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European Journal of Operational Research 163 (2005) 83–93
EUROPEAN JOURNAL OF OPERATIONAL RESEARCH
www.elsevier.com/locate/dsw

Credit risk analysis of mortgage loans: An application to the Italian market

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Available online 27 February 2004

Abstract

The valuation of financial instruments in which both credit risk and interest rate risk are taken into account is an outstanding task for financial institutions. In this paper, we propose an affine-reduced model dealing with this topic. We show that this model offers analytical tractability as well as flexibility. We also show that the parameters of the model can be estimated via maximum likelihood in a straightforward way. To outline the procedure, we estimate the model on Italian data, using zero-coupon bond and historical default probabilities, as provided by the Bank of Italy.
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Keywords: Credit risk; Affine processes; Mortgage loans

1. Introduction

The market for mortgage loans is of primary importance in any developed country, and its quality is directly connected to the quality of the whole economy. The valuation of mortgage loans can be a challenging task for financial institutions whose aim is to devise risk management strategies. It is also a complex problem from both a theoretical and an empirical point of view, since a fair valuation should take into account interest rate risk and credit risk.

In this paper we will analyze mortgage loans by using an arbitrage methodology which has been developed to value defaultable bonds. Indeed, a mortgage loan can be regarded as a portfolio of defaultable zero coupon bond issued by the debt-holders, then valuation can be accomplished via a linear combination of defaultable zero-coupon bond prices. Furthermore we believe that, since the market for mortgage loans is highly competitive, using arbitrage-free models in this context should be a reasonable approximation.

The literature on credit risk has followed two main directions (Ahn et al., 1998; Cooper and Martin, 1996). In the first, the event of default is modeled via an endogenous process which is related to the value of the issuing firm. In the seminal paper by Merton (1974), default can occur only at maturity, if the value of

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doi:10.1016/j.ejor.2003.12.007

the firm is smaller than the nominal value of debt. The valuation of risky debt is then obtained in an arbitrage free context in which the spot rate is assumed to be constant. This approach has been extended in numerous ways. For example, Brennan and Schwartz (1980) included the possibility of a stochastic spot rate process, while Longstaff and Schwartz (1995) considered the possibility that default can occur at any time between issue and maturity. Anyway, this approach does present various difficulties. First, it is hard to define the value of the firm, so that it is very difficult to estimate the stochastic process of the value especially if the firm is not listed in an official market. From this point of view, it may be very difficult to evaluate mortgage loans, whose issuer can be a firm or a family.

Starting out from these considerations, Duffie and Singleton (1999) proposed a different approach for credit risk management. In the so-called reduced models, default is regarded as exogenous and it is modeled, generally speaking, as the first jump of a Poisson process (Lando, 1997; Jarrow et al., 1997; Nielsen and Ronn, 1997; Duffee, 1999). Such models are embedded in a framework which is naturally arbitrage free, and we will show in the following that they are more flexible and appropriate to value mortgage loans.

Following therefore the reduced approach, we propose a model for evaluating mortgage loans, as well as credit risk sensitive instruments, in which both interest rate risk and credit risk are taken into account. Such a model is characterized by the fact that the event of default is modeled as the first jump of a Poisson process with intensity $\lambda(t)$, the instantaneous probability of default. The dynamics of the state variables, namely the spot rate $r(t)$ and the instantaneous probability of default $\lambda(t)$, are described by a two-factor affine model where the spot rate process is chosen according to the celebrated Cox et al. (1985a,b) for the valuation of risk-free bonds. The choice of a two-factor affine model is motivated by the fact that it offers a very interesting compromise between mathematical tractability and financial meaning, thus offering a good starting point for the empirical analysis (Duffie and Kan, 1996). Moreover, following the approach proposed by Singleton (2001), the estimate of affine models can be done by maximum likelihood, i.e. in the most efficient way.

