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Local risk-minimization under restricted information on asset prices

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Abstract

In this paper we investigate the local risk-minimization approach for a semimartingale financial market where there are restrictions on the available information to agents who can observe at least the asset prices. We characterize the optimal strategy in terms of suitable decompositions of a given contingent claim, with respect to a filtration representing the information level, even in presence of jumps. Finally, we discuss an application to a Markovian framework and show that the computation of the optimal strategy leads to filtering problems under the real-world probability measure and under the minimal martingale measure.

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1 Introduction

The paper studies *locally risk-minimizing* hedging strategies (see e.g. [14], [31] and [35] for a deeper discussion on this issue) when there are restrictions on the available information to traders and extends some results of [33], proved in the local martingale case, to a semimartingale market model. Furthermore, we discuss some Markovian models where we compute explicitly the optimal strategy even by means of filtering problems. More precisely, we assume that in our model the agents have a limited knowledge on the market, so that their choices cannot be based on the full information flow described by the filtration $\mathbb{F} := \{\mathcal{F}_t,\ t \in [0,T]\}$, with T denoting a fixed finite time horizon. The available information level is basically given by a smaller filtration $\mathbb{H} := \{\mathcal{H}_t,\ t \in [0,T]\}$. However, since, in general, stock prices are publicly available, we assume that the agents can reasonably observe at least the asset prices.

In this market we consider a European-type contingent claim whose final payoff is given by an $\mathcal{H}_T$-measurable square integrable random variable ξ on a given probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The goal is to study the hedging problem of the payoff ξ via the local risk-minimization approach in the underlying incomplete market, which is driven by

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an $(\mathbb{F}, \mathbf{P})$-semimartingale S representing the stock price process and where there are restrictions on the available information to traders.

The quadratic hedging method of local risk-minimization extends the theory of risk-minimization introduced in [15] and formulated when the price process is a local martingale under the real-word probability measure $\mathbf{P}$, to the semimartingale case. The local martingale case was largely developed both under complete and partial information. One of the pioneer papers in the restricted information setting is represented by [33], where the optimal strategy is constructed via predictable dual projections. More recently, in [8], the authors characterized the risk-minimizing hedging strategy via an orthogonal decomposition of the contingent claim, called the *Galtchouk-Kunita-Watanabe decomposition under restricted information*. Furthermore, a contribution about risk-minimization under partial information in the insurance framework is given by [6], when the underlying price process is expressed in units of the so called numéraire portfolio.

The local risk-minimization method under partial information has been investigated for the first time in [7], where the authors, thanks to existence and uniqueness results for backward stochastic differential equations under partial information, characterized the optimal hedging strategy for an $\mathcal{F}_T$-measurable contingent claim ξ, via a suitable version of the Föllmer-Schweizer decomposition working in the case of restricted information, by means of the new concept of weak orthogonality introduced in [8]. More precisely, they proved that the $\mathbb{H}$-predictable integrand with respect to the stock price process in the Föllmer-Schweizer decomposition gives the $\mathbb{H}$-locally risk minimizing strategy; nevertheless, they did not furnish any operational method to represent explicitly the optimal strategy. Our contribution, in this context, is to provide a full description of the optimal strategy for an $\mathcal{H}_T$-measurable contingent claim, under the additional hypothesis that the information available to investors is, at least, given by the stock prices. This scenario is characterized by the following condition on filtrations:

$$\mathcal{F}_t^S \subseteq \mathcal{H}_t \subseteq \mathcal{F}_t, \quad t \in [0,T],$$

where $\mathcal{F}_t^S$ is the σ-field generated by the stock price process S up to time t.
In this paper, the key point is that the risky asset price process S satisfying the *structure condition* with respect to $\mathbb{F}$, see (2.1), turns out to be an $(\mathbb{H}, \mathbf{P})$-semimartingale in virtue of the condition above. Indeed, since the payoff of a given contingent claim is always supposed to be an $\mathcal{H}_T$-measurable random variable, this allows one to reduce the hedging problem under partial information to an equivalent problem in the case of full information. We will see that S also satisfies the structure condition with respect to $\mathbb{H}$, see Proposition 3.2, and then the optimal strategy can be characterized by extending the results of [12] to the partial information framework, see Proposition 4.8. The Galtchouk-Kunita-Watanabe decomposition under restricted information, with respect to the *minimal martingale measure* $\mathbf{P}^*$, represents an essential tool to get the achievement.

We also pay attention to the relation between the optimal strategy under complete information and that under restricted information. In Proposition 4.6 the result is stated under the assumption that the stock price process has continuous trajectories, and then generalized in Proposition 4.8.

Finally, we consider a Markovian jump-diffusion driven market model affected by an unobservable stochastic factor given by a correlated jump-diffusion process having common jump times with S. Here, we characterize the structure conditions of the underlying price process S with respect to both $\mathbb{F}$ and $\mathbb{H}$ and compute the optimal strategy when the information flow coincides with the natural filtration of the stock price process. Moreover, we discuss a simplified model where we compute the optimal strategy for a European put option. As remarks we also deduce the optimal strategy for diffusion and pure jump driven market models. In all of these cases, the computation of the optimal value process leads to a filtering problem with respect to the minimal martingale measure $\mathbf{P}^*$ and the historical probability measure $\mathbf{P}$. We derive the filtering equations for the above mentioned models in Appendix A, by extending the results proved in [4]. Other results concerning filtering problems in a mixed diffusion and jump observation framework can be found in [16, 17, 18, 3, 20, 5].

The paper is organized as follows. In Section 2 we describe the financial market model and formulate the hedging problem under partial information according to the local risk-minimization approach. Section 3 is devoted to prove that the underlying price process satisfies the structure condition under the subfiltration $\mathbb{H}$. The characterization of the optimal strategy can be found in Section 4. An application to a Markovian setting is discussed in Section 5. Finally, the computation of the filter dynamics and some proofs are gathered in Appendix.

2 Hedging problem formulation under partial information

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space endowed with a filtration $\mathbb{F} := \{\mathcal{F}_t,\ t \in [0,T]\}$ that satisfies the usual conditions of right-continuity and completeness, where $T > 0$ is a fixed and finite time horizon; furthermore, we assume that $\mathcal{F} = \mathcal{F}_T$. We consider a simple financial market model where we can find one riskless asset with (discounted) price 1 and a risky asset whose (discounted) price S is represented by an $\mathbb{R}$-valued square integrable càdlàg $(\mathbb{F}, \mathbf{P})$-semimartingale satisfying the following structure condition (see e.g. [35] for further details):

$$S_t = S_0 + M_t + \int_0^t \alpha_u^{\mathcal{F}} \mathrm{d}\langle M \rangle_u, \quad t \in [0,T], \tag{2.1}$$

where $S_0 \in L^2(\mathcal{F}_0, \mathbf{P})^1$, $M = \{M_t,\ t \in [0,T]\}$ is an $\mathbb{R}$-valued square integrable (càdlàg) $(\mathbb{F}, \mathbf{P})$-martingale starting at null, $\langle M \rangle = \{\langle M, M \rangle_t,\ t \in [0,T]\}$ denotes its $\mathbb{F}$-predictable quadratic variation process and $\alpha^{\mathcal{F}} = \{\alpha_t^{\mathcal{F}},\ t \in [0,T]\}$ is an $\mathbb{R}$-valued $\mathbb{F}$-predictable process such that $\int_0^T \left(\alpha_s^{\mathcal{F}}\right)^2 \mathrm{d}\langle M \rangle_s < \infty$ $\mathbf{P}$-a.s..

Remark 2.1. It is quite natural to assume that S is a semimartingale under the real-world probability measure $\mathbf{P}$. Indeed, this is implied by the existence of an equivalent martingale measure, and equivalently by the absence of arbitrage opportunities. Moreover, according to the results proved in [1, page 24] and [34, Theorem 1], if in addition, S has continuous trajectories or càdlàg paths and the following condition holds:

$$\mathbb{E}\left[\sup_{t \in [0,T]} S_t^2\right] < \infty,$$

then, S satisfies the structure condition with respect to $\mathbb{F}$ given in (2.1).

Without further mention, all subsequently appearing quantities will be expressed in discounted units. At any time $t \in [0,T]$, market participants can trade in order to reallocate their wealth. We assume that they have a limitative knowledge on the market, then their choices cannot be based on the full information flow $\mathbb{F}$. To describe this scenario, we consider the filtration $\mathbb{F}^S := \{\mathcal{F}_t^S,\ t \in [0,T]\}$ generated by the risky asset price process S, i.e. $\mathcal{F}_t^S = \sigma\{S_u,\ 0 \le u \le t \le T\}$, and the filtration $\mathbb{H} := \{\mathcal{H}_t,\ t \in [0,T]\}$, representing the available information to traders; both filtrations are supposed to satisfy the usual hypotheses of completeness and right-continuity, and since the information on asset prices is announced to the public, it is reasonable to assume that the stock price

[1] The space $L^2(\mathcal{F}_t, \mathbf{P})$, $t \in [0,T]$, denotes the set of all $\mathcal{F}_t$-measurable random variables H such that $\mathbb{E}\left[|H|^2\right] = \int_\Omega |H|^2 \mathrm{d}\mathbf{P} < \infty$.

process S is adapted to both filtrations $\mathbb{F}$ and $\mathbb{H}$, that is

$$\mathcal{F}_t^S \subseteq \mathcal{H}_t \subseteq \mathcal{F}_t, \quad t \in [0,T]. \tag{2.2}$$

Condition (2.2) implies that agents can observe at least the market prices of negotiated assets.

In this market we consider a European-type contingent claim whose final payoff is given by an $\mathcal{H}_T$-measurable random variable ξ such that $\mathbb{E}\left[|\xi|^2\right] < \infty$ (or equivalently, $\xi \in L^2(\mathcal{H}_T, \mathbf{P})$).

Then, the goal is to study the hedging problem of the given contingent claim ξ in the incomplete market driven by S where there are restrictions on the available information to traders, via the local risk-minimization approach (see e.g. [14], [31] and [35]).

It is important to stress that the risky asset price process S turns out to be an $(\mathbb{H}, \mathbf{P})$-semimartingale in virtue of condition (2.2) on filtrations. Then it admits a semimartingale decomposition with respect to $\mathbb{H}$, i.e.

$$S_t = S_0 + N_t + R_t, \quad t \in [0,T], \tag{2.3}$$

where $N = \{N_t,\ t \in [0,T]\}$ is an $\mathbb{R}$-valued square integrable $(\mathbb{H}, \mathbf{P})$-martingale with $N_0 = 0$ and $R = \{R_t,\ t \in [0,T]\}$ is an $\mathbb{R}$-valued $\mathbb{H}$-predictable process of finite variation with $R_0 = 0$. Moreover, since R is $\mathbb{H}$-predictable this decomposition is unique (see e.g. [29, Chapter III, Theorem 34]) and will be called the *canonical* $\mathbb{H}$*-decomposition* of S.

On the other hand, the payoff of a given contingent claim is always supposed to be an $\mathcal{H}_T$-measurable random variable. We observe that all the processes involved are then $\mathbb{H}$-adapted, and this allows to reduce the hedging problem under partial information to an equivalent one in the case of full information.

We now briefly recall the main concepts and results about the local risk-minimization approach (with respect to $\mathbb{H}$).

Since we work with both the decompositions of S, in the sequel we refer to M as the $\mathbb{F}$-martingale part of S, and N as the $\mathbb{H}$-martingale part of S.

Firstly, we introduce the definition of (hedging) strategy and assume some minimal requirements to make it admissible.

