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Single factor models with Markovian spot interest rate: an analytical treatment

Carlo Mari

Dipartimento di Scienze, Università di Chieti
e-mail: `mari@sci.unich.it`

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Abstract. In the spirit of the Heath–Jarrow–Morton methodology, we provide an analytical characterization of bond prices within the context of single factor term structure models in which the spot rate follows a Markov process and the volatility structure of zero coupon bond returns is stochastic. Also, a perturbative analysis of the extended Cox–Ingersoll–Ross model is proposed.

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1. Introduction

Non-Markov spot interest rate dynamics are allowed within the framework of the Heath–Jarrow–Morton (HJM) (1990, 1992) methodology, and they represent a technical difficulty to performing calculations. In the recent financial literature some authors addressed the problem by finding constraints on the volatility structure of zero-coupon bond returns in order to avoid this phenomenon. Carverhill (1994), in a multifactor HJM model with deterministic volatility structure, derives a necessary and sufficient condition to imply a Markovian spot rate process. This condition generalizes the result obtained by Hull and White (1990, 1993) in the case of single factor models.

If the volatility structure of zero coupon bond returns is allowed to be stochastic via possible dependence on the spot rate, Jeffrey (1995) derives, in a single factor HJM context, a constraint on the volatility structure of the forward rate to assure Markov spot rate dynamics. From this point of view, the extended Cox–Ingersoll–Ross (CIR) (1985 a,b) model proposed by Hull and White (1990) can be viewed as a particular case in which Jeffrey's condition is satisfied.

The purpose of this paper is to provide analytical solutions of the bond pricing problem in the class of single factor HJM models with Markovian spot rate and stochastic volatility satisfying Jeffrey's constraint.

Let us state the problem. We denote by

$$P(r(t), t, T) = \exp\left(- \int_t^T f(r(t), t, u)du \right), \tag{1}$$

the price at time t of a pure discount bond maturing at time T, by $f(r(t), t, T)$ the forward rate curve, and by $r(t) = f(r(t), t, t)$ the spot rate. Under the risk-neutral measure, the stochastic dynamics of the term structure can be cast in the form

$$\begin{aligned} \frac{dP}{P}(r(t), t, T) &= r(t)dt - \sigma_p(r(t), t, T)dw(t), \\ P(r(0), 0, T) &= P^*(0, T), \end{aligned} \tag{2}$$

where $P^*(0, T)$ is the initial term structure accounting for market data and $\sigma_p(r(t), t, T)$ is the volatility structure; here $w(t)$ is a standard unidimensional Brownian motion.

It is staightforward to derive the dynamics of the forward rate. In fact, we can use Equation (1) and Itô' s Lemma to obtain

$$df(r(t), t, T) = \sigma(r(t), t, T)\sigma_p(r(t), t, T)dt + \sigma(r(t), t, T)dw(t),$$

where

$$\sigma(r(t), t, T) = \frac{\partial \sigma_p(r(t), t, T)}{\partial T}.$$

The following assumptions are made:

(i) $P(r(t), t, T)$, or equivalently $f(r(t), t, T)$, are smooth functions of their arguments, and $P^*(0, T)$ is a smooth function of maturity T[1];

(ii) the volatility structure of the forward rate $\sigma(r(t), t; T)$ satisfies Jeffrey's condition

$$\sigma(r(t), t, T) = \sigma(r(t), t)\xi(t, T),$$

where

$$\sigma^2(r, t) = h(t) + k(t)r$$

[1] The term smooth is used here to mean that the discount function $P(r, t, T)$, or equivalently the forward rate $f(r, t, T)$, and $P^*(0, T)$ are continuous and differentiable with respect to all their arguments as many times as necessary.

is the diffusion coefficient of the spot rate dynamics, and $h(t)$, $k(t)$ are arbitrary functions of time, and

$$\xi(t,T) = \frac{2A(T)(C'(t) - A(t))}{k(t)\Big(\int_t^T A(u)du + C(t)\Big)^2},$$

with

$$A(t) = \frac{1}{2}\big(C'(t) \pm \sqrt{C'^2(t) - 2k(t)C^2(t)}\big),$$

and $C(t)$ an arbitrary function of time such that $C'(t)^2 \geq 2k(t)C^2(t)$.