In the second part of the paper, we exploit this estimation technique in order to illustrate an application of the model for valuing mortgage loans in the Italian market. We propose therefore an estimate of the model on Italian data, namely on time series of three months BOT prices and default probabilities of bank loans for different economic sectors. It is important to note that our estimation technique differs substantially from the typical approach. Typically, credit risk models are estimated on corporate bond yield spreads (see e.g. Duffee, 1999). By estimating our model directly on default probabilities, we illustrate a procedure which could be useful for estimating models for bank loans to different economic sectors, or geographical areas, or internal rating classes. Banks do have historical data on default probabilities in different classes, and could use these data to estimate their credit risk model. This is again possible thanks to the affine structure of the proposed model.

The remainder of the paper is organized as follows. In Section 2, the model is presented and discussed. Section 3 provides a brief description of the estimation technique used in Section 4 to determine the parameters of the model. Finally, some comments conclude the paper.

2. The model

We start with the analysis of defaultable zero-coupon bonds since, as it will be shown in subsequent analysis, the value of a loan can be obtained as a portfolio of these instruments.

Let us denote by τ the stopping time at which default occurs, and by $v(t,T)$ the value at time t $(t<\tau)$ of a defaultable zero coupon bond paying one unit of money in T. If we assume that in the market there are no arbitrage opportunities, then there exists a risk neutral probability $\mathscr{Q}$ equivalent to the probability of nature $\mathscr{P}$ under which the bond price is a martingale, and the value of the bond is given by

$$v(t,T) = \boldsymbol{E}_t^{\mathcal{Q}}\left[\mathrm{e}^{-\int_t^T r(u)\,\mathrm{d}u} 1_{\{\tau>T\}} + \mathrm{e}^{-\int_t^\tau r(u)\,\mathrm{d}u}\theta_\tau 1_{\{\tau\leqslant T\}}\right], \tag{1}$$

where $\boldsymbol{E}_t$ denotes expectation conditional on information at time t, 1_A denotes the indicator function of the set A and θ_τ is the so-called recovery rate, i.e. the amount (typically smaller than unity) which is recovered in the event of default. Clearly, one needs to make assumptions about the distribution of τ and θ_τ to specify the model. We will therefore adopt the following assumptions.

Assumption 1. The default time τ is the time of the first jump of a Poisson process with intensity $\lambda(t)$ under $\mathcal{P}$.

The intensity $\lambda(t)$ of a Poisson process is known as the instantaneous probability of default, since $\lambda(t)\,\mathrm{d}t$ is the probability to jump, then to default, between time t and $t + \mathrm{d}t$. Consequently, the probability not to default between time t and T will be given by

$$P(t,T) = \boldsymbol{E}_t^{\mathcal{P}}\left[\mathrm{e}^{-\int_t^T \lambda(s)\,\mathrm{d}s}\right]. \tag{2}$$

We choose a time-dependent probability of default, since the hypothesis of a constant intensity, which is adopted for instance in Demchak (2000), is too restrictive. Moreover, according to most empirical results, it is counter-factual.

Assumption 2. The recovery rate is proportional to the value of the bond,

$$\theta_\tau = [1 - L(\tau)]v(\tau^-, T). \tag{3}$$

The quantity $L(t)$ is the fraction of the debt which gets lost and it will be called fractional loss.

Under the above assumptions, Duffie and Singleton (1999) proved the following result:

$$v(t,T) = \boldsymbol{E}_t^{\mathcal{Q}}\left[\exp\left(-\int_t^T R(s)\,\mathrm{d}s\right)\right], \tag{4}$$

where $R(t) = r(t) + \lambda^*(t)L(t)$ and $\lambda^*(t)$ denotes the instantaneous probability of default under the risk-neutral probability. [1]

This result tells us that discounted (at rate $R(t)$) prices of defaultable zero-coupon bonds are martingales under the probability measure $\mathcal{Q}$ and that the valuation formula for a defaultable claim is the same as in the no-default case, upon augmenting the spot rate with a spread given by the product of the risk-neutral intensity and the fractional loss. [2] We make therefore an assumption regarding the instantaneous probability of default under the equivalent martingale measure $\mathcal{Q}$, assuming that it is proportional to the instantaneous probability of default under $\mathcal{P}$:

Assumption 3. The intensity process under $\mathcal{Q}$ is proportional to that under $\mathcal{P}$,

$$\lambda^*(t) = k\lambda(t). \tag{5}$$

[1] If an intensity process is defined under $\mathcal{P}$, then there exists an intensity process under any equivalent measure (see Artzner and Delbaen, 1995).