Definition 2.2. *The space* $\Theta(\mathbb{H})$ *(respectively* $\Theta(\mathbb{F})$*) consists of all* $\mathbb{R}$*-valued* $\mathbb{H}$*-predictable (respectively* $\mathbb{F}$*-predictable) processes* $\theta = \{\theta_t,\ t \in [0,T]\}$ *satisfying the following integrability condition:*

$$\mathbb{E}\left[\int_0^T \theta_u^2 \mathrm{d}\langle N\rangle_u + \left(\int_0^T |\theta_u| \mathrm{d}|R_u|\right)^2\right] < \infty$$

$$\left(\textit{respectively } \ \mathbb{E}\left[\int_0^T \theta_u^2 \mathrm{d}\langle M\rangle_u + \left(\int_0^T |\theta_u||\alpha_u^{\mathcal{F}}| \mathrm{d}\langle M\rangle_u\right)^2\right] < \infty\right).$$

Definition 2.3. *An* $\mathbb{H}$-admissible strategy *is a pair* $\psi = (\theta, \eta)$*, where* $\theta \in \Theta(\mathbb{H})$ *and* $\eta = \{\eta_t,\ t \in [0,T]\}$ *is an* $\mathbb{R}$*-valued* $\mathbb{H}$*-adapted process such that the value process* $V(\psi) = \{V_t(\psi),\ t \in [0,T]\} := \theta S + \eta$ *is right-continuous and square integrable, i.e.* $V_t(\psi) \in L^2(\mathcal{H}_t, \mathbf{P})$*, for each* $t \in [0,T]$*.*

Note that θ and η describe the amount of wealth invested in the risky asset and in the riskless asset, respectively.

For any $\mathbb{H}$-admissible strategy ψ, we can define the associated *cost process* $C(\psi) = \{C_t(\psi),\ t \in [0,T]\}$, which is the $\mathbb{R}$-valued $\mathbb{H}$-adapted process given by

$$C_t(\psi) = V_t(\psi) - \int_0^t \theta_u \mathrm{d}S_u,$$

for every $t \in [0,T]$.

In our framework the market is incomplete, then perfect replication of a given contingent claim by a self-financing $\mathbb{H}$-admissible strategy is not guaranteed. However, even if $\mathbb{H}$-admissible strategies ψ with $V_T(\psi) = \xi$ will in general not be self-financing, it turns out that good $\mathbb{H}$-admissible strategies are still self-financing on average in the following sense.

Definition 2.4. *An $\mathbb{H}$-admissible strategy ψ is called* mean-self-financing *if the associated cost process $C(\psi)$ is an $(\mathbb{H},\mathbf{P})$-martingale.*

Similarly to [35], we introduce the concept of *pseudo optimal strategy*.

Definition 2.5. *Let $\xi \in L^2(\mathcal{H}_T,\mathbf{P})$ be a contingent claim. An $\mathbb{H}$-admissible strategy ψ such that $V_T(\psi) = \xi$ $\mathbf{P}-$a.s. is called* $\mathbb{H}$-pseudo optimal *for ξ if and only if ψ is mean-self-financing and the $(\mathbb{H},\mathbf{P})$-martingale $C(\psi)$ is strongly orthogonal to the $\mathbb{H}$-martingale part, N, of S, see* (2.3)*.*

We have skipped the original definition of *locally risk-minimizing strategy*, given in [31], since it is rather technical and delicate. Moreover, since in our setting S satisfies the structure condition (3.2) with respect to $\mathbb{H}$, see Proposition 3.2 below, if $R := \left\{\int_0^t \alpha_u^{\mathcal{H}} \mathrm{d}\langle N\rangle_u,\ t \in [0,T]\right\}$ is continuous, $\langle N\rangle$ is $\mathbf{P}$-almost surely strictly increasing and $\mathbb{E}\left[\int_0^T \left(\alpha_t^{\mathcal{H}}\right)^2 \mathrm{d}\langle N\rangle_t\right] < \infty$, then $\mathbb{H}$-locally risk minimizing and $\mathbb{H}$-pseudo optimal strategies coincide, see [35, Theorem 3.3]. The advantage of working with pseudo optimal strategies is that they can be characterized through an appropriate decomposition of the contingent claim ξ, as we will see in Proposition 2.8.

Definition 2.6. *Let $\xi \in L^2(\mathcal{H}_T,\mathbf{P})$ be the payoff of a European-type contingent claim. We say that ξ admits a Föllmer-Schweizer decomposition with respect to S and $\mathbb{H}$, if there exist a random variable $U_0 \in L^2(\mathcal{H}_0,\mathbf{P})$, a process $\beta^{\mathcal{H}} \in \Theta(\mathbb{H})$ and a square integrable $(\mathbb{H},\mathbf{P})$-martingale $A = \{A_t,\ t \in [0,T]\}$ with $A_0 = 0$ strongly orthogonal to the $\mathbb{H}$-martingale part of S, N, such that*

$$\xi = U_0 + \int_0^T \beta_t^{\mathcal{H}} \mathrm{d}S_t + A_T \quad \mathbf{P}-\text{a.s.}. \tag{2.4}$$

Remark 2.7. Some classes of sufficient conditions for the existence of the Föllmer-Schweizer decomposition are given for example in [32, 34, 27, 12, 7].

The following result enables us to characterize the $\mathbb{H}$-pseudo optimal strategy via the Föllmer-Schweizer decomposition.

Proposition 2.8. *A contingent claim $\xi \in L^2(\mathcal{H}_T,\mathbf{P})$ admits an $\mathbb{H}$-pseudo optimal strategy $\psi^* = (\theta^*,\eta^*)$ with $V_T(\psi^*) = \xi$ $\mathbf{P}-$a.s. if and only if decomposition* (2.4) *holds. The strategy ψ^* is explicitly given by*

$$\theta_t^* = \beta_t^{\mathcal{H}} \quad \mathbf{P}-a.s., \quad t \in [0,T],$$

with minimal cost

$$C_t(\psi^*) = U_0 + A_t \quad \mathbf{P}-a.s., \quad t \in [0,T];$$

its value process is

$$V_t(\psi^*) = \mathbb{E}\left[\xi - \int_t^T \beta_u^{\mathcal{H}} \mathrm{d}S_u \middle| \mathcal{H}_t\right] = U_0 + \int_0^t \beta_u^{\mathcal{H}} \mathrm{d}S_u + A_t \quad \mathbf{P}-a.s., \quad t \in [0,T],$$

so that $\eta_t^* = V_t(\psi^*) - \beta_t^{\mathcal{H}} S_t$ $\mathbf{P}-a.s.$, for every $t \in [0,T]$.

Proof. For the proof see [35, Proposition 3.4]. □

The problem is then how to compute such a decomposition. We address this issue to Section 4.

3 Structure condition of the stock price S with respect to $\mathbb{H}$

In the sequel we will use the notation ${}^oX$ (respectively, ${}^pX$) to indicate the optional (respectively, predictable) projection with respect to $\mathbb{H}$ under $\mathbf{P}$ of a given process $X = \{X_t,\ t \in [0,T]\}$ satisfying $\mathbb{E}\left[|X_t|\right] < \infty$ for every $t \in [0,T]$, defined as the unique $\mathbb{H}$-optional (respectively, $\mathbb{H}$-predictable) process such that ${}^oX_\tau = \mathbb{E}\left[X_\tau|\mathcal{H}_\tau\right]$ $\mathbf{P}$-a.s. on $\{\tau < \infty\}$ for every $\mathbb{H}$-stopping time τ (respectively, ${}^pX_\tau = \mathbb{E}\left[X_\tau|\mathcal{H}_{\tau^-}\right]$ $\mathbf{P}$-a.s. on $\{\tau < \infty\}$ for every $\mathbb{H}$-predictable stopping time τ).

We also denote by $B^{p,\mathbb{H}}$ the $(\mathbb{H},\mathbf{P})$-predictable dual projection of an $\mathbb{R}$-valued càdlàg $\mathbb{F}$-adapted process $B = \{B_t,\ t \in [0,T]\}$ of integrable variation, defined as the unique $\mathbb{R}$-valued $\mathbb{H}$-predictable process $B^{p,\mathbb{H}} = \{B^{p,\mathbb{H}}_t,\ t \in [0,T]\}$ of integrable variation, such that

$$\mathbb{E}\left[\int_0^T \varphi_t \mathrm{d}B^{p,\mathbb{H}}_t\right] = \mathbb{E}\left[\int_0^T \varphi_t \mathrm{d}B_t\right],$$

for every $\mathbb{R}$-valued $\mathbb{H}$-predictable (bounded) process $\varphi = \{\varphi_t,\ t \in [0,T]\}$. See e.g. Section 4.1 of [8] for further details.

When the risky asset price process S has continuous trajectories, the classical decomposition of S with respect to the filtration $\mathbb{H}$ has the form (see, e.g. [24] or [26]):

$$S_t = S_0 + N_t + \int_0^t {}^p\alpha^{\mathcal{F}}_u \mathrm{d}\langle N\rangle_u, \quad t \in [0,T],$$

where the process $N = \{N_t,\ t \in [0,T]\}$ given by

$$N_t = M_t + \int_0^t [\alpha^{\mathcal{F}}_u - {}^p\alpha^{\mathcal{F}}_u] \mathrm{d}\langle M\rangle_u, \quad t \in [0,T],$$

is an $(\mathbb{H},\mathbf{P})$-martingale. Recall that M denotes the martingale part of S under $\mathbb{F}$, see (2.1). Since the quadratic variation process $[S]$ of S is defined by

$$[S]_t = S_t^2 - 2\int_0^t S_{u^-} \mathrm{d}S_u, \quad t \in [0,T],$$

it turns out to be $\mathbb{F}^S$-adapted, while in general the predictable quadratic variation $\langle S\rangle$ of S depends on the choice of the filtration. Clearly, if S is continuous, we have that ${}^{\mathbb{H}}\langle N\rangle = {}^{\mathbb{F}}\langle M\rangle$ and these sharp brackets are $\mathbb{F}^S$-predictable. Here, the notations ${}^{\mathbb{H}}\langle\cdot\rangle$ and ${}^{\mathbb{F}}\langle\cdot\rangle$ just stress the fact that the predictable quadratic variations are computed with respect to the filtrations $\mathbb{H}$ and $\mathbb{F}$, respectively. However, if it does not create ambiguity, we will always write $\langle M\rangle = {}^{\mathbb{F}}\langle M\rangle$ and $\langle N\rangle = {}^{\mathbb{H}}\langle N\rangle$ to simplify the notation.

In presence of jumps these relations are no longer true, since ${}^{\mathbb{F}}\langle M^d\rangle \neq {}^{\mathbb{H}}\langle N^d\rangle$, where M^d and N^d denote the discontinuous parts of the martingales M and N, respectively. To compute explicitly the predictable quadratic variations, we introduce the integer-valued random measure associated to the jumps of S:

$$m(\mathrm{d}t,\mathrm{d}z) = \sum_{s:\Delta S_s \neq 0} \delta_{(s,\Delta S_s)}(\mathrm{d}t,\mathrm{d}z),$$

where δ_a denotes the Dirac measure at point a. In the sequel we make the following assumption.

Assumption 3.1. *The process S has only $(\mathbb{F},\mathbf{P})$-totally inaccessible jump times.*

Denote by $\nu^{\mathbb{F}}(\mathrm{d}t, \mathrm{d}z)$ and $\nu^{\mathbb{H}}(\mathrm{d}t, \mathrm{d}z)$ the predictable dual projections of $m(\mathrm{d}t, \mathrm{d}z)$ under $\mathbf{P}$ with respect to $\mathbb{F}$ and $\mathbb{H}$ respectively (we refer the reader to [21] or [22] for the definition). Then, by [22, Chapter II, Corollary 2.38] and Assumption 3.1 we get the following representations of the martingales M and N:

$$M_t = M_t^c + \int_0^t \int_{\mathbb{R}} z(m(\mathrm{d}t, \mathrm{d}z) - \nu^{\mathbb{F}}(\mathrm{d}t, \mathrm{d}z)), \quad t \in [0, T],$$
$$N_t = N_t^c + \int_0^t \int_{\mathbb{R}} z(m(\mathrm{d}t, \mathrm{d}z) - \nu^{\mathbb{H}}(\mathrm{d}t, \mathrm{d}z)), \quad t \in [0, T],$$

where M^c and N^c denote the continuous parts of M and N respectively, and we have $\langle M^c \rangle = \langle N^c \rangle$ as just observed before. Hence

$$\langle M \rangle_t = \langle M^c \rangle_t + \int_0^t \int_{\mathbb{R}} z^2 \nu^{\mathbb{F}}(\mathrm{d}t, \mathrm{d}z), \quad t \in [0, T],$$
$$\langle N \rangle_t = \langle M^c \rangle_t + \int_0^t \int_{\mathbb{R}} z^2 \nu^{\mathbb{H}}(\mathrm{d}t, \mathrm{d}z), \quad t \in [0, T].$$

Now, we can derive the structure condition of S with respect to the filtration $\mathbb{H}$.

