ are its covering transformations, and there is a Kähler metric on K which is conformally equivalent to a Γ-invariant metric". Due to the 1-1 correspondence existing between locally conformal Kähler manifolds and minimal presentations, we will often abuse of this language by saying "the locally conformal Kähler manifold (K, Γ)".

In a homothetic Kähler manifold K we denote by $\mathrm{Hmt}(K)$ the group of its biholomorphic homotheties, and by

$$\rho_K : \mathrm{Hmt}(K) \to \mathbb{R}^+$$

the group homomorphism associating to a homothety its dilation factor. For any locally conformal Kähler manifold M, LCK-presented as (K, Γ), the rank of the free abelian group $\rho_K(\Gamma)$ depends only on M [3, Proposition 2.10].

Definition 2.3 The rank of $\rho_K(\Gamma)$ is called *the rank of M*, and is denoted by r_M.

Remark 2.4 The rank r_M measures "how much" the locally conformal Kähler manifold is far from the Kähler geometry.

3 The Lee form

Let M be a locally conformal Kähler manifold LCK-presented as (K, Γ). The question if the de Rham class of any Lee form of M can be completely described in terms of LCK-presentations has been left open in [3]. In this Section we fill this gap.

From the following exact sequence:

$$1 \to \pi_1(K) \to \pi_1(M) \to \Gamma \to 1$$

we get:

$$H_1(K, \mathbb{Z}) \to H_1(M, \mathbb{Z}) \to \frac{\Gamma}{[\Gamma, \Gamma]} \to 0 \tag{3.1}$$

so in de Rham cohomology we have:

$$0 \to \mathrm{Hom}\left(\frac{\Gamma}{[\Gamma, \Gamma]}, \mathbb{R}\right) \xrightarrow{i} H^1_{dR}(M) \to H^1_{dR}(K) \tag{3.2}$$

On the other hand, since $\Gamma \subset \mathrm{Hmt}(K)$ (by definition) we have a natural homomorphism from $\frac{\Gamma}{[\Gamma,\Gamma]}$ to $\mathbb{R}$ by

$$\frac{\Gamma}{[\Gamma, \Gamma]} \xrightarrow{\rho_K} \mathbb{R}^+ \xrightarrow{\log} \mathbb{R} \tag{3.3}$$

For the sake of simplicity, we still denote by ρ_K this element of $\mathrm{Hom}\left(\frac{\Gamma}{[\Gamma,\Gamma]}, \mathbb{R}\right)$. We are now ready to state the following Theorem.

Theorem 3.1 *Let M be a locally conformal Kähler manifold LCK-presented as (K, Γ), and let $[\omega] \in H^1_{dR}(M)$ be its Lee form. Let i be the map given by (3.2), and ρ_K the element of $\mathrm{Hom}\left(\frac{\Gamma}{[\Gamma,\Gamma]}, \mathbb{R}\right)$ given by (3.3). Then*

$$[\omega] = i(\rho_K)$$

Proof Denote by p the projection from K to M, and by g the Riemannian metric on M associated to ω. Thus, p^*g is a Γ-invariant metric on K, $p^*\omega = df$ is an exact 1-form on K, and the metric $g_K = e^{-f} p^*g$ on K is Kähler.

For any $\gamma \in \Gamma$, denote by $[\gamma]$ the corresponding element of $\frac{\Gamma}{[\Gamma,\Gamma]}$. We then have:

$$\gamma^* g_K = \gamma^* e^{-f} p^* g = e^{-f\circ\gamma} \gamma^* p^* g = e^{-f\circ\gamma} p^* g = e^{-f\circ\gamma + f} e^{-f} p^* g = e^{-f\circ\gamma+f} g_K$$

and thus [remember that ρ_K is defined as in (3.3)]:

$$\rho_K([\gamma]) = -f \circ \gamma + f \tag{3.4}$$

Remark in particular that since $\gamma \in \mathrm{Hmt}(K)$, then $-f \circ \gamma + f$ is constant.

Fix a point x_0 of M, and a loop $\alpha\colon [0, 1] \to M$ with base point x_0. Choose a point y_0 in the fibre $p^{-1}(x_0)$, and lift α to a path $\tilde{\alpha}_{y_0}$ on K starting from y_0. The map γ_α sending y_0 to $\tilde{\alpha}_{y_0}(1)$ depends only on the homotopy class $[\alpha]$ of α, and defines a right action of $\pi_1(M, x_0)$ on the fibre $p^{-1}(x_0)$.