[2] Starting out from these considerations, Duffie and Singleton (1999) propose to directly model $R(t)$ instead of $r(t)$ when managing credit risk.

We can interpret k as the market price of risk for default risk (Blauer and Wilmott, 1997; Duffie and Singleton, 1999). We will also adopt the following:

Assumption 4. The process governing the fractional loss is a constant independent of time, $L(t) = L$.

In such a case we get

$$R(t) = r(t) + kL\lambda(t) \tag{6}$$

and as a consequence, we cannot identify k and L separately, but only the product kL. One can then circumvent the complicated issue of estimating the recovery rate L, concentrating on the estimate of kL.

In our model, the crucial quantities are the spot rate $r(t)$ and the instantaneous probability of default $\lambda(t)$. We will model these quantities via the following stochastic processes:

Assumption 5

$$\begin{aligned} \mathrm{d}r(t) &= \alpha[\gamma - r(t)]\,\mathrm{d}t + \sigma\sqrt{r(t)}\mathrm{d}w_r(t), \\ \mathrm{d}\lambda(t) &= a[b - \lambda(t) + cr(t)]\,\mathrm{d}t + s\sqrt{\lambda(t)}\mathrm{d}w_\lambda(t), \end{aligned} \tag{7}$$

where $\mathrm{d}w_r(t)$, $\mathrm{d}w_\lambda(t)$ are independent Brownian motions under the probability $\mathscr{P}$.

The process of the spot rate $r(t)$ is chosen according to the well known CIR model (Cox et al., 1985a,b). The process of the instantaneous probability of default λ is characterized by mean-reversion, square root diffusion and correlation with r in the long-run mean. We force c to be non-negative, since we expect that default probabilities increase with interest rate level. In the affine model classification of Dai and Singleton (2000), this is a (maximal) $A_{2,2}$ model. [3] For default-free valuation, the model coincides with the celebrated Cox–Ingersoll–Ross model.

When passing from the probability $\mathscr{P}$ to the risk neutral probability $\mathscr{Q}$, we have to include the market prices of risk. In this way, we provide a parametric form of the equivalent martingale measure. We will follow Cox et al. (1985a,b) and Consiglio and Mari (2001) by choosing

Assumption 6. The market prices of risk are of the form

$$q^r = \frac{\pi}{\sigma}\sqrt{r}, \tag{8}$$

$$q^\lambda = \frac{\eta}{s}\sqrt{\lambda}, \tag{9}$$

where π and η are assumed to be constant.

Under the hypothesis that the price of a defaultable zero-coupon bond $v(t,T) = v(t, r(t), \lambda(t); T)$ is a smooth function of the state variables, [4] Eq. (4) can be regarded as the Feynman–Kac solution of the following partial differential equation of the parabolic type:

[3] A similar model has already been proposed by Blauer and Wilmott (1997) for the valuation of Latin American Brady bonds. In Duffee (1999) a similar model is estimated on corporate bonds, using the two-factor model of Pearson and Sun (1994) for the spot rate process.

[4] By smooth function in this context we mean a continuously differentiable function of its arguments, once with respect to time t, twice with respect to r and λ.

$$v_t + [\alpha(\gamma - r) + \pi r]v_r + \frac{1}{2}\sigma^2 r v_{rr} + [a(b - \lambda + cr) + \eta\lambda]v_\lambda + \frac{1}{2}s^2\lambda v_{\lambda\lambda} = (r + kL\lambda)v, \tag{10}$$

subject to the boundary condition $v(T, r, \lambda; T) = 1$.