Proposition 3.2. *Assume that*

$$\mathbb{E}\left[\int_0^T \left(\alpha_u^{\mathcal{F}}\right)^2 \mathrm{d}\langle M \rangle_u\right] < \infty. \tag{3.1}$$

Then, under Assumption 3.1, the $(\mathbb{F}, \mathbf{P})$*-semimartingale* S *satisfies the structure condition with respect to* $\mathbb{H}$*, i.e.*

$$S_t = S_0 + N_t + \int_0^t \alpha_s^{\mathcal{H}} \mathrm{d}\langle N \rangle_s, \quad t \in [0, T], \tag{3.2}$$

where $\langle N \rangle$ *coincides with the* $(\mathbb{H}, \mathbf{P})$*-predictable dual projection of* $\langle M \rangle$*, that is,* $\langle N \rangle = \langle M \rangle^{p,\mathbb{H}}$ *and the* $\mathbb{R}$*-valued* $\mathbb{H}$*-predictable process* $\alpha^{\mathcal{H}} = \{\alpha_t^{\mathcal{H}},\ t \in [0, T]\}$ *given by*

$$\alpha_t^{\mathcal{H}} := \frac{\mathrm{d}\left(\int_0^t \alpha_u^{\mathcal{F}} \mathrm{d}\langle M \rangle_u\right)^{p,\mathbb{H}}}{\mathrm{d}\langle M \rangle_t^{p,\mathbb{H}}} \quad \mathbf{P} - a.s., \quad t \in [0, T],$$

satisfies an integrability condition analogous to (3.1).

Proof. By [21, Proposition 9.24] we get that the process $R = \{R_t,\ t \in [0, T]\}$ in decomposition (2.3) is given by

$$R_t = \left(\int_0^t \alpha_s^{\mathcal{F}} \mathrm{d}\langle M \rangle_s\right)^{p,\mathbb{H}}, \quad t \in [0, T].$$

Now, by applying [8, Proposition 4.9] we deduce that R is absolutely continuous with respect to $\langle M \rangle^{p,\mathbb{H}}$ and as a consequence, it can be written as $R_t = \int_0^t \alpha_s^{\mathcal{H}} \mathrm{d}\langle M \rangle_s^{p,\mathbb{H}}$, for every $t \in [0, T]$, where the process $\alpha^{\mathcal{H}}$ is the Radon-Nikodym derivative of $\left(\int \alpha_t^{\mathcal{F}} \mathrm{d}\langle M \rangle_t\right)^{p,\mathbb{H}}$ with respect to $\langle M \rangle^{p,\mathbb{H}}$.

To prove that $\langle N \rangle = \langle M \rangle^{p,\mathbb{H}}$ we notice that $\langle M^c \rangle = \langle N^c \rangle$, which is $\mathbb{H}$-predictable, and then we only need to show that $\langle N^d \rangle = \langle M^d \rangle^{p,\mathbb{H}}$, that is

$$\int_0^t \int_{\mathbb{R}} z^2 \nu^{\mathbb{H}}(\mathrm{d}s, \mathrm{d}z) = \left(\int_0^t \int_{\mathbb{R}} z^2 \nu^{\mathbb{F}}(\mathrm{d}s, \mathrm{d}z)\right)^{p,\mathbb{H}}, \quad t \in [0, T].$$

To this aim, we observe that by definitions of $\nu^{\mathbb{F}}(\mathrm{d}t,\mathrm{d}z)$ and $\nu^{\mathbb{H}}(\mathrm{d}t,\mathrm{d}z)$, for every $\mathbb{H}$-predictable (bounded) process $\varphi = \{\varphi_t,\ t \in [0,T]\}$ we have

$$\mathbb{E}\left[\int_0^T \varphi_s \int_{\mathbb{R}} z^2 \nu^{\mathbb{F}}(\mathrm{d}s,\mathrm{d}z)\right] = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} \varphi_s z^2 m(\mathrm{d}s,\mathrm{d}z)\right]$$
$$= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} \varphi_s z^2 \nu^{\mathbb{H}}(\mathrm{d}s,\mathrm{d}z)\right] = \mathbb{E}\left[\int_0^T \varphi_s \int_{\mathbb{R}} z^2 \nu^{\mathbb{H}}(\mathrm{d}s,\mathrm{d}z)\right].$$

Finally, it remains to check that $\alpha^{\mathcal{H}}$ satisfies the required integrability condition, i.e.

$$\mathbb{E}\left[\int_0^T \left(\alpha_u^{\mathcal{H}}\right)^2 \mathrm{d}\langle N\rangle_u\right] < \infty.$$

Since for every $\mathbb{H}$-predictable process φ we have

$$\mathbb{E}\left[\int_0^T \varphi_u \alpha_u^{\mathcal{H}} \mathrm{d}\langle M\rangle_u\right] = \mathbb{E}\left[\int_0^T \varphi_u \alpha_u^{\mathcal{H}} \mathrm{d}\langle N\rangle_u\right]$$
$$= \mathbb{E}\left[\int_0^T \varphi_u (\alpha_u^{\mathcal{F}} \mathrm{d}\langle M\rangle_u)^{p,\mathbb{H}}\right] = \mathbb{E}\left[\int_0^T \varphi_u \alpha_u^{\mathcal{F}} \mathrm{d}\langle M\rangle_u\right],$$

by choosing $\varphi = \alpha^{\mathcal{H}}$ and applying the Cauchy-Schwarz inequality, we have

$$\mathbb{E}\left[\int_0^T \left(\alpha_u^{\mathcal{H}}\right)^2 \mathrm{d}\langle N\rangle_u\right] \le \mathbb{E}\left[\int_0^T \left(\alpha_u^{\mathcal{F}}\right)^2 \mathrm{d}\langle M\rangle_u\right] < \infty.$$

□

Notice that under Assumption 3.1, the processes $\left\{\int_0^t \alpha_u^{\mathcal{F}} \mathrm{d}\langle M\rangle_u,\ t \in [0,T]\right\}$ and $\left\{\int_0^t \alpha_u^{\mathcal{H}} \mathrm{d}\langle N\rangle_u,\ t \in [0,T]\right\}$ are both continuous.

In the special case where the $\mathbb{F}$-predictable quadratic variation of the $(\mathbb{F},\mathbf{P})$-martingale M is absolutely continuous with respect to the Lebesgue measure, that is, $\langle M\rangle_t = \int_0^t a_s\ \mathrm{d}s$, $t \in [0,T]$, for some $\mathbb{R}$-valued $\mathbb{F}$-predictable process $a = \{a_t,\ t \in [0,T]\}$, we get that $\langle N\rangle_t = \langle M\rangle_t^{p,\mathbb{H}} = \int_0^t {}^p a_s\ \mathrm{d}s$ and $\left(\int_0^t \alpha_s^{\mathcal{F}}\ \mathrm{d}\langle M\rangle_s\right)^{p,\mathbb{H}} = \int_0^t {}^p(\alpha_s^{\mathcal{F}}\ a_s)\mathrm{d}s$ for each $t \in [0,T]$. Hence

$$\alpha_t^{\mathcal{H}} = \frac{{}^p(\alpha_t^{\mathcal{F}} a_t)}{{}^p a_t}\mathbf{1}_{\{{}^p a_t \neq 0\}}, \quad t \in [0,T].$$

4 The $\mathbb{H}$-pseudo optimal strategy

In the case of full information, when the stock price process S has continuous trajectories, it is proved in [35, Theorem 3.5], that there exists the $\mathbb{H}$-pseudo optimal strategy which can be obtained by switching to the minimal martingale measure $\mathbf{P}^*$, see Definition 4.1 below, and computing the Galtchouk-Kunita-Watanabe decomposition of the contingent claim ξ with respect to S under $\mathbf{P}^*$. Indeed, in the case of continuous trajectories, the minimal martingale measure preserves orthogonality, and then the Galtchouk-Kunita-Watanabe decomposition of the contingent claim under $\mathbf{P}^*$ provides the Föllmer-Schweizer decomposition of the contingent claim under the historical probability measure $\mathbf{P}$. Obviously, this does not work if the $(\mathbb{F},\mathbf{P})$-semimartingale S exhibits jumps. However, also in presence of jumps, the minimal martingale measure and the Galtchouk-Kunita-Watanabe decomposition of the contingent claim ξ still represent the key tools to

compute the $\mathbb{H}$-pseudo optimal strategy (we refer to [12] for the full information case). Here, we provide a similar criterion to characterize the pseudo optimal strategy in the partial information setting, see equation (4.16).
For reader's convenience, firstly we recall the definition of the minimal martingale measure with respect to the filtration $\mathbb{F}$.

Definition 4.1. *An equivalent martingale measure* $\mathbf{P}^*$ *for* S *with square integrable density* $\frac{\mathrm{d}\mathbf{P}^*}{\mathrm{d}\mathbf{P}}$ *is called* minimal martingale measure *(for* S*) if* $\mathbf{P}^* = \mathbf{P}$ *on* $\mathcal{F}_0$ *and if every square integrable* $(\mathbb{F}, \mathbf{P})$*-martingale, strongly orthogonal to the* $\mathbb{F}$*-martingale part of* S*,* M*, is also an* $(\mathbb{F}, \mathbf{P}^*)$*-martingale.*

Analogously, to define the minimal martingale measure $\mathbf{P}^0$ with respect to the filtration $\mathbb{H}$ it is sufficient to replace $\mathbb{F}$ by $\mathbb{H}$ and M by N.

We assume that

$$1 - \alpha_t^{\mathcal{F}} \Delta M_t > 0 \quad \mathbf{P} - \text{a.s.}, \quad t \in [0, T],$$

and

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T \left(\alpha_t^{\mathcal{F}}\right)^2 \mathrm{d}\langle M^c \rangle_t + \int_0^T \left(\alpha_t^{\mathcal{F}}\right)^2 \mathrm{d}\langle M^d \rangle_t\right\}\right] < \infty, \tag{4.1}$$

where M^c and M^d denote the continuous and the discontinuous parts of the $(\mathbb{F}, \mathbf{P})$-martingale M respectively and $\alpha^{\mathcal{F}}$ is given in (2.1), and define the process $L = \{L_t,\ t \in [0, T]\}$ by setting

$$L_t := \mathcal{E}\left(-\int \alpha_u^{\mathcal{F}} \mathrm{d}M_u\right)_t, \quad t \in [0, T], \tag{4.2}$$

where $\mathcal{E}(Y)$ refers to the Doléans-Dade exponential of an $(\mathbb{F}, \mathbf{P})$-semimartingale Y. Under condition (4.1), the nonnegative $(\mathbb{F}, \mathbf{P})$-local martingale L is indeed an $(\mathbb{F}, \mathbf{P})$-martingale, see e.g. [30], and also that (3.1) holds true. In addition, we assume that L is square integrable. Then, by the Ansel-Stricker Theorem (see [1]) there exists the minimal martingale measure $\mathbf{P}^*$ for S, which is defined by

$$L_T = \left.\frac{\mathrm{d}\mathbf{P}^*}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_T}. \tag{4.3}$$

Similarly, we assume that

$$1 - \alpha_t^{\mathcal{H}} \Delta N_t > 0 \quad \mathbf{P} - \text{a.s.} \quad \forall\, t \in [0, T],$$

and

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T \left(\alpha_t^{\mathcal{H}}\right)^2 \mathrm{d}\langle N^c \rangle_t + \int_0^T \left(\alpha_t^{\mathcal{H}}\right)^2 \mathrm{d}\langle N^d \rangle_t\right\}\right] < \infty,$$

where, as usual N^c and N^d denote the continuous and the discontinuous parts of the $(\mathbb{H}, \mathbf{P})$-martingale N respectively. Then, we define the process $L^0 = \{L_t^0,\ t \in [0, T]\}$ by setting

$$L_t^0 := \mathcal{E}\left(-\int \alpha_u^{\mathcal{H}} \mathrm{d}N_u\right)_t, \quad t \in [0, T]. \tag{4.4}$$

We notice that L^0 is an $(\mathbb{H}, \mathbf{P})$-martingale and as before we assume that L^0 is square integrable. Then, we can define $\mathbf{P}^0$ as the probability measure on $(\Omega, \mathcal{H}_T)$ such that

$$L_T^0 = \left.\frac{\mathrm{d}\mathbf{P}^0}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{H}_T}. \tag{4.5}$$

We are now in the position to state the following result.