To prove the claim, we thus need to show that $\omega([\alpha]) = \rho_K(\gamma_\alpha^{-1})$ for every loop α in M. As for $\omega([\alpha])$, we have:

$$\omega([\alpha]) = \int_\alpha \omega = \int_{\tilde{\alpha}_{y_0}} p^*\omega = \int_{\tilde{\alpha}_{y_0}} df = f(\tilde{\alpha}_{y_0}(1)) - f(\tilde{\alpha}_{y_0}(0))$$

As for $\rho_K(\gamma_\alpha^{-1})$, we use (3.4):

$$\begin{aligned}\rho_K(\gamma_\alpha^{-1}) &= -f \circ \gamma_\alpha^{-1} + f = -f \circ \gamma_\alpha^{-1}(\tilde{\alpha}_{y_0}(1)) + f(\tilde{\alpha}_{y_0}(1))\\ &= -f(\tilde{\alpha}_{y_0}(0)) + f(\tilde{\alpha}_{y_0}(1))\end{aligned}$$

and this proves the claim. □

Remark 3.2 The rank r_M satisfies $0 \le \mathrm{r}_M \le b_1(M)$, and $\mathrm{r}_M = 0$ if and only if M is globally conformal Kähler.

4 Oeljeklaus–Toma manifolds

In their beautiful paper [6], the authors construct locally conformal Kähler manifolds using tools from Algebraic Number Theory. For the sake of completeness, we recall their construction.

We will denote by F an algebraic number field, by $\mathcal{O}_F$ the ring of algebraic integers of F and by $\mathcal{O}_F^*$ the multiplicative group of units of $\mathcal{O}_F$. If $[F : \mathbb{Q}] = n = s + 2t$ is the degree of F over $\mathbb{Q}$, we denote by $\{\sigma_i\colon F \to \mathbb{C}\}_{i=1,\dots,n}$ the complex embeddings of F, where the first s embeddings are real, and the last $2t$ satisfy $\sigma_{s+i} = \bar{\sigma}_{s+i+t}$. The units which are positive in all real embeddings of F are denoted by $\mathcal{O}_F^{*,+}$.

The embeddings σ_i give the natural map

$$\begin{gathered}F \to \mathbb{C}^{s+t}\\ \sigma(x) \stackrel{\text{def}}{=} (\sigma_1(x) \dots, \sigma_{s+t}(x))\end{gathered}$$

In this way we get an action of $\mathcal{O}_F$ on $\mathbb{C}^{s+t}$ given by translations. We denote this action by T: if $a \in \mathcal{O}_F$ and $(z_1, \dots, z_{s+t}) \in \mathbb{C}^{s+t}$, then

$$T_a(z_1, \dots, z_{s+t}) \stackrel{\text{def}}{=} (z_1 + \sigma_1(a), \dots, z_{s+t} + \sigma_{s+t}(a)) \tag{4.1}$$

We also have an action of $\mathcal{O}_F^*$ on $\mathbb{C}^{s+t}$, denoted by R (as in "rotation"): if $u \in \mathcal{O}_F^*$, then

$$R_u(z_1, \dots, z_{s+t}) \stackrel{\text{def}}{=} (\sigma_1(u)z_1, \dots, \sigma_{s+t}(u)z_{s+t}) \tag{4.2}$$

Next, letting $\mathbb{H}$ be the upper complex half-plane of complex numbers with strictly positive imaginary part, we see that $\mathcal{O}_F \rtimes \mathcal{O}_F^{*,+}$ acts freely on $\mathbb{H}^s \times \mathbb{C}^t$. However, this action is not properly discontinuous in general. Still, Oeljeklaus and Toma show that one can always find a suitable subgroup $U \subset \mathcal{O}_F^{*,+}$ such that the action of $\mathcal{O}_F \rtimes U$ is properly discontinuous and moreover, the quotient is compact. Subgroups of these kind are called *admissible*, and it is furthermore shown that when $t = 1$ every subgroup of finite index of $\mathcal{O}_F^{*,+}$ is admissible.