Since the model belongs to the affine class (Duffie and Kan, 1996), the solution of the PDE (10) can be easily determined. It can be cast in the following exponential-affine form:

$$v(t, r, \lambda; T) = \exp[A(t, T) - B(t, T)r - C(t, T)\lambda], \tag{11}$$

where A, B, C are solutions of the system of ordinary differential equations,

$$\begin{aligned} A'(t,T) &= \alpha\gamma B(t,T) + abC(t,T), \\ B'(t,T) &= (\alpha - \pi)B(t,T) + \frac{1}{2}\sigma^2 B^2(t,T) - acC(t,T) - 1, \\ C'(t,T) &= (a - \eta)C(t,T) + \frac{1}{2}s^2 C^2(t,T) - kL \end{aligned} \tag{12}$$

with boundary conditions,

$$A(T,T) = 0, \quad B(T,T) = 0, \quad C(T,T) = 0. \tag{13}$$

Solving a system of ordinary differential equation is a simple numerical task. Algorithms like Runge–Kutta allow fast and arbitrarily accurate solution, so we will not distinguish between analytical and numerical solution of first order differential equations. We notice that the third equation in (12) is a Riccati equation which can be solved exactly, obtaining,

$$C(t,T) = \frac{2kL[e^{d_1(T-t)} - 1]}{(a - \eta + d_1)[e^{d_1(T-t)} - 1] + 2d_1}, \tag{14}$$

where

$$d_1 = \sqrt{(a - \eta)^2 + 2s^2 kL}, \tag{15}$$

as can be verified by substituting (14) into (12).

Within this framework, it is also simple to compute the spreads on the yields due to credit risk. Denoting the yield to maturity by

$$y(t,T) \equiv -\frac{\log v(t,T)}{T - t}, \tag{16}$$

from (11) we have

$$y(t,T) = \frac{1}{T - t}[-A(t,T) + r(t)B(t,T) + \lambda(t)C(t,T)] \tag{17}$$

and the spreads due to credit risk are given by

$$\Delta y(t,T) \equiv y(t,T) - y_0(t,T) = \frac{\{A_0(t,T) - A(t,T) + r[B(t,T) - B_0(t,T)] + \lambda C(t,T)\}}{T - t}, \tag{18}$$

being y_0, v_0 as in the CIR model,

$$y_0(t,T) = -\frac{\ln v_0(t,T)}{T - t}, \tag{19}$$

$$v_0(t,T) = \exp[A_0(t,T) - r(t)B_0(t,T)] \tag{20}$$

with

$$A_0(t,T) = \frac{2\alpha\gamma}{\sigma^2}\log\left(\frac{2d\mathrm{e}^{(\alpha-\pi+d)\frac{T-t}{2}}}{(\alpha-\pi+d)[\mathrm{e}^{d(T-t)}-1]+2d}\right), \tag{21}$$

$$B_0(t,T) = \frac{2[\mathrm{e}^{d(T-t)}-1]}{(\alpha-\pi+d)[\mathrm{e}^{d(T-t)}-1]+2d}, \tag{22}$$

$$d = \sqrt{(\alpha-\pi)^2+2\sigma^2}. \tag{23}$$

Finally, we point out that the valuation of a loan can simply be achieved via a linear combination of defaultable zero-coupon bonds. Indeed, denoting by $\mathbf{R} = \{R_1, R_2, \ldots, R_m\}$ the cash-flow payments at times $\{t_1, t_2, \ldots, t_m\}$, then under the assumption of proportional recovery we get (Duffie and Singleton, 1999)

$$V(t,\mathbf{R}) = \sum_{k=1}^{m} R_k v(t,t_k), \tag{24}$$

where $v(t,t_k)$ is the price at time t of a defaultable zero-coupon with maturity t_k.