Proposition 4.2. *Let $\xi \in L^2(\mathcal{H}_T, \mathbf{P})$ be a contingent claim that admits a Föllmer-Schweizer decomposition with respect to $\mathbb{H}$ and S, and $\psi^* = (\theta^*, \eta^*)$ be the associated $\mathbb{H}$-pseudo optimal strategy. Then, the optimal value process $V(\psi^*) = \{V_t(\psi^*),\ t \in [0,T]\}$ is given by*

$$V_t(\psi^*) = \mathbb{E}^{\mathbf{P}^0}\left[\xi|\mathcal{H}_t\right], \quad t \in [0,T],$$

where $\mathbb{E}^{\mathbf{P}^0}\left[\cdot|\mathcal{H}_t\right]$ denotes the conditional expectation with respect to $\mathcal{H}_t$ computed under $\mathbf{P}^0$; moreover the first component θ^* of the $\mathbb{H}$-pseudo optimal strategy ψ^* is given by

$$\theta_t^* = \frac{\mathrm{d}^{\mathbb{H}}\langle V^m(\psi^*), N\rangle_t}{\mathrm{d}^{\mathbb{H}}\langle N\rangle_t} \quad \mathbf{P}-a.s., \quad t \in [0,T], \tag{4.6}$$

where $V^m(\psi^*)$ is the $(\mathbb{H}, \mathbf{P})$-martingale part of the process $V(\psi^*)$ and here the sharp brackets are computed under $\mathbf{P}$ (with the convention $\theta^* = 0$ for the indeterminate form $\frac{0}{0}$).

Proof. Since L^0 given in (4.5) is a square integrable $(\mathbb{H}, \mathbf{P})$-martingale, by Cauchy-Schwarz inequality we get that

$$\mathbb{E}^{\mathbf{P}^0}[|\xi|] = \mathbb{E}\left[|\xi| L_T^0\right] \leq \mathbb{E}\left[\xi^2\right]^{1/2} \mathbb{E}\left[(L_T^0)^2\right]^{1/2} < \infty,$$

which means that $\xi \in L^1(\mathcal{H}_T, \mathbf{P}^0)$.
Consider the Föllmer-Schweizer decomposition of ξ with respect to S and $\mathbb{H}$, see (2.4), and let $\psi^* = (\theta^*, \eta^*)$ be the $\mathbb{H}$-pseudo optimal strategy. Then, by Proposition 2.8 we get $\theta^* = \beta^{\mathcal{H}}$ and the optimal value process $V(\psi^*)$ satisfies

$$V_t(\psi^*) = U_0 + \int_0^t \beta_u^{\mathcal{H}} \mathrm{d}S_u + A_t, \quad t \in [0,T].$$

Observe that $\int \beta_t^{\mathcal{H}} \mathrm{d}S_t$ is an $(\mathbb{H}, \mathbf{P}^0)$-martingale since $\int \beta_t^{\mathcal{H}} \mathrm{d}N_t$ and L are $(\mathbb{H}, \mathbf{P})$-martingales (see the proof of Theorem 3.14 in [14]) and A turns out to be an $(\mathbb{H}, \mathbf{P}^0)$-martingale by definition of the minimal martingale measure with respect to the filtration $\mathbb{H}$. Then, the optimal value process $V(\psi^*)$ is an $(\mathbb{H}, \mathbf{P}^0)$-martingale, and as a consequence it can be written as

$$V_t(\psi^*) = \mathbb{E}^{\mathbf{P}^0}\left[V_T(\psi^*)|\mathcal{H}_t\right] = \mathbb{E}^{\mathbf{P}^0}\left[\xi|\mathcal{H}_t\right], \quad t \in [0,T].$$

Finally, to compute the $\mathbb{H}$-pseudo optimal strategy we consider the $(\mathbb{H}, \mathbf{P})$-martingale part of the process $V(\psi^*)$ given by

$$V_t^m(\psi^*) = U_0 + \int_0^t \beta_u^{\mathcal{H}} \mathrm{d}N_u + A_t, \quad t \in [0,T].$$

Then, taking the predictable quadratic covariation with respect to the $\mathbb{H}$-martingale part N of S computed under $\mathbf{P}$ into account, we get that

$$\mathrm{d}^{\mathbb{H}}\langle V^m(\psi^*), N\rangle_t = \beta_t^{\mathcal{H}} \mathrm{d}^{\mathbb{H}}\langle N\rangle_t, \quad t \in [0,T],$$

since A is strongly orthogonal to N under $\mathbf{P}$. Then, we obtain equation (4.6). □

When the stock price process S has continuous trajectories, the optimal value process and the $\mathbb{H}$-pseudo optimal strategy can be characterized in terms of the minimal martingale measure $\mathbf{P}^*$ with respect to the filtration $\mathbb{F}$ as proved in Corollary 4.4 below. We start with a useful lemma.

Lemma 4.3. *Assume that S has continuous trajectories. Then the minimal martingale measure $\mathbf{P}^0$ with respect to the filtration $\mathbb{H}$ coincides with the restriction on the filtration $\mathbb{H}$ of the minimal martingale measure $\mathbf{P}^*$ with respect to the filtration $\mathbb{F}$.*

Proof. The proof is postponed to Appendix B. □

Corollary 4.4. *Assume that S has continuous trajectories and let $\xi \in L^2(\mathcal{H}_T, \mathbf{P})$ be a contingent claim that admits a Föllmer-Schweizer decomposition with respect to $\mathbb{H}$ and S, and $\psi^* = (\theta^*, \eta^*)$ be the associated $\mathbb{H}$-pseudo optimal strategy. Then, the optimal value process $V(\psi^*) = \{V_t(\psi^*),\ t \in [0,T]\}$ is given by*

$$V_t(\psi^*) = \mathbb{E}^{\mathbf{P}^*}\left[\xi|\mathcal{H}_t\right], \quad t \in [0,T], \tag{4.7}$$

where $\mathbb{E}^{\mathbf{P}^*}[\cdot|\mathcal{H}_t]$ denotes the conditional expectation with respect to $\mathcal{H}_t$ computed under $\mathbf{P}^*$; moreover the first component θ^* of the $\mathbb{H}$-pseudo optimal strategy ψ^* is given by

$$\theta^*_t = \frac{\mathrm{d}^{\mathbb{H}}\langle V(\psi^*), S\rangle_t}{\mathrm{d}^{\mathbb{H}}\langle S\rangle_t} \quad \mathbf{P} - a.s., \quad t \in [0,T], \tag{4.8}$$

where the sharp brackets are computed under $\mathbf{P}$ (with the convention $\theta^* = 0$ for the indeterminate form $\frac{0}{0}$).

Proof. The proof follows by Proposition 4.2 observing that, in virtue of Lemma 4.3, the optimal value process $V(\psi^*)$ can be written as

$$V_t(\psi^*) = \mathbb{E}^{\mathbf{P}^0}\left[\xi|\mathcal{H}_t\right] = \mathbb{E}^{\mathbf{P}^*}\left[\xi|\mathcal{H}_t\right], \quad t \in [0,T].$$

Finally, since the finite variation part of S is continuous we get that $\mathrm{d}^{\mathbb{H}}\langle N\rangle = \mathrm{d}^{\mathbb{H}}\langle S\rangle$ and $\mathrm{d}^{\mathbb{H}}\langle V^m(\psi^*), N\rangle = \mathrm{d}^{\mathbb{H}}\langle V(\psi^*), S\rangle$, which leads to (4.8). □

When S has also jumps it is not possible to provide a characterization of the optimal value process analogous to (4.7). This is essentially due to the fact that in general the minimal martingale measure $\mathbf{P}^0$ with respect to the filtration $\mathbb{H}$ does not coincide with the restriction of $\mathbf{P}^*$ over $\mathbb{H}$. Then, to compute explicitly the $\mathbb{H}$-pseudo-optimal strategy we follow the approach suggested by [12] in the full information framework.

Assume now that ξ admits a Föllmer-Schweizer decomposition of ξ with respect to S and $\mathbb{F}$, i.e.

$$\xi = \widetilde{U}_0 + \int_0^T \beta^{\mathcal{F}}_t \mathrm{d}S_t + \widetilde{A}_T \quad \mathbf{P} - \text{a.s.}, \tag{4.9}$$

where $U_0 \in L^2(\mathcal{F}_0, \mathbf{P})$, $\beta^{\mathcal{F}} \in \Theta(\mathbb{F})$ and $\widetilde{A} = \{\widetilde{A}_t,\ t \in [0,T]\}$ is a square integrable $(\mathbb{F}, \mathbf{P})$-martingale with $\widetilde{A}_0 = 0$ strongly orthogonal to the $\mathbb{F}$-martingale part M of S under $\mathbf{P}$. By applying Proposition 2.8 with the choice $\mathbb{H} = \mathbb{F}$, we know that $\beta^{\mathcal{F}}$ provides the pseudo-optimal strategy under full information.

In the sequel, we characterize the $\mathbb{H}$-pseudo-optimal strategy $\beta^{\mathcal{H}}$ and discuss the relation between $\beta^{\mathcal{H}}$ and $\beta^{\mathcal{F}}$.

Denote by $\Theta(\mathbb{F}, \mathbf{P}^*)$ ($\Theta(\mathbb{H}, \mathbf{P}^*)$, respectively) the set of all $\mathbb{R}$-valued $\mathbb{F}$-predictable (respectively, $\mathbb{H}$-predictable) processes $\delta = \{\delta_t,\ t \in [0,T]\}$ satisfying the following integrability condition:

$$\mathbb{E}^{\mathbf{P}^*}\left[\int_0^T \delta_u^2 \mathrm{d}\langle S\rangle_u\right] < \infty.$$

For the rest of the section we assume ξ to be square integrable with respect to $\mathbf{P}^*$.

Let us observe that since S is a $\mathbf{P}^*$-martingale with respect to both the filtrations $\mathbb{F}$ and $\mathbb{H}$, the random variable ξ admits the Galtchouk-Kunita-Watanabe decomposition with respect to S and both the filtrations $\mathbb{F}$ and $\mathbb{H}$ under $\mathbf{P}^*$, i.e.

$$\xi = \widetilde{U}_0 + \int_0^T \widetilde{\beta}^{\mathcal{F}}_u \mathrm{d}S_u + \widetilde{G}_T \quad \mathbf{P}^* - \text{a.s.}, \tag{4.10}$$

$$\xi = U_0 + \int_0^T \widetilde{\beta}_u^{\mathcal{H}} \mathrm{d}S_u + G_T \quad \mathbf{P}^* - \text{a.s.}, \tag{4.11}$$

where $\widetilde{U}_0 \in L^2(\mathcal{F}_0, \mathbf{P}^*)$, $U_0 \in L^2(\mathcal{H}_0, \mathbf{P}^*)$, $\widetilde{\beta}^{\mathcal{F}} \in \Theta(\mathbb{F}, \mathbf{P}^*)$, $\widetilde{\beta}^{\mathcal{H}} \in \Theta(\mathbb{H}, \mathbf{P}^*)$, $\widetilde{G} = \{\widetilde{G}_t,\ t \in [0,T]\}$ and $G = \{G_t,\ t \in [0,T]\}$ are square integrable $(\mathbb{F}, \mathbf{P}^*)$ and $(\mathbb{H}, \mathbf{P}^*)$-martingales respectively with $\widetilde{G}_0 = G_0 = 0$, strongly orthogonal to S under $\mathbf{P}^*$.
On the other hand, if S turns out to be square integrable with respect to $\mathbf{P}^*$, the $\mathbf{P}^*$-martingale property of S with respect to both the filtrations $\mathbb{F}$ and $\mathbb{H}$ also ensures that we can apply [8, Theorem 3.2] which provides the Galtchouk-Kunita-Watanabe decomposition of a square integrable random variable under partial information with respect to $\mathbf{P}^*$. More precisely, every $\xi \in L^2(\mathcal{F}_T, \mathbf{P}^*)$ can be uniquely written as

$$\xi = U_0^{'} + \int_0^T H_u^{\mathcal{H}} \mathrm{d}S_u + G_T^{'} \quad \mathbf{P}^* - \text{a.s.}, \tag{4.12}$$

where $U_0^{'} \in L^2(\mathcal{F}_0, \mathbf{P}^*)$, $H^{\mathcal{H}} = \{H_t^{\mathcal{H}},\ t \in [0,T]\} \in \Theta(\mathbb{H}, \mathbf{P}^*)$ and $G^{'} = \{G_t^{'},\ t \in [0,T]\}$ is a square integrable $(\mathbb{F}, \mathbf{P}^*)$-martingale with $G_0^{'} = 0$ weakly orthogonal[2] to S under $\mathbf{P}^*$, according to Definition 2.1 given in [8].