Under the above assumptions, it has been shown (e.g., Jeffrey (1995)) that this model is consistent with any initial term structure and that the spot rate follows a Markov process in which both the drift and the diffusion coefficient are affine functions of the spot rate,

$$dr(t) = \big[a(t) - b(t)r(t)\big]dt + \sqrt{h(t) + k(t)r(t)}dw(t), \tag{3}$$

where

$$b(t) = \frac{\partial \xi(t,T)}{\partial t}|_{T=t},$$

and $a(t)$ is a function of time accounting for the initial term structure of interest rates[2].

Within this context, a stochastic model of the term structure can be constructed specifying the initial discount function $P^*(0,T)$, and three arbitrary functions of time, h, k, C, to account for the volatility structure. Most of the single factor models of the term structure proposed in the literature belong to this class. Gaussian models can be recovered in the limit $k(t) = 0$; the extended Cox–Ingersoll–Ross model can be obtained if we choose $h(t) = 0$, $k(t) = k$ and $C(t) = e^{dt}$, where k and d are constant (e.g., Mari (1999)). Starting from these considerations, we extend the analysis in two main directions.

First, we provide solutions to the Cauchy problem (2) when Jeffrey's condition on the volatility of the forward rate is satisfied, thus deriving a bond pricing formula (see Theorem 1) which can be very useful for applications. It will be explicitly shown that such a formula depends on the solution of an associated Volterra integral equation of the first kind.

Exact solutions of the Volterra equation are not straightforward, but well-established numerical methods are available. In the paper, and this is the second result of our analysis, we provide the solution of the Volterra equation

[2] $a(t)$ can be determined in a very involved way as a solution of a Volterra integral equation of the first kind (Jeffrey (1995)).

by using a perturbation technique within the class of term structure models characterized by *generalized* CIR volatility structure (see Section 4).

As a consequence of the bond pricing formula, the dynamics of the spot interest rate can be easily obtained (see Corollary 1). In particular it will be shown that $a(t)$ is completely described by the initial term structure and by the solution of the Volterra equation[3]. The analytical expression of the dynamics of the spot rate simplifies the valuation of interest rate contingent claims in the sense that, under the Markovian assumption, we can use a partial differential equation representation of asset prices.

The remainder of the paper is organized as follows. In Section 2, the functional form of the term structure is derived and discussed. Section 3 presents some meaningful examples of term structure models in which exact solutions of the Volterra equation can be found. In Section 4, we derive the perturbative solution of the Volterra equation in the case of generalized CIR volatility structure. An appendix, containing the detailed proofs of the propositions stated in the text, concludes the paper.

2. The term structure functional form

The purpose of this section is to provide a closed form characterization of bond prices in the class of term structure models described in the introduction. We provide the solution of the following problem:

$$\begin{aligned} \frac{dP}{P}(r(t), t, T) &= r(t)dt - \sigma_p(r(t), t, T)dw(t), \\ P(r(0), 0, T) &= P^*(0, T), \end{aligned} \tag{4}$$

where the volatility structure of zero coupon bond returns $\sigma_p(r(t), t, T)$ satisfies the following condition (*) (which can be easily obtained from Jeffrey's condition).

Condition (*).

$$\sigma_p(r(t), t, T) = \int_t^T \sigma(r(t), t, u)du = \sigma(r(t), t)B(t, T),$$

where

$$B(t, T) = 2\frac{C'(t) - A(t)}{k(t)}\left[\frac{1}{C(t)} - \frac{1}{\int_t^T A(u)du + C(t)}\right],$$

[3] The functional form of the Volterra equation looks more appealing and very different from that proposed in Jeffrey's paper. It will be shown in the paper that this fact increases the mathematical tractability of the model.