3. Evaluating the transition density via the characteristic function

In a recent paper, Singleton (2001) proposed to use the characteristic function to estimate the transition probability density of a stochastic process. Given an R^N-valued Markov stochastic process X_t, its characteristic function is defined by

$$\varphi_{X_t}(u;t,T) = \boldsymbol{E}_t^{\mathscr{P}}[\mathrm{e}^{\mathrm{i}u\cdot X_T}], \tag{25}$$

where $u \in R^N$. The characteristic function is the Fourier transform of the transition probability density, so that the latter can be obtained via the inversion formula

$$f(X_{t+1}|X_t) = \frac{1}{\pi^N}\int_{\mathbb{R}_+^N} \mathrm{Re}[\mathrm{e}^{-\mathrm{i}u\cdot X_{t+1}}\varphi_{X_t}(u)]\mathrm{d}u, \tag{26}$$

where R_+^N is the set of all vectors with non-negative components.

In the affine model developed so far, the characteristic function can be written in the exponential-affine form (Duffie et al., 2002),

$$\varphi(u_1,u_2,r,\lambda;t,T) = \mathrm{e}^{\chi(t,T)+\beta_1(t,T)r+\beta_2(t,T)+\lambda}, \tag{27}$$

where χ, β_1 and β_2 solve the complex-valued system of ordinary differential equations,

$$\begin{aligned}
\chi'(t,T) &= -\alpha\gamma\beta_1(t,T) - ab\beta_2(t,T),\\
\beta_1'(t,T) &= \alpha\beta_1(t,T) - ac\beta_2(t,T) - \frac{1}{2}\sigma^2\beta_1^2(t,T),\\
\beta_2'(t,T) &= \alpha\beta_2(t,T) - \frac{1}{2}s^2\beta_2^2(t,T)
\end{aligned} \tag{28}$$

with the boundary conditions,

$$\chi(T,T) = 0, \quad \beta_1(T,T) = \mathrm{i}u_1, \quad \beta_2(T,T) = \mathrm{i}u_2. \tag{29}$$

From the probability density and $T+1$ observations of X_t, one can extract the log-likelihood,

$$\log \mathscr{L} = \sum_{t=1}^{T} \log f(X_{t+1}|X_t) \tag{30}$$

or, for example, infer measure of riskiness of its financial position (Value at Risk). In this context, this issue has been assessed by Duffie and Pan (2001). Examples of this estimation technique can be found in Singleton (2001), who fits a CIR model on simulated data, in Mari and Renò (2002), on an extended version of the CIR model which allows arbitrary initial term structure, and in Das (2002), in a model in which a constant intensity process is used to model jumps in the interest rate process.

4. Estimating the model on Italian data

In this section we illustrate an application of our model, estimating it on Italian data, namely on default rates of bank loans for different economic sectors. The problem of modeling bank loans is an open topic, since one has to deal with very different regulatory frameworks across countries. We will refer to a stylized context, in which a bank grants debts and observes defaults. We will show that in the framework described in this paper, it is relatively easy to estimate the parameters of the proposed model.

Suppose that the observation set consists of N observation $(v_{0i},\ P_i)$ at equally spaced times t_i, $i = 1, \ldots, N$, where v_{0i} is the price of a default-free bond with a given time to maturity (e.g. Treasury Bills) and P_i is the observed probability of receiving back the debt in a given time window ($1 - P_i$ is the default probability). Both these variables are easily observed in the operative practice. In our framework, the value of a default-free bond with time to maturity $T - t$ is given by (20), while the probability $P(t, T)$ of meeting the obligation between time t and T is given by (2). Then, $P(t, T)$ is the Feynman–Kac solution of the PDE,

$$P_t + \alpha(\gamma - r)P_r + \frac{1}{2}\sigma^2 r P_{rr} + a(b - \lambda + cr)P_\lambda + \frac{1}{2}s^2\lambda P_{\lambda\lambda} = \lambda P, \tag{31}$$

subject to the boundary condition $P(T, T) = 1$.