Both $\mathcal{O}_F$ and $\mathcal{O}_F^{*,+}$ act holomorphically on $\mathbb{H}^s \times \mathbb{C}^t$, so the quotient inherits a complex structure.

Definition 4.1 Given a finite field extension F of $\mathbb{Q}$ and an admissible subgroup $U \subset \mathcal{O}_F^{*,+}$, the compact complex manifold

$$M(F, U) \stackrel{\text{def}}{=} \frac{\mathbb{H}^s \times \mathbb{C}^t}{\mathcal{O}_F \rtimes U}$$

is called an *Oeljeklaus–Toma manifold*.

The first Betti number of $M(F, U)$ is computed in [6]. Since we also need this fact, we include here a different and less technical proof than the original one.

Theorem 4.2 [6, Proposition 2.3] *Let $M = M(F, U)$ be an Oeljeklaus–Toma manifold. Then $b_1(M) = s$.*

Proof We identify $\pi_1(M)$ with the deck transformation group $\mathcal{O}_F \rtimes U$, which is generated by $\{T_a, R_u\}_{a \in \mathcal{O}_F, u \in U}$, given by (4.1), (4.2). Since $\pi_1(M)/\mathcal{O}_F \simeq U$ is abelian, and $H_1(M, \mathbb{Z})$ is the maximal abelian quotient of $\pi_1(M)$, we have a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \to & [\pi_1(M), \pi_1(M)] & \to & \pi_1(M) & \to & H_1(M, \mathbb{Z}) & \to & 1 \\ & & i \uparrow & & \| & & p \downarrow & & \\ 1 & \to & \mathcal{O}_F & \to & \pi_1(M) & \to & U & \to & 1 \end{array}$$

From p we get that $\mathrm{Rank}(H_1(M, \mathbb{Z})) = \mathrm{Rank}(U) - \mathrm{Rank}(\ker p)$. But the Snake Lemma gives $\ker p \simeq \operatorname{coker} i$, thus it is enough to show that $\operatorname{coker} i = \mathcal{O}_F/[\pi_1(M), \pi_1(M)]$ is finite.

By direct computation we see $[T_a, R_u] = T_{(1-u)a}$, for any $a \in \mathcal{O}_F$ and any $u \in U$. In particular, this shows that for any $u \in U$, the principal ideal $(1 - u)\mathcal{O}_F$ is a subgroup of $[\pi_1(M), \pi_1(M)]$. But if $u \neq 1$, then $(1 - u)\mathcal{O}_F$ has finite index, as $\mathcal{O}_F$ is a Dedekind ring. □

Since for the rest of this paper we will be concerned only with the case $t = 1$ and $U = \mathcal{O}_F^{*,+}$, we skip the details about U.

5 The rank of Oeljeklaus–Toma manifolds

The following result was the starting point for this paper.

Theorem 5.1 [6, Page 7] *Consider an Oeljeklaus-Toma manifold $M(F, \mathcal{O}_F^{*,+})$, with $t = 1$ and $s > 0$.*

(1) *The real function*

$$\phi(z) \stackrel{\text{def}}{=} \prod_{j=1}^{s} \frac{i}{z_j - \bar{z}_j} + |z_{s+1}|^2 \qquad (5.1)$$

is a global Kähler potential on $\mathbb{H}^s \times \mathbb{C}$.

(2) *When* $\mathbb{H}^s \times \mathbb{C}$ *is equipped with the Kähler metric* $i\partial\bar{\partial}\phi$ *given above, the pair* $(\mathbb{H}^s \times \mathbb{C}, \mathcal{O}_F \rtimes \mathcal{O}_F^{*,+})$ *is a* LCK*-presentation for* $M(F, \mathcal{O}_F^{*,+})$.

Remark 5.2 If $u \in \mathcal{O}_F^{*,+}$ is a unit, then its norm

$$\mathrm{Nm}(u) = \prod_{i=1}^{s+2} \sigma_i(u)$$

is a unit of $\mathbb{Z} = \mathcal{O}_\mathbb{Q}$ as well, so $\mathrm{Nm}(u) = \pm 1$ [5, Lemma 5.2]. As $\sigma_i(u) > 0$ for all $i = 1, \ldots, s$, we see that $\mathrm{Nm}(u) = 1$. In particular, from the expression of the potential above, we get that the rank of $M(F, \mathcal{O}_F^{*,+})$ is the rank of the multiplicative subgroup of $\mathbb{R}^+$ given by $\{|\sigma_{s+1}(u)|^2$ such that $u \in \mathcal{O}_F^{*,+}\}$ (see also the proof of [6, Proposition 2.9]).