and

$$\sigma^2(r,t) = h(t) + k(t)r.$$

The solution of problem (4) can be cast in the form

$$P(r(t),t,T) = A(t,T)e^{-r(t)B(t,T)}, \tag{5}$$

where the unknown function $A(t,T)$ must be consistent with the initial term structure $P^*(0,T)$, and must satisfy the following differential equation[4]

$$\frac{d\ln A(t,T)}{dt} = a(t)B(t,T) - \frac{1}{2}h(t)B^2(t,T),$$

where

$$a(t) = -\frac{\partial^2 \ln A(t,T)}{\partial T^2}_{|T=t},$$

with the boundary condition $A(T,T) = 1$ (see Appendix).

We provide the solution of the above differential equation subject to all the quoted constraints, thus deriving a bond pricing formula which can be very useful for applications.

Theorem 1. *If the volatility structure of zero-coupon bond returns satisfies Condition* (*)*, then the solution of the Cauchy problem (4) is given by*

$$P(r(t),t,T) = \frac{P^*(0,T)}{P^*(0,t)}\exp\left[f^*(0,t)B(t,T) - \int_0^t H(u)B(u,T)du \right. \\ \left. - \frac{1}{2}\int_0^t \sigma^2(f^*(0,u),u)B^2(u,T)du\right]e^{-r(t)B(t,T)}, \tag{6}$$

where $H(t)$ is the solution of the following Volterra integral equation of the first kind

$$\int_0^t H(u)B(u,t)du = G(t), \tag{7}$$

with

$$G(t) = -\frac{1}{2}\int_0^t \sigma^2(f^*(0,u),u)B^2(u,t)du, \tag{8}$$

and $f^*(0,T)$ is the initial forward rate curve.

[4] It is well-known that the functional form (5) characterizes the discount function in the so-called affine term structure models (e.g., Duffie–Kan (1996)).

Proof. See Appendix.

This functional form of the bond pricing formula looks quite interesting. First, it is very similar to the expression found by Jamshidian (1990) in the Gaussian case. Second, it explicitly shows the consistency with any observable term structure and with the boundary condition $P(r(T), T; T) = 1$ (to verify the first statement it suffices to put $t = 0$ in Equation (6); the second can be obtained putting $t = T$ in (6) and using Equations (7) and (8)). Third, it depends on the solution of an associated Volterra integral equation of the first kind. Since the initial term structure affects the right-hand side of Equation (7), exact solutions cannot be easily found. In some cases however the Volterra equation can be solved exactly. We will prove in the next section that in Gaussian models (i.e., in the limit $k(t) = 0$, but for arbitrary $h(t)$ and $C(t)$), the Volterra equation is independent of the initial term structure, and the exact solution can be easily found. The exact solution can be also calculated in the case of the celebrated CIR model.

Furthermore, as a consequence of Theorem 1, the dynamics of the spot rate is explicitly derived.

Corollary 1. *The dynamics of the spot rate is described by*

$$dr(t) = \left[a(t) - b(t)r(t)\right]dt + \sqrt{h(t) + k(t)r(t)}\,dw(t),$$

where

$$a(t) = \frac{\partial f^*(0,t)}{\partial t} + b(t)f^*(0,t) - H(t), \tag{9}$$

$$b(t) = -\frac{\partial^2 B(t,T)}{\partial T^2}\Big|_{T=t}.$$

Proof. See Appendix.

The dynamics of the spot rate confirms the well-known fact that, in order to make the model consistent with any initial term structure, the drift coefficient cannot be time independent. This result generalizes those obtained by Hull–White (1990) in the case of the extended Vasicek model and the extended CIR model. To fit in with any observable term structure, the drift term cannot be arbitrary, but it must depend on the initial term structure as specified in (9). Furthermore, since $H(t)$ depends in general on the initial term structure, $a(t)$ is very strongly related to market data. In the class of Gaussian models, in which $H(t)$ does not depend on the initial data, nevertheless $a(t)$ depends on $f^*(0,t)$ via the first two terms in (9). The remaining terms in the spot rate dynamics depend on the volatility structure of the model.