$P(t, T)$ can be written therefore in the exponential-affine form,

$$P(t, T) = \mathrm{e}^{A_1(t,T)+B_1(t,T)r+B_2(t,T)\lambda}, \tag{32}$$

where A_1, B_1, B_2 satisfy the following system of ordinary differential equations,

$$\begin{aligned} A_1'(t, T) &= -\alpha\gamma B_1(t, T) - abB_2(t, T), \\ B_1'(t, T) &= \alpha B_1(t, T) - acB_2(t, T) - \frac{1}{2}\sigma^2 B_1^2(t, T), \\ B_2'(t, T) &= aB_2(t, T) - \frac{1}{2}s^2 B_2^2(t, T) + 1 \end{aligned} \tag{33}$$

with boundary conditions,

$$A_1(T, T) = 0,\ B_1(T, T) = 0,\ B_2(T, T) = 0. \tag{34}$$

Since we do not observe directly r, λ we cannot use formula (26) to find the transition probability density. On the other hand, our model is formulated in terms of r, λ, so that the solution for the characteristic function (27) is easily expressed in the variables r, λ. We can circumvent this difficulty by a simple change of variable. Denoting by

$$q(t_i) = \log v_{0i}, \tag{35}$$

$$p(t_i) = \log P_i, \tag{36}$$

Table 1
Summary statistics of the default probabilities over a time horizon of one month, for the different economic sectors considered

Economic sector	Mean default probability	Variance
Insurance companies	0.0054900	0.0000014
Financial companies	0.028519	0.000058
Individual enterprises	0.1927	0.0016
Consumer families	0.13086	0.00010
Public administration	0.005442	0.000015
Not financial companies	0.116547	0.000065

from Eqs. (20) and (32) it is clear that q, p are affine functions of r, λ so that it is simple to write down the characteristic function of q, p given that of r, λ,

$$\varphi(u_1, u_2, q(t_i), p(t_i)) = e^{i[u_1 A_0(t,T) + u_2 A_1(t,T)]}\varphi(-B_0 u_1 + B_1 u_2, B_2 u_2, r(t_i), \lambda(t_i)). \tag{37}$$

From (37) we can compute the transition probability density,

$$f[q(t_{i+1}), p(t_{i+1})|q(t_i), p(t_i)] = \frac{1}{\pi^2}\int_0^{+\infty} du_1 \int_0^{+\infty} du_2 \mathrm{Re}\{e^{-i[u_1 q(t_{i+1}) + u_2 p(t_{i+1})]}\varphi(u_1, u_2, q(t_i), p(t_i))\}, \tag{38}$$

then maximum likelihood can run via usual algorithms. A practical difficulty of this procedure is given by the computation of the two-dimensional integral in (38), which is computational intensive even when adopting efficient quadrature techniques (Gauss–Legendre).

We estimated our model on Italian data. We propose an alternative set of data. Typically, credit risk models are estimated on corporate bond prices. We used zero-coupon bond prices and default probabilities instead. The zero-coupon bond used are the three-months BOT, which are issued by the Italian Treasury every two weeks via auction. The default probabilities are inferred from the data collected monthly by the Bank of Italy. These probabilities are computed on a one-month period and are separated into their different economic sectors. For both the time series we use monthly data from October 1995 to December 1999 for a total of 51 observations. Table 1 offers a summary statistics of default probabilities while Fig. 1 displays their time evolution. As it is well known, loans to individual enterprises are more risky: nearly 20% of the loans in this sector suffers default. Other quite risky sectors are consumer families and non-financial companies, while financial and insurance companies are more reliable. The less risky sector is public administration, which defaults only in the 0.5% of the cases.

Estimation is achieved in two steps: at first we estimate the CIR model via maximum likelihood using only BOT data, thus inferring the parameters α, γ, σ, π. In the simple case of the CIR diffusion, the characteristic function can be written in a closed form (see Singleton, 2001). Results are provided in Table 2 and they are in good agreement with the extant literature (see e.g. Gentile and Renò, 2002). We remark that we are able to estimate the market price of risk, since the risk-free bond prices carry information on both the risk-neutral and the natural probability.