Lemma 4.5. *Assume $\xi \in L^2(\mathcal{H}_T, \mathbf{P}^*)$ and that S is square integrable with respect to $\mathbf{P}^*$. Let $\widetilde{\beta}^{\mathcal{H}} \in \Theta(\mathbb{H}, \mathbf{P}^*)$ and $H^{\mathcal{H}} \in \Theta(\mathbb{H}, \mathbf{P}^*)$ be the integrands in decompositions* (4.11) *and* (4.12) *respectively. Then*

$$H_t^{\mathcal{H}} = \widetilde{\beta}_t^{\mathcal{H}} \quad \mathbf{P} - a.s., \quad \forall t \in [0,T]. \tag{4.13}$$

Proof. Let $\xi \in L^2(\mathcal{H}_T, \mathbf{P}^*)$ and consider decomposition (4.12). By taking the conditional expectation with respect to $\mathcal{H}_T$ under $\mathbf{P}^*$, we get

$$\xi = \mathbb{E}^{\mathbf{P}^*}\left[U_0^{'} \middle| \mathcal{H}_T\right] + \int_0^T H_u^{\mathcal{H}} \mathrm{d}S_u + \mathbb{E}^{\mathbf{P}^*}\left[G_T^{'} \middle| \mathcal{H}_T\right] = \widehat{U}_0 + \int_0^T H_u^{\mathcal{H}} \mathrm{d}S_u + \widehat{G}_T, \tag{4.14}$$

where $\widehat{U}_0 := \mathbb{E}^{\mathbf{P}^*}\left[U_0^{'} \middle| \mathcal{H}_0\right]$ and $\widehat{G}_t := \mathbb{E}^{\mathbf{P}^*}\left[G_t^{'} \middle| \mathcal{H}_t\right] + \mathbb{E}^{\mathbf{P}^*}\left[U_0^{'} \middle| \mathcal{H}_t\right] - \mathbb{E}^{\mathbf{P}^*}\left[U_0^{'} \middle| \mathcal{H}_0\right]$, for every $t \in [0,T]$, so that $\widehat{G} = \{\widehat{G}_t,\ t \in [0,T]\}$ turns out to be a square integrable $(\mathbb{H}, \mathbf{P}^*)$-martingale with $\widehat{G}_0 = 0$ weakly orthogonal to S under $\mathbf{P}^*$. Indeed, $\mathbb{E}^{\mathbf{P}^*}\left[U_0^{'} \middle| \mathcal{H}_t\right] - \mathbb{E}^{\mathbf{P}^*}\left[U_0^{'} \middle| \mathcal{H}_0\right] \in L^2(\mathcal{H}_t, \mathbf{P}^*)$, for every $t \in [0,T]$, is clearly weakly orthogonal to S under $\mathbf{P}^*$ thanks to the martingale property of S with respect to both the filtrations $\mathbb{F}$ and $\mathbb{H}$. Furthermore, for every $\varphi \in \Theta(\mathbb{H})$ we have

$$\begin{aligned}
\mathbb{E}^{\mathbf{P}^*}\left[\mathbb{E}^{\mathbf{P}^*}\left[G_T^{'} \middle| \mathcal{H}_T\right] \int_0^T \varphi_u \mathrm{d}S_u\right] &= \mathbb{E}^{\mathbf{P}^*}\left[\mathbb{E}^{\mathbf{P}^*}\left[G_T^{'} \int_0^T \varphi_u \mathrm{d}S_u \middle| \mathcal{H}_T\right]\right] \\
&= \mathbb{E}^{\mathbf{P}^*}\left[G_T^{'} \int_0^T \varphi_u \mathrm{d}S_u\right] = 0,
\end{aligned}$$

since $G^{'}$ is weakly orthogonal to S under $\mathbf{P}^*$. Moreover, $\widehat{U}_0 \in L^2(\mathcal{H}_0, \mathbf{P}^*)$ and since $\widehat{G}$ is $\mathbb{H}$-adapted, it is also strongly orthogonal to S under $\mathbf{P}^*$, see [8, Remark 2.4]. Then, by uniqueness of the Galtchouk-Kunita-Watanabe decomposition, representations (4.14) and (4.11) for ξ coincide, and in particular this implies (4.13). □

[2] We say that a square integrable $(\mathbb{F}, \mathbf{P}^*)$-martingale O is *weakly orthogonal* to the square integrable $(\mathbb{F}, \mathbf{P}^*)$-martingale S if the following condition

$$\mathbb{E}\left[O_T \int_0^T \varphi_t \mathrm{d}S_t\right] = 0,$$

holds for all processes $\varphi \in \Theta(\mathbb{H}, \mathbf{P}^*)$.

The following result furnishes the relation between the strategies $\beta^{\mathcal{F}}$ and $\beta^{\mathcal{H}}$ when S has continuous trajectories.

Proposition 4.6. *Let $\xi \in L^2(\mathcal{H}_T, \mathbf{P}^*)$ be a contingent claim and assume that S is continuous and square integrable with respect to $\mathbf{P}^*$. Then, the following relation between the $\mathbb{H}$-pseudo optimal strategy $\beta^{\mathcal{H}}$ and the $\mathbb{F}$-pseudo optimal strategy $\beta^{\mathcal{F}}$ holds*

$$\beta_t^{\mathcal{H}} = {}^{p,*}\beta_t^{\mathcal{F}} \quad \mathbf{P} - a.s., \quad t \in [0,T]. \tag{4.15}$$

Here, the notation ${}^{p,*}D$ refers to the $(\mathbb{H}, \mathbf{P}^*)$-predictable projection of an $\mathbb{R}$-valued integrable process $D = \{D_t,\ t \in [0,T]\}$.

Proof. When S has continuous trajectories, decompositions (4.9) (with respect to $\mathbb{F}$) and (2.4) (with respect to $\mathbb{H}$) coincide to the corresponding Galtchouk-Kunita-Watanabe decompositions under $\mathbf{P}^*$, see (4.10) and (4.11) above. Lemma 4.5 implies that $\beta^{\mathcal{H}} = H^{\mathcal{H}}$ and since $\langle S \rangle$ is $\mathbb{H}$-predictable, due to the fact that in this case $\langle S \rangle = [S]$, which is $\mathbb{F}^S$-adapted by definition, under $\mathbf{P}^*$, by applying [8, Proposition 4.1] we get (4.15). □

Remark 4.7. Note that the characterization of the optimal strategy in terms of the $(\mathbb{H}, \mathbf{P}^*)$-predictable projection of the integrand $\beta^{\mathcal{F}}$ in (4.9), also holds thanks to relation (4.2) of [33], since the Galtchouk-Kunita-Watanabe decomposition under $\mathbf{P}^*$ coincides with the Föllmer-Schweizer decomposition under $\mathbf{P}$ when S has continuous trajectories.

In presence of jumps in the underlying process S, the relation between $\beta^{\mathcal{H}}$ and $\beta^{\mathcal{F}}$ is more complicated. In [12], the relation between $\widetilde{\beta}^{\mathcal{F}}$ and $\beta^{\mathcal{F}}$, given in (4.10) and (4.9) respectively, is written in terms of the local characteristics associated to $\widetilde{G}$ under $\mathbf{P}^*$. A similar result can be applied to derive the relation between $\widetilde{\beta}^{\mathcal{H}}$ and $\beta^{\mathcal{H}}$, given in (4.11) and (2.4) respectively, in terms of the local characteristics associated to G under $\mathbf{P}^*$.

We are now in the position to state the following result.

Proposition 4.8. *Let $\xi \in L^2(\mathcal{H}_T, \mathbf{P}^*)$ be a contingent claim that admits a Föllmer-Schweizer decomposition with respect to $\mathbb{H}$ and S, and assume that S is square integrable with respect to $\mathbf{P}^*$. The first component of the associated $\mathbb{H}$-pseudo optimal strategy $\psi^* = (\beta^{\mathcal{H}}, \eta^*)$ is given by*

$$\beta_t^{\mathcal{H}} = H_t^{\mathcal{H}} + \phi_t^{\mathcal{H}}, \quad \mathbf{P} - a.s., \quad t \in [0,T]. \tag{4.16}$$

In other terms,

$$\beta_t^{\mathcal{H}} = \frac{\mathrm{d}(\int_0^t \widetilde{\beta}_u^{\mathcal{F}}\, \mathrm{d}\langle S \rangle_u)^{p,\mathbb{H},*}}{\mathrm{d}\langle S \rangle_t^{p,\mathbb{H},*}} + \phi_t^{\mathcal{H}} = \frac{\mathrm{d}(\int_0^t \beta_u^{\mathcal{F}}\, \mathrm{d}\langle S \rangle_u)^{p,\mathbb{H},*}}{\mathrm{d}\langle S \rangle_t^{p,\mathbb{H},*}} + \phi_t^{\mathcal{H}} - \frac{\mathrm{d}(\int_0^t \phi_u^{\mathcal{F}}\, \mathrm{d}\langle S \rangle_u)^{p,\mathbb{H},*}}{\mathrm{d}\langle S \rangle_t^{p,\mathbb{H},*}} \quad \mathbf{P} - a.s., \tag{4.17}$$

for every $t \in [0,T]$, where $D^{p,\mathbb{H},*}$ denotes the $(\mathbb{H}, \mathbf{P}^*)$-predictable dual projection of an $\mathbb{R}$-valued process $D = \{D_t,\ t \in [0,T]\}$ of finite variation, and the processes $\phi^{\mathcal{F}} = \{\phi_t^{\mathcal{F}},\ t \in [0,T]\}$ and $\phi^{\mathcal{H}} = \{\phi_t^{\mathcal{H}},\ t \in [0,T]\}$ are respectively given by

$$\phi_t^{\mathcal{F}} = \frac{\mathrm{d}^{\mathbb{F}}\langle [\widetilde{G}, S], \int_0^\cdot \alpha_r^{\mathcal{F}} \mathrm{d}M_r \rangle_t}{\mathrm{d}^{\mathbb{F}}\langle S \rangle_t}, \quad \phi_t^{\mathcal{H}} = \frac{\mathrm{d}^{\mathbb{H}}\langle [G, S], \int_0^\cdot \alpha_r^{\mathcal{H}} \mathrm{d}N_r \rangle_t}{\mathrm{d}^{\mathbb{H}}\langle S \rangle_t} \quad \mathbf{P} - a.s., \tag{4.18}$$

for every $t \in [0,T]$, where the sharp brackets are computed under $\mathbf{P}$.

Proof. Taking Lemma 4.5 into account, by [8, Proposition 4.9] we obtain

$$H_t^{\mathcal{H}} = \widetilde{\beta}_t^{\mathcal{H}} = \frac{\mathrm{d}(\int_0^t \widetilde{\beta}_u^{\mathcal{F}} \mathrm{d}\langle S \rangle_u)^{p,\mathbb{H},*}}{\mathrm{d}\langle S \rangle_t^{p,\mathbb{H},*}}, \quad t \in [0,T].$$

Then, by applying [12, Theorem 3.2], we get $\beta^{\mathcal{H}} = \widetilde{\beta}^{\mathcal{H}} + \phi^{\mathcal{H}}$ and $\beta^{\mathcal{F}} = \widetilde{\beta}^{\mathcal{F}} + \phi^{\mathcal{F}}$, and then equalities (4.17). Finally, the expressions in (4.18) follow by [12, Remark on page 8]. □

5 Application to Markovian models

In this section we wish to apply the results of Section 3 and Section 4 to a Markovian setting. We assume that the dynamics of the risky asset price process S depends on some unobservable process X, which may represent the activity of other markets, macroeconomics factors or microstructure rules that drive the market.