3. Examples

3.1. Gaussian models

Gaussian models are term structure models in which the volatility structures of zero coupon bond returns are deterministic and the spot interest rate follows a Markov process. Hull and White (1990) showed that a necessary and sufficient condition for a Gaussian model is,

$$\sigma_p(t,T) = \sigma(t)B(t,T),$$

with

$$B(t,T) = \frac{y(T) - y(t)}{y'(t)}, \tag{10}$$

for an arbitrary function $y(t)$ (we set $h(t) = \sigma^2(t)$) [5]. In such models the solution of the Volterra equation (7) can be easily obtained.

As has been pointed out in the previous section, the Volterra equation does not depend on the initial term structure because $k = 0$, and therefore $\sigma^2(f^*(0,t),t) = \sigma^2(t)$. The solution can be found by differentiating twice both sides of Equation (7) with respect to time t.

After the first differentiation we get

$$\int_0^t \frac{H(u)}{y'(u)} du = -\int_0^t \frac{\sigma^2(u)}{y'^2(u)} \big[y(t) - y(u)\big] du,$$

and after the second we obtain the solution

$$H(t) = -y'^2(t) \int_0^t \frac{\sigma^2(u)}{y'^2(u)} du. \tag{11}$$

If we substitute (11) into (6), after some algebraic manipulations, we find that

$$P(r(t),t,T) = \frac{P^*(0,T)}{P^*(0,t)} \exp\left[f^*(0,t)B(t,T) - \frac{1}{2}B^2(t,T) \int_0^t \sigma^2(u) \left(\frac{\partial B(u,t)}{\partial t} \right)^2 du \right] e^{-r(t)B(t,T)}. \tag{12}$$

We note that the functional form (12) coincides with that found by Jamshidian (1991) by using a different approach.

[5] Equation (10) can be derived as a particular case of Jeffrey's constraint in the limit $k(t) = 0$. It is easy to show that in such a case $y(t) = -\frac{1}{C(t)}$.

3.2. Generalizing the CIR volatility structure

We say that the model is characterized by a *generalized* CIR volatility structure if

$$\sigma^2(r(t), t) = h + kr(t), \tag{13}$$

$$C(t) = e^{dt} \quad d^2 \geq 2k, \tag{14}$$

where h, k, and d are assumed constant. The definition is motivated by the fact that, if we substitute (14) in condition (*), we obtain, as in the CIR model,

$$B(t, T) = \frac{e^{d(T-t)} - 1}{\phi\left[e^{d(T-t)} - 1\right] + d},$$

where

$$\phi = \frac{1}{2}\left[d + \sqrt{d^2 - 2k}\right].$$

Under the generalized CIR volatility assumption, the dynamics of the spot rate is given by

$$dr(t) = \left[a(t) - (2\phi - d)r(t)\right]dt + \sqrt{h + kr(t)}dw(t),$$

where

$$a(t) = \frac{\partial f^*(0, t)}{\partial t} + (2\phi - d)f^*(0, t) - H(t).$$

The parameter of mean reversion, $2\phi - d$, is constant and coincides with the CIR mean reversion parameter; the deterministic term $a(t)$ accounts for the (arbitrary) initial term structure.

In the CIR model, $h = 0$, and the initial discount function is given by

$$f^*(0, T) = \nu\phi\left[\frac{de^{dT}}{\phi(e^{dT} - 1) + d} - 1\right] + r(0)\left[\frac{de^{dT/2}}{\phi(e^{dT} - 1) + d}\right]^2,$$

where ν is a third parameter. In this case the Volterra equation can be solved exactly and the solution reads

$$H(t) = \frac{\partial f^*(0, t)}{\partial t} + (2\phi - d)f^*(0, t) - \nu\phi(d - \phi).$$