In the second step, we estimate the diffusion parameters of the instantaneous probability of default on the joint set of bond prices and default probabilities. [5] In this case we cannot estimate the risk-neutral parameters η and kL, since the default probability $1 - P(t, T)$ is computed in the nature probability. For the estimate of η, kL one should resort to risk-neutral evaluated financial instruments, such as loan internal rates, defaultable bonds or credit derivatives.

Results for the different economic sectors are given in Table 3. As expected, the long-run mean b is higher for individual enterprises and lower for insurance companies and public administration, reflecting

[5] For the individual enterprises we used the first 32 observations only, because of the abrupt change which occurs thereafter, see Fig. 1.

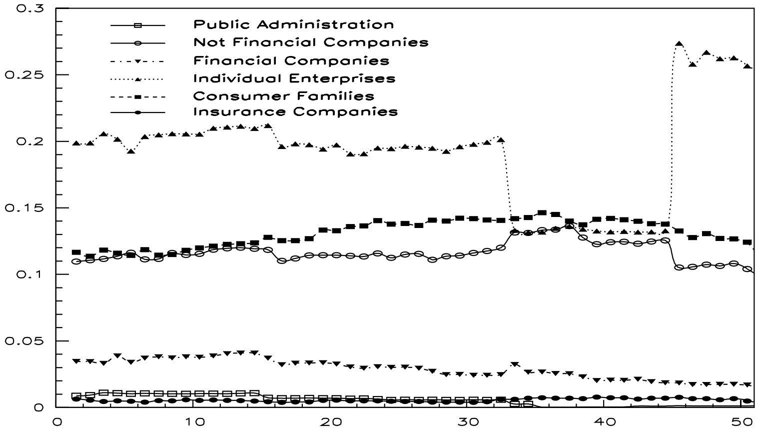

Fig. 1. Time evolution of the default probabilities in the sample, computed on a one-month period, for different economic sectors.

Table 2
Parameter estimates of the CIR model, obtained with the method described in the text, on 181 observations of the three-months BOT (two observations per month), from 1994 to 2001

Parameter	Estimate	Standard error
$\mathscr{L} = 396.53$		
α	0.106	(0.023)
γ	0.021	(0.006)
σ	0.0607	(0.0059)
π	0.01	(0.18)

Standard errors are estimated by $\text{diag}(\sqrt{-H^{-1}})$, where H is the Hessian as computed by numerical derivatives.

Table 3
Parameter estimates of the model, obtained via maximum likelihood on 51 observations of the monthly default probabilities for different economic sectors, from October 1995 to December 1999

Parameter	Insurance companies	Consumer families	Financial companies	Individual enterprises	Not financial companies	Public administration
$\mathscr{L}/T$	11.1202	7.2507	7.7454	7.9525	6.7162	9.7287
a	0.83	0.088	0.05	0.269	0.246	0.079
	(0.23)	(0.002)	(0.33)	(0.012)	(0.049)	(0.006)
b	0.0026	0.357	0.307	1.14	0.53	0.038
	(0.0005)	(0.072)	(0.010)	(0.10)	(0.11)	(0.012)
s	0.119	0.181	0.230	0.176	0.141	0.083
	(0.042)	(0.049)	(0.052)	(0.034)	(0.028)	(0.005)
c	0.100	0.115	0.475	1.49	0.089	0.88
	(0.019)	(0.023)	(0.077)	(0.30)	(0.018)	(0.12)

Standard errors are estimated by $\text{diag}(\sqrt{-H^{-1}})$, where H is the Hessian, and are reported in parenthesis.

the riskiness of the different sectors. The coefficient s is in the range 0.1–0.2 for all sectors. Then, since the volatility λ is proportional to the square root of λ itself, riskier sectors are also more volatile. Estimates of c are all significant, pointing out the relation between the interest rate level and the probability of default: low interest rates are a signal of healthy economies, in which default is less likely. The estimate of a, which is the mean-reversion parameter, is more difficult given the smallness of the data-sample.