We consider a European-type contingent claim whose payoff $\xi \in L^2(\mathcal{H}_T, \mathbf{P}) \cap L^2(\mathcal{H}_T, \mathbf{P}^*)$ is of the form

$$\xi = H(T, S_T),$$

where $H(t, s)$ is a deterministic function. We define the processes $V^{\mathcal{F}}$ and $V^{\mathcal{H}}$ by setting

$$V_t^{\mathcal{F}} := \mathbb{E}^{\mathbf{P}^*}\left[H(T, S_T)|\mathcal{F}_t\right], \quad V_t^{\mathcal{H}} := \mathbb{E}^{\mathbf{P}^*}\left[H(T, S_T)|\mathcal{H}_t\right], \quad t \in [0, T].$$

If the pair (X, S) is an $(\mathbb{F}, \mathbf{P}^*)$-Markov process, then there exists a measurable function $g(t, x, s)$ such that

$$V_t^{\mathcal{F}} = \mathbb{E}^{\mathbf{P}^*}\left[H(T, S_T)|\mathcal{F}_t\right] = g(t, X_t, S_t) \tag{5.1}$$

for every $t \in [0, T]$ and

$$V_t^{\mathcal{H}} = \mathbb{E}^{\mathbf{P}^*}\left[\mathbb{E}^{\mathbf{P}^*}\left[H(T, S_T)|\mathcal{F}_t\right]|\mathcal{H}_t\right] = \mathbb{E}^{\mathbf{P}^*}\left[g(t, X_t, S_t)|\mathcal{H}_t\right], \quad t \in [0, T]. \tag{5.2}$$

We denote by $\mathcal{L}^*_{X,S}$ the $(\mathbb{F}, \mathbf{P}^*)$-Markov generator of the pair (X, S). Then, by [13, Chapter 4, Proposition 1.7] the process

$$\left\{f(t, X_t, S_t) - \int_0^t \mathcal{L}^*_{X,S} f(u, X_u, S_u)\mathrm{d}u, \quad t \in [0, T]\right\}$$

is an $(\mathbb{F}, \mathbf{P}^*)$-martingale for every function $f(t, x, s)$ in the domain of the operator $\mathcal{L}^*_{X,S}$, denoted by $D(\mathcal{L}^*_{X,S})$. Then the following result, which allows to compute the function $g(t, x, s)$, holds.

Lemma 5.1. *Let $\widetilde{g}(t, x, s) \in D(\mathcal{L}^*_{X,S})$ such that*

$$\begin{cases} \mathcal{L}^*_{X,S}\widetilde{g}(t, x, s) = 0, & t \in [0, T) \\ \widetilde{g}(T, x, s) = H(T, s). \end{cases} \tag{5.3}$$

Then $\widetilde{g}(t, X_t, S_t) = g(t, X_t, S_t)$ $\mathbf{P}$-a.s., for every $t \in [0, T]$, with $g(t, x, s)$ satisfying (5.1).

Proof. Let $\widetilde{g}(t, x, s) \in D(\mathcal{L}^*_{X,S})$ be the solution of problem (5.3). Then the process $\{\widetilde{g}(t, X_t, S_t),\ t \in [0, T]\}$ is an $(\mathbb{F}, \mathbf{P}^*)$-martingale. Since $\widetilde{g}(T, X_T, S_T) = H(T, S_T)$, by the martingale property we get that $\widetilde{g}(t, X_t, S_t) = \mathbb{E}^{\mathbf{P}^*}[H(T, S_T)|\mathcal{F}_t]$. □

In the computation of the $\mathbb{H}$-pseudo optimal strategies we consider the case where the information available to traders is represented by the filtration generated by the stock price process S; in other terms, we assume that

$$\mathcal{H}_t = \mathcal{F}_t^S \qquad \forall\ t \in [0, T]. \tag{5.4}$$

We define the filter $\pi(f) = \{\pi_t(f),\ t \in [0, T]\}$, by setting for each $t \in [0, T]$

$$\pi_t(f) := \mathbb{E}^{\mathbf{P}^*}[f(t, X_t, S_t)|\mathcal{F}_t^S]$$

for any measurable function $f(t, x, s)$ such that $\mathbb{E}^{\mathbf{P}^*}\left[|f(t, X_t, S_t)|\right] < \infty$, for every $t \in [0, T]$. It is known that $\pi(f)$ is a probability measure-valued process with càdlàg trajectories (see [25]), which provides the $\mathbf{P}^*$-conditional law of X given the information flow.

Then, by (5.2) the process $V^{\mathcal{H}}$ can be written in terms of the filter as

$$V_t^{\mathcal{H}} = \pi_t(g) \qquad \forall\ t \in [0,T], \tag{5.5}$$

where the function $g(t,x,s)$ is the solution of the problem with final value (5.3). Therefore we can characterize the integrand $\widetilde{\beta}^{\mathcal{H}}$ in the Galtchouk-Kunita-Watanabe decomposition (4.11) of ξ under partial information as

$$\widetilde{\beta}_t^{\mathcal{H}} = H_t^{\mathcal{H}} = \frac{\mathrm{d}\langle \pi(g), S\rangle_t^{*,\mathbb{H}}}{\mathrm{d}\langle S\rangle_t^{*,\mathbb{H}}}, \quad t \in [0,T],$$

where $\langle\ \rangle^{*,\mathbb{H}}$ denotes the sharp bracket computed with respect to $\mathbb{H}$ and $\mathbf{P}^*$.

Finally, assume that ξ admits a Föllmer-Schweizer decomposition with respect to $\mathbb{H}$ and S, then by Proposition 4.8 we get that the first component of the corresponding $\mathbb{H}$-pseudo optimal strategy is given by

$$\beta_t^{\mathcal{H}} = \widetilde{\beta}_t^{\mathcal{H}} + \phi_t^{\mathcal{H}} = \frac{\mathrm{d}\langle \pi(g), S\rangle_t^{*,\mathbb{H}}}{\mathrm{d}\langle S\rangle_t^{*,\mathbb{H}}} + \frac{\mathrm{d}^{\mathbb{H}}\langle [G,S], \int_0^{\cdot} \alpha_s^{\mathcal{H}} \mathrm{d}N_s\rangle_t}{\mathrm{d}^{\mathbb{H}}\langle S\rangle_t}, \quad t \in [0,T], \tag{5.6}$$

where G is the $(\mathbb{H}, \mathbf{P}^*)$-martingale in the Galtchouk-Kunita-Watanabe decomposition (4.11) of ξ, given by

$$G_t = -U_0 + \pi_t(g) - \int_0^t \widetilde{\beta}_u^{\mathcal{H}} \mathrm{d}S_u, \quad t \in [0,T].$$

In the following, we compute explicitly the process $\widetilde{\beta}^{\mathcal{H}}$ and provide the $\mathbb{H}$-pseudo optimal strategy $\psi^* = (\beta^{\mathcal{H}}, \eta^*)$ for a jump-diffusion market model by characterizing the process $\phi^{\mathcal{H}}$. Finally we deduce the corresponding $\mathbb{H}$-pseudo optimal strategy for diffusion and pure jump market models.

5.1 A jump-diffusion market model

Jump-diffusion models are widely used in practice to describe asset prices dynamics and in the last years they also have been applied to energy finance to represent the behaviour of spot electricity prices. Indeed, stocks dynamics in general take into account two effects: the normal price changes, described by diffusion processes, which are due to the interaction between supply and demand, and sudden changes modeled by jump processes, representing updates when new information arrives. Accordingly, we consider the following application. Assume that the risky asset price dynamics is described by a geometric jump-diffusion, which depends on an unobservable stochastic factor X modeled by a Markovian jump-diffusion that may have common jump times with S. Precisely, we have the following system of stochastic differential equations (in short SDEs):

$$\begin{cases} \mathrm{d}X_t = \mu_0(t,X_t)\mathrm{d}t + \sigma_0(t,X_t)\mathrm{d}W_t^0 + \displaystyle\int_Z K_0(\zeta;t,X_{t^-})\mathcal{N}(\mathrm{d}t,\mathrm{d}\zeta), \\ \mathrm{d}S_t = S_{t^-}\left(\mu_1(t,X_t,S_t)\mathrm{d}t + \sigma_1(t,S_t)\mathrm{d}W_t^1 + \displaystyle\int_Z K_1(\zeta;t,X_{t^-},S_{t^-})\mathcal{N}(\mathrm{d}t,\mathrm{d}\zeta)\right), \end{cases} \tag{5.7}$$

with $X_0 = x \in \mathbb{R}$ and $S_0 = s > 0$. Here $\mathcal{N}(\mathrm{d}t,\mathrm{d}\zeta)$, $(t,\zeta) \in [0,T] \times Z$, with $Z \subseteq \mathbb{R}$, is an $(\mathbb{F}, \mathbf{P})$-Poisson random measure having nonnegative intensity $\eta(\mathrm{d}\zeta)\mathrm{d}t$. The measure $\eta(\mathrm{d}\zeta)$, defined on the measurable space $(Z, \mathcal{Z})$, is σ-finite. The corresponding $(\mathbb{F}, \mathbf{P})$-compensated random measure is given by

$$\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}\zeta) = \mathcal{N}(\mathrm{d}t,\mathrm{d}\zeta) - \eta(\mathrm{d}\zeta)\mathrm{d}t.$$

The processes $W^0 = \{W^0_t,\ t \in [0,T]\}$ and $W^1 = \{W^1_t,\ t \in [0,T]\}$ are $(\mathbb{F}, \mathbf{P})$-Brownian motions independent of $\mathcal{N}(\mathrm{d}t, \mathrm{d}\zeta)$ such that $\langle W^0, W^1\rangle_t = \rho t$, for every $t \in [0,T]$, with $\rho \in [-1,1]$. The coefficients $\mu_0(t,x), \mu_1(t,x,s)$, $\sigma_0(t,x) > 0, \sigma_1(t,x,s) > 0$, $K_0(\zeta;t,x)$ and $K_1(\zeta;t,x,s)$ are $\mathbb{R}$-valued measurable functions of their arguments such that a unique strong solution for the system (5.7) exists, see for instance [28]. In particular, this implies that the pair (X,S) is an $(\mathbb{F}, \mathbf{P})$-Markov process.

Note that if the set $\{\zeta \in Z : K_1(\zeta;t,X_{t^-},S_{t^-}) \neq 0 \ \text{ and } \ K_0(\zeta;t,X_{t^-}) \neq 0\}$ is not empty, S and X have common jump times. This feature may describe, for example, catastrophic events that affect at the same time the stock price and the hidden state variable that influences it.

For simplicity we take

$$|\mu_1(t,X_t,S_t)| < c_1,\ 0 < c_2 < \sigma_1(t,S_t) < c_3 \ \text{ and } \ |K_1(\zeta;t,X_t,S_t)| < c_4, \tag{5.8}$$

for every $t \in [0,T]$, $\zeta \in Z$ and for some constants c_1,c_2,c_3,c_4. Moreover, to ensure nonnegativity of S we also assume that $K_1(\zeta;t,X_t,S_t) + 1 > 0$ $\mathbf{P}$-a.s. for every $t \in [0,T]$ and $\zeta \in Z$.