The Gaussian limit of the model specified by Equations (13) and (14) is also interesting. In fact, if we put $k = 0$ and $h = \sigma^2$, the volatility structure of zero-coupon bond returns becomes

$$\sigma_p(t, T) = \sigma B_V(t, T),$$

where

$$B_V(t,T) = \frac{1}{d}\big[1 - e^{-d(T-t)}\big]. \tag{15}$$

It is easy to verify that $\sigma_p(t,T)$ coincides with the volatility structure of the Vasicek model (1977). In such a case the exact solution of the Volterra equation can be found and reads

$$H(t) = \frac{\sigma^2}{2d}\big(e^{-2dt} - 1\big). \tag{16}$$

If the initial term structure is chosen according to

$$f^*(0,T) = \theta(1 - e^{-dT}) + \frac{\sigma^2}{2d^2}(e^{-dT} - e^{-2dT}) + r(0)e^{-dT},$$

where θ is a third parameter, then the Vasicek model is recovered.

4. The perturbative solution of the Volterra equation

Under the generalized CIR volatility assumption, and with arbitrary initial term structures, we will prove that the Volterra equation can be solved by using perturbation methods.

Since the Vasicek volatility structure can be viewed as the limiting case ($k = 0$) of the generalized CIR volatility structure, the extended Vasicek model can be considered as the zeroth-order approximation of the general model defined by Equations (13) and (14). From this point of view, the perturbative solution of the Volterra equation can be used to build higher order corrections in the case $k \neq 0$.

We assume therefore that $H(t)$ can be expanded in a power series in the parameter k around the value $k = 0$ as follows:

$$H(t) = H_0(t) + H_1(t)k + H_2(t)k^2 + \cdots = \sum_j H_j(t)k^j, \tag{17}$$

where H_0 coincides with the solution of the Volterra equation in the Vasicek model and is given in (16).

The expansion of $B(t,T)$ can be easily derived to obtain

$$\begin{aligned} B(t,T) = B_V(t,T) &+ \frac{B_V^2(t,T)}{2d}k \\ &+ \frac{B_V^2(t,T) + dB_V^3(t,T)}{4d^3}k^2 + \cdots, \end{aligned} \tag{18}$$

where B_V is given in (15).

Substituting Equations (17) and (18) into (7), and equating the coefficients order by order in the expansion, we get

$$\int_0^t H_j(u) B_V(u,t)du = G_j(t), \qquad j = 0, 1, 2, \dots, \tag{19}$$

where

$$G_0(t) = -\frac{h}{2}\int_0^t B_V^2(u,t)du,$$

and

$$G_1(t) = -\frac{1}{2d}\int_0^t \big[H_0(u) + df^*(0,u) + hB_V(u,t)\big]B_V^2(u,t)du.$$

Equations (19) are characterized by the same analytical structure. At any order in the perturbative expansion, the right-hand side $G_j(t)$ is a known function, depending on the solutions $H_i(t)$, $i = 0, 1, 2, \dots, j-1$, of the previous equations. Although Equations (19) are Volterra integral equations of the first kind, the solutions can be calculated exactly by differentiating both sides of (19) twice with respect to t. Since $B_V(t,t) = 0$, after the first differentiation we get

$$\int_0^t e^{du} H_j(u)du = e^{dt} G_j'(t), \qquad j = 0, 1, 2, \dots,$$

and after the second we obtain the solution

$$H_j(t) = dG_j'(t) + G_j''(t), \qquad j = 0, 1, 2, \dots .$$

After some algebraic manipulations, we get

$$H_1(u) = -\frac{e^{-2dt}}{d}\int_0^t e^{2du}\big[H_0(u) + df^*(0,u) + 3hB_V(u,t)\big]du.$$

Up to the first order in k, the solution of the Volterra equation, under the generalized CIR volatility assumption, is given therefore by

$$H(t) = H_0(t) - \frac{hk}{d^3}\left[\frac{1}{4} - \left(\frac{5}{4} - \frac{dt}{2}\right)e^{-2dt} + e^{-3dt}\right] - ke^{-2dt}\int_0^t e^{2du} f^*(0,u)du.$$

The above solution is still valid in the case $h = 0$, i.e., in the case of the extended CIR model, and reads

$$H(t) = -ke^{-2dt}\int_0^t e^{2du} f^*(0,u)du.$$

5. Conclusions

To summarize, in the class of single factor HJM models with Markovian spot interest rate, we have provided an analytical treatment for the solutions of the term structure of interest rates. The dynamics of the spot rate has also been derived. Both these expressions depend on the solution of a well-defined Volterra integral equation of the first kind. In some cases (see, for example, Gaussian models or the CIR model), the exact solution is available; otherwise numerical methods have to be used. In the case of the generalized CIR volatility structure, the solution can be found at any order in the perturbative expansion, as proposed in the paper.