The following result describes the rank of Oeljeklaus–Toma manifolds.

Theorem 5.3 *Let* F *be a number field with* $s > 0$ *real embeddings* $\sigma_1, \ldots, \sigma_s : F \to \mathbb{R}$ *and exactly two non-real embeddings* $\sigma_{s+1}, \bar{\sigma}_{s+1} : F \to \mathbb{C}$. *Let* $M = M(F, \mathcal{O}_F^{*,+})$ *be an Oeljeklaus–Toma manifold with the locally conformal Kähler structure given by Theorem 5.1. Let* $n = [F : \mathbb{Q}]$, *so that* $n = s + 2$, $\dim_\mathbb{C} M = n - 1$ *and* $b_1(M) = n - 2$.

(1) *If* n *is odd, then* M *has maximal rank, that is to say,* $\mathrm{r}_M = b_1(M) = n - 2$.
(2) *If* n *is even, then either* M *has again maximal rank or* $\mathrm{r}_M = \frac{b_1(M)}{2}$; *the last situation occurs if and only if* F *is a quadratic extension of a totally real number field.*

Proof (1) By Remark 5.2, it is enough to show that the map

$$u \mapsto |\sigma_{s+1}(u)|^2 \tag{5.2}$$

is injective. Let $u \in \mathcal{O}_F^{*,+}$ be a unit with $|\sigma_{s+1}(u)|^2 = 1$ (that is to say, $|\sigma_{s+1}(u)| = 1$), and let P_u be its minimal polynomial over $\mathbb{Q}$. Since $p \overset{\text{def}}{=} \deg P_u = [\mathbb{Q}(u) : \mathbb{Q}]$ divides $n = [F : \mathbb{Q}]$ and n is odd, we see that also p is odd. So P_u is given by

$$P_u(X) = X^p + a_{p-1}X^{p-1} + \cdots + a_1 X + a_0$$

Hence

$$a_0^{n/p} = -\mathrm{Nm}(u) = -1$$

and since $a_0 \in \mathbb{Z}$, we get $a_0 = -1$. Now observe that $|\sigma_{s+1}(u)| = 1$ implies $\bar{\sigma}_{s+1}(u) = \frac{1}{\sigma_{s+1}(u)}$. But $\bar{\sigma}_{s+1}(u)$ is a root of P_u, hence $P_u\left(\frac{1}{\sigma_{s+1}(u)}\right) = 0$. This means that $\sigma_{s+1}(u)$ satisfies the equation of degree p

$$1 + a_{p-1}X + \cdots + a_1 X^{p-1} - X^p = 0$$

Thus, the uniqueness of the minimal polynomial forces $a_k = -a_{p-k}$ for all k: but then $P_u(1) = 0$, hence $u = 1$.

(2) We consider the case when M is not of maximal rank. This means that the map (5.2) is not injective, so that there exists a unit $u \in \mathcal{O}_F^{*,+}$, $u \neq 1$, such that $|\sigma_{s+1}(u)| = 1$. We claim that $\deg P_u$ is even. In fact, if $\deg P_u$ was odd we would get

$$a_0^{n/p} = \mathrm{Nm}(u) = 1$$

that is, $a_0 = \pm 1$. If $a_0 = -1$, arguing the same as in point (1) above, we would have $u = 1$, whereas if $a_0 = 1$, we would have $P_u(-1) = 0$, that is, $u = -1 \notin \mathcal{O}_F^{*,+}$, which is a contradiction in both cases. Thus, $\deg P_u$ is even.