We introduce the integer-valued random measure that describes the jumps of S,

$$m(\mathrm{d}t, \mathrm{d}z) = \sum_{r:\Delta S_r \neq 0} \delta_{(r,\Delta S_r)}(\mathrm{d}t, \mathrm{d}z),$$

where δ_a denotes as usual the Dirac measure at point a. Note that the following equality holds

$$\int_0^t \int_{\mathbb{R}} z\, m(\mathrm{d}u, \mathrm{d}z) = \int_0^t S_{u^-} \int_Z K_1(\zeta;u,X_{u^-},S_{u^-}) \mathcal{N}(\mathrm{d}u, \mathrm{d}\zeta)$$

and, in general, for any measurable function $\gamma : \mathbb{R} \to \mathbb{R}$, we get that

$$\int_0^t \int_{\mathbb{R}} \gamma(z) m(\mathrm{d}s, \mathrm{d}z) = \int_0^t \int_Z \mathbf{1}_{D_u}(\zeta) \gamma\left(S_{u^-} K_1(\zeta;u,X_{u^-},S_{u^-})\right) \mathcal{N}(\mathrm{d}u, \mathrm{d}\zeta),$$

where $D_t := \{\zeta \in Z : K_1(\zeta;t,X_{t^-},S_{t^-}) \neq 0\}$. From now on we assume that

$$\mathbb{E}\left[\int_0^T \eta(D_t)\mathrm{d}t\right] < \infty, \quad \forall t \in [0,T]. \tag{5.9}$$

Remark 5.2. Recall that $\nu^{\mathbb{F}}(\mathrm{d}t, \mathrm{d}z)$ denotes the $(\mathbb{F}, \mathbf{P})$-predictable dual projection of the random measure $m(\mathrm{d}t, \mathrm{d}z)$. Under condition (5.9), it is proved in [10] and [2] that $\nu^{\mathbb{F}}(\mathrm{d}t, \mathrm{d}z)$, is absolutely continuous with respect to the Lebesgue measure, that is, $\nu^{\mathbb{F}}(\mathrm{d}t, \mathrm{d}z) = \nu^{\mathbb{F}}_t(\mathrm{d}z)\mathrm{d}t$ where, for any $\mathcal{A} \in \mathcal{B}(\mathbb{R})$, $\nu^{\mathbb{F}}_t(\mathcal{A}) = \eta(D^{\mathcal{A}}_t)$ with $D^{\mathcal{A}}_t := \{\zeta \in Z : K_1(\zeta;t,X_{t^-},S_{t^-}) \in \mathcal{A} \setminus \{0\}\}$.
In particular, $\nu^{\mathbb{F}}_t(\mathbb{R}) = \eta(D_t)$, where $D_t = D^{\mathbb{R}}_t$, for every $t \in [0,T]$, provides the $(\mathbb{F}, \mathbf{P})$-intensity of the point process $m((0,t] \times \mathbb{R})$ which counts the total number of jumps of S up to time t.

5.1.1 Structure conditions of the stock price S with respect to $\mathbb{F}$ and $\mathbb{H}$.

The canonical $\mathbb{F}$-decomposition of S with respect to $\mathbb{F}$ is given by

$$S_t = S_0 + M_t + \Gamma_t, \quad t \in [0,T],$$

where M is the square integrable $(\mathbb{F}, \mathbf{P})$-martingale given by

$$\begin{aligned}\mathrm{d}M_t &= S_t\sigma_1(t,S_t)\mathrm{d}W^1_t + S_{t^-} \int_Z K_1(\zeta;t,X_{t^-},S_{t^-}) \widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}\zeta) \\ &= S_t\sigma_1(t,S_t)\mathrm{d}W^1_t + \int_{\mathbb{R}} z(m(\mathrm{d}t, \mathrm{d}z) - \nu^{\mathbb{F}}_t(\mathrm{d}z)\mathrm{d}t)\end{aligned}$$

and Γ is the following $\mathbb{R}$-valued $\mathbb{F}$-predictable finite variation process

$$\begin{aligned}\mathrm{d}\Gamma_t &= S_{t^-}\left\{\mu_1(t, X_{t^-}, S_{t^-}) + \int_Z K_1(\zeta; t, X_{t^-}, S_{t^-})\eta(\mathrm{d}\zeta)\right\}\mathrm{d}t\\ &= \left\{S_{t^-}\mu_1(t, X_{t^-}, S_{t^-}) + \int_{\mathbb{R}} z\nu_t^{\mathbb{F}}(\mathrm{d}z)\right\}\mathrm{d}t.\end{aligned}$$

We note that the $\mathbb{F}$-predictable quadratic variation of M is absolutely continuous with respect to the Lebesgue measure, that is, $\mathrm{d}\langle M\rangle_t = a_t\ \mathrm{d}t$ with

$$a_t = S_{t^-}^2\left(\sigma_1^2(t, S_{t^-}) + \int_Z K_1^2(\zeta; t, X_{t^-}, S_{t^-})\eta(\mathrm{d}\zeta)\right) = S_{t^-}^2\sigma_1^2(t, S_{t^-}) + \int_{\mathbb{R}} z^2\nu_t^{\mathbb{F}}(\mathrm{d}z),$$

for every $t \in [0,T]$. Then, the semimartingale S satisfies the structure condition with respect to $\mathbb{F}$ given by

$$S_t = S_0 + M_t + \int_0^t \alpha_s^{\mathcal{F}}\mathrm{d}\langle M\rangle_s, \quad t \in [0,T]$$

where

$$\alpha_t^{\mathcal{F}} = \frac{\mu_1(t, X_{t^-}, S_{t^-}) + \int_Z K_1(\zeta; t, X_{t^-}, S_{t^-})\eta(\mathrm{d}\zeta)}{S_{t^-}\left(\sigma_1^2(t, S_{t^-}) + \int_Z K_1^2(\zeta; t, X_{t^-}, S_{t^-})\eta(\mathrm{d}\zeta)\right)} = \frac{S_{t^-}\mu_1(t, X_{t^-}, S_{t^-}) + \int_{\mathbb{R}} z\ \nu_t^{\mathbb{F}}(\mathrm{d}z)}{S_{t^-}^2\sigma_1^2(t, S_{t^-}) + \int_{\mathbb{R}} z^2\nu_t^{\mathbb{F}}(\mathrm{d}z)}, \tag{5.10}$$

for every $t \in [0,T]$.

Remark 5.3. Notice that, under the assumptions on the coefficients of the dynamics of S, $\alpha^{\mathcal{F}}$ is well defined and because of (5.9) also $\mathbb{E}\left[\int_0^T(\alpha_t^{\mathcal{F}})^2\mathrm{d}\langle M\rangle_t\right] < \infty$ is fulfilled. Indeed,

$$\begin{aligned}\mathbb{E}\left[\int_0^T(\alpha_t^{\mathcal{F}})^2\mathrm{d}\langle M\rangle_t\right] &\le \mathbb{E}\left[\int_0^T \frac{\mu_1(t, X_{t^-}, S_{t^-})}{\sigma_1^2(t, S_{t^-})}\mathrm{d}t + \int_0^T \frac{\left(\int_{\mathbb{R}} z\ \nu_t^{\mathbb{F}}(\mathrm{d}z)\right)^2}{\int_{\mathbb{R}} z^2\ \nu_t^{\mathbb{F}}(\mathrm{d}z)}\mathrm{d}t\right]\\ &\le T\frac{c_1^2}{c_2^2} + \mathbb{E}\left[\int_0^T \eta(D_t)\mathrm{d}t\right] < \infty.\end{aligned}$$

Now, define the process $I = \{I_t,\ t \in [0,T]\}$ by setting

$$I_t := W_t^1 + \int_0^t \frac{\mu_1(u, X_u, S_u) - {}^p\mu_1(u, X_u, S_u)}{\sigma_1(u, S_u)}\mathrm{d}u \tag{5.11}$$

for each $t \in [0,T]$. It is known that I is an $(\mathbb{H}, \mathbf{P})$-Brownian motion (see e.g. [19] and [23]). Moreover the $(\mathbb{H}, \mathbf{P})$-predictable dual projection of the measure $m(\mathrm{d}t, \mathrm{d}z)$ is given by $\nu_t^{\mathbb{H}}(\mathrm{d}t, \mathrm{d}z) = \nu_t^{\mathbb{H}}(\mathrm{d}z)\mathrm{d}t$; then, according to Proposition 3.2, S admits the structure condition with respect to $\mathbb{H}$ which is given by

$$S_t = S_0 + N_t + \int_0^t \alpha_s^{\mathcal{H}}\mathrm{d}\langle N\rangle_s, \quad t \in [0,T],$$

where

$$\begin{aligned}N_t &= \int_0^t S_r\sigma_1(r, S_r)\mathrm{d}I_r + \int_0^t\int_{\mathbb{R}} z(m(\mathrm{d}r, \mathrm{d}z) - \nu_r^{\mathbb{H}}(\mathrm{d}z)\mathrm{d}r), \quad t \in [0,T],\\ \alpha_t^{\mathcal{H}} &= \frac{S_{t^-}{}^p\mu_1(t, X_{t^-}, S_{t^-}) + \int_{\mathbb{R}} z\nu_t^{\mathbb{H}}(\mathrm{d}z)}{S_{t^-}^2\sigma_1^2(t, S_{t^-}) + \int_{\mathbb{R}} z^2\nu_t^{\mathbb{H}}(\mathrm{d}z)}, \quad t \in [0,T].\end{aligned}$$

5.1.2 The $\mathbb{H}$-pseudo optimal strategy

To introduce the minimal martingale measure $\mathbf{P}^*$ for the underlying market model, we assume (4.1) and

$$\alpha_t^{\mathcal{F}} \Delta M_t = K_1(\zeta; t, X_{t^-}, S_{t^-}) \frac{\mu_1(t, X_{t^-}, S_{t^-}) + \int_Z K_1(\zeta; t, X_{t^-}, S_{t^-}) \eta(\mathrm{d}\zeta)}{\sigma_1^2(t, S_{t^-}) + \int_Z K_1^2(\zeta; t, X_{t^-}, S_{t^-}) \eta(\mathrm{d}\zeta)} < 1, \quad t \in [0, T]. \tag{5.12}$$

Remark 5.4. A sufficient condition for the validity of (4.1) is given by

$$\mathbb{E}\left[\exp\left\{2 \int_0^T \eta(D_t)\mathrm{d}t\right\}\right] < \infty.$$

Indeed,

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T (\alpha_t^{\mathcal{F}})^2 \mathrm{d}\langle M^c\rangle_t + \int_0^T (\alpha_t^{\mathcal{F}})^2 \mathrm{d}\langle M^d\rangle_t\right\}\right] \\
&\quad \le \mathbb{E}\left[\exp\left\{\int_0^T \frac{\left(\mu_1(t, X_{t^-}, S_{t^-}) + \int_Z K_1(\zeta; t, X_{t^-}, S_{t^-}) \eta(\mathrm{d}\zeta)\right)^2}{\sigma_1^2(t, S_{t^-}) + \int_Z K_1^2(\zeta; t, X_{t^-}, S_{t^-}) \eta(\mathrm{d}\zeta)} \mathrm{d}t\right\}\right] \\
&\quad \le \mathbb{E}\left[\exp\left\{2\int_0^T \left(\frac{\mu_1^2(t, X_{t^-}, S_{t^-})}{\sigma_1^2(t, S_{t^-})} + \eta(D_t)\right)\mathrm{d}t\right\}\right] \\
&\quad \le \exp\left\{2\,\frac{c_1^2}{c_2^2}\,T\right\} \mathbb{E}\left[\exp\left\{2\int_0^T \eta(D_t)\mathrm{d}t\right\}\right].
\end{aligned}$$

Define the process L by setting $L_t = \mathcal{E}\left(-\int \alpha_r^{\mathcal{F}} \mathrm{d}M_r\right)_t$ for every $t \in [0, T]$. Under (5.12) and (4.1), L is an $(\mathbb{F}, \mathbf{P})$-martingale, and if in addition L is square integrable, then we can apply the Ansel-Stricker Theorem and define the change of probability measure $\left.\frac{\mathrm{d}\mathbf{P}^*}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_T} = L_T$.