Table 4
Defaultable zero-coupon bond prices with principal equal to one and time to maturity one year, computed with the parameter values in Table 3, for different values of kL

Economic sector	1-year bond price	Yield (%)	Spread (%)
Default-free	0.9789	2.12	0
Insurance companies	0.9785	2.17	0.05
Consumer families	0.9443	5.72	3.60
Financial companies	0.9484	5.30	3.18
Public administration	0.9735	2.69	0.57
Not financial companies	0.9282	7.45	5.33
Individual enterprises	0.8711	13.80	11.68

We also report the yield to maturity (16) and the spread over the risk-free rate (18).

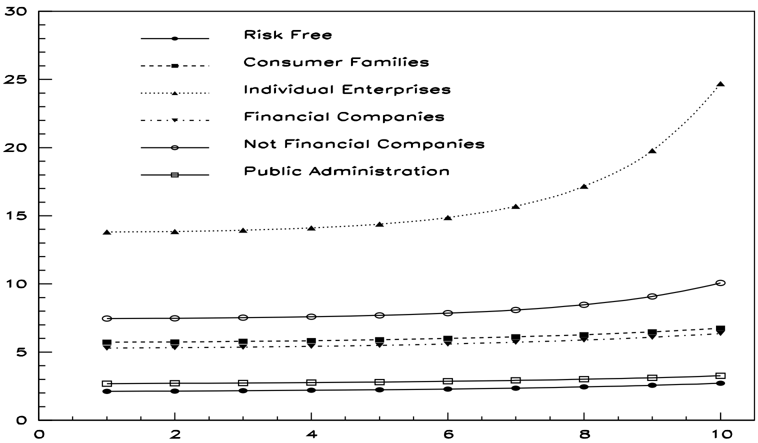

Fig. 2. Term structure of defaultable bond yields for different economic sectors.

To clarify the economic relevance of our estimates, Table 4 reports the prices of a defaultable zero-coupon bond with time to maturity equal to one year, computed via (11) with the parameter estimates in Table 3, after setting $kL = 0.1$, $\eta = 0$, $r(0) = 0.025$, $\lambda(0) = b + c\gamma$. For less risky sectors, like public administration or insurance companies, the spread over the risk-free rate ranges from 5 to about 60 basis points. This spread can be as high as nearly 14% for risky sectors as individual enterprises. This result is not surprising, since economic sectors which experienced higher default probabilities are expected to be charged with a higher spread. Similar results on yield spreads are found in the different rating classes for corporate bonds (see e.g. Crouhy et al., 2000).

In a similar fashion, we computed the term structure of defaultable bond yields, which is plotted in Fig. 2. We can see that the effect of modeling the instantaneous probability of default as a stochastic process is not only to increase the level of the yield curve, but also to modify its steepness.

5. Concluding remarks

In this paper we proposed an affine-reduced model for managing credit risk. This model can be used for evaluating credit risk sensitive bonds, as well as credit derivatives. In particular, it is well suited to the valuation of mortgage loans, a topic which is quite neglected in the financial literature. We provided estimation methods via maximum likelihood which makes use of easily observable variables, such as risk-free bond prices and default probabilities. We illustrated these results by estimating the model on Italian data.

Our results can, in principle, be extended in numerous ways. First of all the risk-free model specification can be extended in the affine class, following e.g. Dai and Singleton (2000) or Jeffrey (1995) to account for the observed initial term structure. Second, many assumptions could be relaxed. For example, when extending the model for the term structure, we can also extend the specification of market prices of risk (Dai and Singleton, 2002), moreover correlations in the Brownian motions could be introduced. Finally the EMM method of Gallant and Tauchen (1996), which is less computational intensive, but which can be as efficient as maximum likelihood, can be employed for the estimation and to provide diagnostics of the model specification.

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