The possibility of obtaining the functional form of the dynamics of the spot rate (i.e., in a perturbative way in the cases in which the Volterra integral equation cannot be solved exactly), greatly simplifies the problem of valuing interest rate contingent claims. In fact, under the Markovian property, we can use a partial differential equation representation of asset prices.

Appendix. Proofs of Theorem 1 and Corollary 1

By proving Theorem 1, we also obtain a proof of Corollary 1. To prove Theorem 1 we need to recall some well-known results in the Heath–Jarrow–Morton context. In particular, the spot rate dynamics in the risk-neutral measure can be written as (cf. Heath–Jarrow–Morton (1992))

$$dr(t) = \frac{\partial f(t,T)}{\partial T}|_{T=t}dt + \sigma(r(t),t)dw(t).$$

By Itô' s lemma, the solution of the Cauchy problem (4) can be cast in the following affine form:

$$P(r(t),t,T) = A(t,T)e^{-r(t)B(t,T)},$$

where $A(t,T)$ and $B(t,T)$ satisfy the coupled equations

$$\frac{\partial \ln A(t,T)}{\partial t} = a(t)B(t,T) - \frac{1}{2}h(t)B^2(t,T), \tag{20}$$

$$\frac{\partial B(t,T)}{\partial t} = b(t)B(t,T) + \frac{1}{2}k(t)B^2(t,T) - 1, \tag{21}$$

with

$$a(t) = -\frac{\partial^2 \ln A(t,T)}{\partial T^2}_{|T=t}, \tag{22}$$

$$b(t) = -\frac{\partial^2 B(t,T)}{\partial T^2}_{|T=t}, \tag{23}$$

and subject to the boundary conditions $A(T,T)=1$ and $B(T,T)=0$. The proof is almost straightforward; it suffices to note that

$$\frac{\partial f(t,T)}{\partial T}|_{T=t} = -\frac{\partial^2 \ln A(t,T)}{\partial T^2}_{|T=t} + r(t)\frac{\partial^2 B(t,T)}{\partial T^2}_{|T=t}.$$

Since the drift of the spot rate dynamics is an affine function of the spot rate, Equations (22) and (23) follow from (3).

It is easy to verify that $B(t,T)$ as given in the text,

$$B(t,T) = 2\frac{C'(t)-A(t)}{k(t)}\left[\frac{1}{C(t)} - \frac{1}{\int_t^T A(u)du + C(t)}\right],$$

is the solution of Equation (21) satisfying all the quoted constraints.

In this appendix we solve Equation (20) by looking for a solution, $A(t,T)$, that must be consistent with the initial term structure $P^*(0,T)$, and must satisfy the boundary condition $A(T,T)=1$. To do this, we choose $A(t,T)$ in the following functional form

$$A(t,T) = \frac{P^*(0,T)}{P^*(0,t)}\exp\left[f^*(0,t)B(t,T) - \int_t^T dv\int_0^t H(u,v)du\right], \tag{24}$$

where the unknown function $H(\cdot\,,\cdot)$ will be determined below. It is easy to verify that $A(t,T)$ given in (24) satisfies the conditions $A(0,T) = P^*(0,T)\exp(r(0)B(0,T))$ and $A(T,T)=1$.