Lemma 5.4 *If F admits exactly 2 complex non-real embeddings, then any proper intermediate field extension $F \supsetneq E \supsetneq \mathbb{Q}$ is totally real.*

Proof Assume E is not totally real, and let ς be a complex non-real embedding. Let $d \stackrel{\text{def}}{=} [F : E]$. Then F admits d embeddings fixing E pointwise, and their composition with ς gives $2d$ complex non-real embeddings of F. Thus $d = 1$, and $F = E$. □

In what follows, we need to assume that u is non-real: since we can always replace u with $\sigma_{s+1}(u)$, this assumption is not restrictive. Then, using Lemma 5.4, we get $F = \mathbb{Q}(u)$.

Consider now the intermediate field extension $E \stackrel{\text{def}}{=} \mathbb{Q}(u + \frac{1}{u})$. Clearly, $F \supsetneq E \supsetneq \mathbb{Q}$. Using again Lemma 5.4, we get that E is totally real, whereas from $\left(u + \frac{1}{u}\right) u = u^2 + 1$ we get $[F = \mathbb{Q}(u) : E] = 2$.

It remains to check that the rank in this case is $\frac{b_1(M)}{2}$. For any unit $u \in \mathcal{O}_F^{*,+}$ we have $|\sigma_{s+1}(u)|^2 = \mathrm{Nm}_{F/E}(u)$. This means that the rank of the group

$$\left\{ |\sigma_{s+1}(u)|^2 \text{ such that } u \in \mathcal{O}_F^{*,+} \right\}$$

is the rank of the image of the norm map $\mathrm{Nm}_{F/E} : \mathcal{O}_F^{*,+} \to \mathcal{O}_E^{*,+}$. But $(\mathcal{O}_E^{*,+})^2 \subset \mathrm{Im}(\mathrm{Nm}_{F/E})$, thus $\mathrm{Nm}_{F/E}$ has the same rank as $\mathcal{O}_E^{*,+}$, which is $n/2 + 0 - 1 = n/2 - 1$ by Dirichlet Unit Theorem. □

Remark 5.5 Theorem 5.3 holds for the general Oeljeklaus–Toma manifold $M(F, U)$, whenever $U \subset \mathcal{O}_F^*$ has finite index.

Remark 5.6 Theorem 5.3 shows that [3, Example 2.13] holds only for some Oeljeklaus–Toma manifolds.

The following two examples describe the case $[F : \mathbb{Q}]$ even.

Example 5.7 Pick monic polynomials f_1, f_2 and f_3 in $\mathbb{Z}[X]$ of degree $2n$ such that:

- f_1 is irreducible modulo 2;
- f_2 splits as a product of a linear factor and an irreducible polynomial modulo 3;
- f_3 is a product of an irreducible polynomial of degree 2 and of two irreducible polynomials of odd degree modulo 5.

Then for every polynomial $g \in \mathbb{Z}[X]$ of degree $2n - 1$ the polynomial

$$f = -15 f_1 + 10 f_2 + 6 f_3 + 30 g$$

is monic, is irreducible (since its reduction modulo 2 is irreducible), and has maximal Galois group S_{2n} (proceed as in [4, Example 4.31], noting that $30 \equiv 0$ modulo 2, 3 and 5). For suitable choices of g we will have that f has exactly 2 non-real roots (proceed as in [6, Remark 1.1], observing that the set of polynomials $\{-15 f_1 + 10 f_2 + 6 f_3 + 30 g, \deg g = 2n - 1\}$ is a lattice in $\mathbb{Q}^{2n}$). Let F_f be the splitting field of f and fix an isomorphism between $\mathrm{Gal}(F_f/\mathbb{Q})$ and S_{2n}; let $H \subset \mathrm{Gal}(F_f/\mathbb{Q})$ be the subgroup corresponding to S_{2n-1} viewed as the set of all permutations fixing 1. Then F_f^H has no proper subfields as $S_{2n-1} \subset S_{2n}$ is a maximal subgroup, and by Theorem 5.3, point 2, the corresponding Oeljeklaus–Toma manifold $M(F_f^H, \mathcal{O}_{F_f^H}^{*,+})$ has maximal rank $\mathrm{r}_M = b_1(M) = 2n - 2$.

Example 5.8 Pick an arbitrary totally real number field E of degree n. Let α be a primitive element of E over $\mathbb{Q}$ and let $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n$ be the conjugates of α: we can assume $\alpha_1 > \alpha_2 > \cdots > \alpha_n$. Let σ_i be the embedding corresponding to α_i, and let $q \in \mathbb{Q}$ such that $\alpha_{n-1} > q > \alpha_n$. Take $F \stackrel{\text{def}}{=} E(\sqrt{\alpha - q})$. Then $[F : E] = 2$ (otherwise, $\alpha - q = e^2$ for some $e \in E$: but then $\sigma_n(\alpha) - q = \sigma_n(e^2)$ so $\alpha_n - q > 0$ since $\sigma_n(e) \in \mathbb{R}$ as E is totally real), and F admits exactly 2 complex non-real embeddings (the $[F : E]$ extensions of σ_n to F). Then by Theorem 5.3, point 2, the corresponding Oeljeklaus–Toma manifold $M(F, \mathcal{O}_F^{*,+})$ has rank $\mathrm{r}_M = \frac{b_1(M)}{2} = n - 1$.

Remark 5.9 Example 5.8 relies on the existence of a totally real number field of degree n, for an arbitrary n. This is a very classical fact, but for the sake of completeness we include a proof below. First, recall that if p is an arbitrary prime number, and ζ is be a primitive root of unity of order p, then $\mathbb{Q} \subset \mathbb{Q}(\zeta)$ is a Galois extension of degree $p - 1$, with Galois group cyclic of order $p - 1$. Moreover, $\mathbb{Q} \subset \mathbb{Q}(\zeta + \frac{1}{\zeta})$ is a totally real Galois extension of $\mathbb{Q}$, with Galois group cyclic of order $\frac{p-1}{2}$. Now, choose a prime p such that n divides $\frac{p-1}{2}$ (Dirichlet's Theorem on prime numbers in arithmetic progressions), and choose a subgroup H of $\mathrm{Gal}(\mathbb{Q}(\zeta + \frac{1}{\zeta})/\mathbb{Q})$ of index n. Then $\mathbb{Q}(\zeta + \frac{1}{\zeta})^H$ is a subfield of $\mathbb{Q}(\zeta + \frac{1}{\zeta})$, hence totally real, and $\mathbb{Q} \subset \mathbb{Q}(\zeta + \frac{1}{\zeta})^H$ is Galois. Thus $[\mathbb{Q}(\zeta + \frac{1}{\zeta})^H : \mathbb{Q}] = n$.

Remark 5.10 We summarize in the following table what we presently know about relations between r_M and properties of M. If $M = (K, \Gamma)$, by "Potential" in the Table we mean there exists a global Kähler potential on K, and by "Automorphic potential" we mean there exists a global Kähler potential on K such that Γ acts on it by homotheties (see [7,8]).

Statement	**True/False**	**Proof or Refutation**
$b_1 = 1 \Rightarrow \mathrm{r}_M = 1$	True	$1 \leq \mathrm{r}_M \leq b_1$
$\mathrm{r}_M = 1 \Rightarrow b_1 = 1$	False	Induced Hopf bundles over curves of large genus in $\mathbb{CP}^2$: $\mathrm{r}_M = 1$, arbitrarily large b_1
Vaisman $\Rightarrow \mathrm{r}_M = 1$	True	[3, Corollary 4.7]
Automorphic potential $\Rightarrow M$ can be deformed to M', with $\mathrm{r}_{M'} = 1$	True	[8, Proposition 2.5] and [7, Proofs of 5.2 and 5.3]
$\mathrm{r}_M = 1 \Rightarrow$ Automorphic potential	False	Inoue surfaces
Potential $\Rightarrow \mathrm{r}_M = 1$	False	Oeljeklaus-Toma manifolds as in Theorem 5.3, with $s > 2$: ϕ is a potential, and $\mathrm{r}_M = b_1$ or $b_1/2$
$\mathrm{r}_M = 1 \Rightarrow$ Potential	False	Diagonal Hopf surface blown up in one point
$\exists M$ with $\mathrm{r}_M = b_1 > 1$	True	Take $n = 2$, $f_1 = X^4 + X + 1 = f_2$, $f_3 = (X^2+2)(X-1)(X+1)$ and $g = 0$ in Example 5.7
$\exists M$ with $b_1 > 1$ and $\mathrm{r}_M = b_1/2$	True	Take $n = 2$, $L = \mathbb{Q}(\sqrt{2})$, $\alpha = \sqrt{2}$ and $q = 0$ in Example 5.8

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