We substitute (24) into (22) and (20) to find, respectively,

$$a(t) = \frac{\partial f^*(0,t)}{\partial t} + b(t)f^*(0,t) + \int_0^t \frac{\partial H(u,t)}{\partial t}du, \tag{25}$$

and

$$\begin{aligned} a(t)B(t,T) = {} & \frac{\partial f^*(0,t)}{\partial t}B(t,T) + b(t)B(t,T)f^*(0,t) \\ & + \frac{1}{2}\sigma^2(f^*(0,t),t)B^2(t,T) + \int_0^t H(u,t)du \\ & - \int_t^T H(t,v)dv, \end{aligned} \tag{26}$$

where

$$\sigma^2(f^*(0,t),t) = h(t) + k(t)f^*(0,t).$$

Our first result is then obtained by putting $T=t$ in (26) to get

$$\int_0^t H(u,t)du = 0. \tag{27}$$

By differentiating both sides of (27) with respect to t, we obtain

$$H(t) = -\int_0^t \frac{\partial H(u,t)}{\partial t} du, \tag{28}$$

where we have put $H(t,t) = H(t)$. To prove Corollary 1 it suffices to substitute Equation (28) into (25).

To prove Theorem 1, first substitute (25) into (26) and then use Equation (27) to get

$$\int_t^T H(t,v)dv = H(t)B(t,T) + \frac{1}{2}\sigma^2(f^*(0,t),t)B^2(t,T). \tag{29}$$

The functional form of the term strucure (6) is then recovered on substituting (29) into (24), and using the relation

$$\int_t^T dv \int_0^t H(u,v)du = \int_0^t du \int_u^T H(u,v)dv,$$

which can be proved by interchanging the order of integration and using (27).

Finally, we must show that $H(t)$ satisfies the Volterra integral equation of the first kind

$$\int_0^t H(u)B(u,t)du = G(t),$$

where

$$G(t) = -\frac{1}{2}\int_0^t \sigma^2(f^*(0,u),u)B^2(u,t)du.$$

To prove this last result it suffices to integrate both sides of Equation (29) with respect to time t on the interval $[0,T]$, interchange the order of integration, and then use Equation (27).

References

Carverhill, A. (1994): When is the short rate Markovian? *Mathematical Finance* **4**, 305–312

Cox, J. C., Ingersoll, J. E., Ross S. A. (1985a): An intertemporal general equilibrium model of asset prices. *Econometrica* **53**, 363–384

Cox, J. C., Ingersoll, J. E., Ross S. A. (1985b): A theory of the term structure of interest rates. *Econometrica* **53**, 385–407

Duffie, D., Kan, R. (1996): A yield-factor model of interest rates. *Mathematical Finance* **6**, 379–406

Heath, D., Jarrow, R., Morton, A. (1990): Bond pricing and the term structure of interest rates: a discrete time approximation. *Journal of Financial and Quantitative Analysis* **25**, 419–440

Heath, D., Jarrow, R., Morton, A. (1992): Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation. *Econometrica* **60**, 77–105

Hull, J., White, A. (1990): Pricing interest-rate-derivative securities. *The Review of Financial Studies* **3**, 573–592

Hull, J., White, A. (1993): One factor interest rate models and the valuation of interest rate derivative securities. *Journal of Financial and Quantitative Analysis* **28**, 235–254

Jamshidian, F. (1991): Bond and option evaluation in the Gaussian interest rate model. In: Chen, A.H. (ed.), Research in finance. Volume 9. JAI Press, Greenwich, CT, pp. 131–170

Jamshidian, F. (1990): The preference-free determination of bond and option prices from the spot interest rate. In: Fabozzi, F.J. (ed.), Advances in futures and options research. Volume 4. JAI Press, Greenwich, CT, pp. 51–67

Jeffrey, A. (1995): Single factor Heath–Jarrow–Morton term structure models based on Markov spot interest rate dynamics. *Journal of Financial and Quantitative Analysis* **30**, 619–642

Mari, C. (1999): Making the CIR model consistent with arbitrary initial term structures: a perturbative approach. Atti del XXIII Convegno AMASES (Rende)

Vasicek, O. (1977): An equilibrium characterization of the term structure. *The Journal of Financial Economics* **5**, 177–188