Under the minimal martingale measure $\mathbf{P}^*$, the dynamics of the pair (X, S) can be written as

$$\begin{cases}
\mathrm{d}X_t = \mu_0(t, X_t)\mathrm{d}t + \sigma_0(t, X_t)\mathrm{d}W_t^0 + \int_Z K_0(\zeta; t, X_{t^-}) \mathcal{N}(\mathrm{d}t, \mathrm{d}\zeta), \quad X_0 = x \in \mathbb{R} \\
\mathrm{d}S_t = S_{t^-}\left\{\sigma_1(t, S_t)\mathrm{d}W_t^* + \int_Z K_1(\zeta; t, X_{t^-}, S_{t^-}) \widetilde{\mathcal{N}}^*(\mathrm{d}t, \mathrm{d}\zeta)\right\}, \quad S_0 = s > 0,
\end{cases}$$

where W^0, W^* are $(\mathbb{F}, \mathbf{P}^*)$-Brownian motions, with

$$W_t^* := W_t^1 + \int_0^t S_u \alpha_u^{\mathcal{F}} \sigma_1(u, S_u)\mathrm{d}u, \ t \in [0, T],$$

whose correlation coefficient is ρ, $\widetilde{\mathcal{N}}^*(\mathrm{d}t, \mathrm{d}\zeta)$ is the compensated Poisson measure under $\mathbf{P}^*$ given by

$$\widetilde{\mathcal{N}}^*(\mathrm{d}t, \mathrm{d}\zeta) := \mathcal{N}(\mathrm{d}t, \mathrm{d}\zeta) - \eta_t^*(\mathrm{d}\zeta)\mathrm{d}t$$

and $\eta_t^*(\mathrm{d}\zeta) = (1 - \alpha_t^{\mathcal{F}} S_t K_1(t, X_t, S_t))\eta(\mathrm{d}\zeta)$ for every $t \in [0, T]$, with $\alpha^{\mathcal{F}}$ given in (5.10).

In the sequel we assume that the following conditions are in force:

$$\mathbb{E}^{\mathbf{P}^*}\left[\int_0^T \left(|\mu_0(t, X_t)| + \sigma_0^2(t, X_t) + \eta_t^*(D_t^0) + \int_Z |K_0(\zeta; t, X_t)| \eta_t^*(\mathrm{d}\zeta)\right)\mathrm{d}t\right] < \infty, \tag{5.13}$$

$$\mathbb{E}^{\mathbf{P}^*}\left[\int_0^T \eta_t^*(D_t)\mathrm{d}t\right] < \infty, \tag{5.14}$$

where $D_t^0 = \{\zeta \in Z : K_0(\zeta; t, X_{t^-}) \neq 0\}$ and $D_t = \{\zeta \in Z : K_1(\zeta; t, X_{t^-}, S_{t^-}) \neq 0\}$ for every $t \in [0,T]$.

Since the change of probability measure is Markovian, the pair (X,S) is still an $(\mathbb{F}, \mathbf{P}^*)$-Markov process and we provide the structure of its $\mathbf{P}^*$-generator in the Proposition below.

Proposition 5.5. *Under conditions* (5.13) *and* (5.14)*, the pair* (X,S) *is an* $(\mathbb{F}, \mathbf{P}^*)$*-Markov process with generator*

$$\mathcal{L}^*_{X,S} f(t,x,s) = \frac{\partial f}{\partial t} + \mu_0(t,x)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma_0^2(t,x)\frac{\partial^2 f}{\partial x^2} + \rho\sigma_0(t,x)\sigma_1(t,s)s\frac{\partial^2 f}{\partial x \partial s}$$
$$+ \frac{1}{2}\sigma_1^2(t,s)\, s^2 \frac{\partial^2 f}{\partial s^2} + \int_Z \Delta f(\zeta; t,x,s)\eta_t^*(\mathrm{d}\zeta) - \frac{\partial f}{\partial s} s \int_Z K_1(\zeta; t,x,s)\eta_t^*(\mathrm{d}\zeta), \tag{5.15}$$

where

$$\Delta f(\zeta; t,x,s) = f\big(t, x + K_0(\zeta;t,x), s(1 + K_1(\zeta;t,x,s))\big) - f(t,x,s).$$

Moreover, the following semimartingale decomposition holds:

$$f(t, X_t, S_t) = f(0, x_0, s_0) + \int_0^t \mathcal{L}^*_{X,S} f(r, X_r, S_r)\mathrm{d}r + M_t^{3,f},\ t \in [0,T]$$

where $M^{3,f} = \{M_t^{3,f},\ t \in [0,T]\}$ *is the* $(\mathbb{F}, \mathbf{P}^*)$*-martingale given by*

$$\mathrm{d}M_t^{3,f} = \frac{\partial f}{\partial x}\sigma_0(t,X_t)\ \mathrm{d}W_t^0 + \frac{\partial f}{\partial s}\sigma_1(t,S_t)S_t\ \mathrm{d}W_t^* + \int_Z \Delta f(\zeta; t, X_{t^-}, S_{t^-})\widetilde{\mathcal{N}}^*(\mathrm{d}t, \mathrm{d}\zeta). \tag{5.16}$$

The proof is postponed to Appendix B.

As pointed out at the beginning of Section 5, we assume (5.4) to compute the $\mathbb{H}$-pseudo optimal strategy for the contingent claim $\xi = H(T, S_T)$. Note that, under the hypotheses on the coefficients of S, the random variable ξ admits a Föllmer-Schweizer decomposition. Therefore, allowing for the filter dynamics given in (A.2) in Appendix A, under (5.8), (5.13), (5.14), the corresponding $\mathbb{H}$-pseudo optimal strategy can be written as

$$\beta_t^{\mathcal{H}} = \widetilde{\beta}_t^{\mathcal{H}} + \phi_t^{\mathcal{H}} \quad \mathbf{P} - a.s., \quad t \in [0,T]$$

where

$$\widetilde{\beta}_t^{\mathcal{H}} = \frac{\mathrm{d}\langle \pi(g), S\rangle_t^{*,\mathbb{H}}}{\mathrm{d}\langle S\rangle_t^{*,\mathbb{H}}} = \frac{S_{t^-}\sigma_1(t,S_{t^-})h_{t^-}(g) + \int_{\mathbb{R}} z\, w^g(t,z)\nu_t^{\mathbb{H},*}(\mathrm{d}z)}{S_{t^-}^2\sigma_1^2(t,S_{t^-}) + \int_{\mathbb{R}} z^2\nu_t^{\mathbb{H},*}(\mathrm{d}z)}, \quad t \in [0,T], \tag{5.17}$$

$$\phi_t^{\mathcal{H}} = \frac{\mathrm{d}^{\mathbb{H}}\langle \sum_{r\le \cdot} \Delta G_r \Delta S_r, \int_0^\cdot \alpha_r^{\mathcal{H}}\mathrm{d}N_r\rangle_t}{\mathrm{d}^{\mathbb{H}}\langle S\rangle_t} = \frac{\alpha_t^{\mathcal{H}}\int_{\mathbb{R}} z^2 \left(w^g(t,z) - \widetilde{\beta}_t^{\mathcal{H}}\, z\right)\nu_t^{\mathbb{H}}(\mathrm{d}z)}{S_{t^-}^2\sigma_1^2(t,S_{t^-}) + \int_{\mathbb{R}} z^2\nu_t^{\mathbb{H}}(\mathrm{d}z)}, \quad t \in [0,T]. \tag{5.18}$$

Here $h_t(g)$ and $w^g(t,z)$ are defined in (A.3) and (A.4) in Appendix A, respectively, with the choice $f = g$ and $g(t,x,s)$ is the solution of (5.3), and $G = \{G_t,\ t \in [0,T]\}$ is the process given by:

$$G_t = \int_0^t \left(h_u(g) - \tilde{\beta}_u^{\mathcal{H}} S_u \sigma_1(u,S_u)\right)\mathrm{d}I_u^* + \int_0^t \left(w^g(u,z) - \widetilde{\beta}_u^{\mathcal{H}}\, z\right)\left(m(\mathrm{d}u,\mathrm{d}z) - \nu_u^{\mathbb{H},*}(\mathrm{d}z)\mathrm{d}u\right),$$

for every $t \in [0,T]$, where I^* is the $(\mathbb{H}, \mathbf{P}^*)$-Brownian motion defined in (A.1) in Appendix A.

It is worth observing that the $(\mathbb{H}, \mathbf{P})$-predictable dual projection $\nu_t^{\mathbb{H}}(\mathrm{d}z)\mathrm{d}t$ of the measure $m(\mathrm{d}t, \mathrm{d}z)$ appearing in (5.18), can be written in terms of the filter under the real-world probability measure $\mathbf{P}$. Indeed, set $\widetilde{\pi}_t(f) := \mathbb{E}\left[f(t,X_t,S_t)|\mathcal{H}_t\right]$, for every $t \in [0,T]$. Then, $\nu_t^{\mathbb{H}}(\mathrm{d}z) = \widetilde{\pi}_{t^-}(\nu^{\mathbb{F}}(\mathrm{d}z))$(see again [2, Proposition 2.2] for the proof). Therefore, in presence of jumps we also need the knowledge of the filter dynamics under $\mathbf{P}$. The Kushner-Stratonovich equation satisfied by $\widetilde{\pi}$ is given by (A.11) in Appendix A.

In the sequel we can easily deduce by (5.18) the $\mathbb{H}$-pseudo optimal strategy in a diffusion and in a pure jump market model.

Remark 5.6 (A diffusion market model)**.** Consider a partially observable diffusion market model described by

$$\begin{cases} \mathrm{d}X_t = \mu_0(t,X_t)\mathrm{d}t + \sigma_0(t,X_t)\mathrm{d}W_t^0, \quad X_0 = x \in \mathbb{R}, \\ \mathrm{d}S_t = S_t\left(\mu_1(t,X_t,S_t)\mathrm{d}t + \sigma_1(t,S_t)\mathrm{d}W_t^1\right), \quad S_0 = s > 0, \end{cases} \tag{5.19}$$

with the same notation introduced above, and analogous assumptions. Then, taking the filtering equation for the diffusion case given by (A.7) in Appendix A into account, the $\mathbb{H}$-pseudo optimal strategy has the following expression:

$$\beta_t^{\mathcal{H}} = \frac{\mathrm{d}^{\mathbb{H}}\langle \pi(g), S\rangle_t}{\mathrm{d}^{\mathbb{H}}\langle S\rangle_t} = \frac{h_{t^-}(g)}{S_{t^-}\sigma_1(t,S_{t^-})} = \frac{\rho\pi_{t^-}\left(\sigma_0\frac{\partial g}{\partial x}\right) + S_{t^-}\sigma_1(t,S_{t^-})\pi_{t^-}\left(\frac{\partial g}{\partial s}\right)}{S_{t^-}\sigma_1(t,S_{t^-})} \quad \mathbf{P}-a.s., \tag{5.20}$$

for every $t \in [0,T]$, where $h_t(g)$ is defined in (A.8) with the choice $f = g$ and $g(t,x,s)$ is the solution of the problem (5.3), with $\mathcal{L}^*_{X,S} = \mathcal{L}^1_{X,S}$ being the Markov generator of the pair (X,S), given by

$$\begin{aligned} \mathcal{L}^1_{X,S}f(t,x,s) =& \frac{\partial f}{\partial t} + \mu_0(t,x)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma_0^2(t,x)\frac{\partial^2 f}{\partial x^2} \\ &+ \rho\sigma_0(t,x)\sigma_1(t,s)s\frac{\partial^2 f}{\partial x\partial s} + \frac{1}{2}\sigma_1^2(t,s)\,s^2\frac{\partial^2 f}{\partial s^2}. \end{aligned} \tag{5.21}$$

Notice that, in this case the $\mathbb{H}$-pseudo-optimal strategy can also be obtained via (4.15); indeed, consider the process $V^{\mathcal{F}} = \{V_t^{\mathcal{F}},\ t \in [0,T]\}$ given by

$$V_t^{\mathcal{F}} = \mathbb{E}^{\mathbf{P}^*}\left[H(T,S_T)|\mathcal{F}_t\right], \qquad \forall t \in [0,T]$$

thanks to Corollary 4.4 with the choice $\mathbb{H} = \mathbb{F}$; consequently, we get that

$$\beta_t^{\mathcal{F}} = \frac{\mathrm{d}^{\mathbb{F}}\langle V^{\mathcal{F}}, S\rangle_t}{\mathrm{d}^{\mathbb{F}}\langle S\rangle_t}, \quad \mathbf{P}-a.s.,\ t \in [0,T].$$

We observe that $V^{\mathcal{F}}$ coincides with the process $\{g(t,X_t,S_t),\ t \in [0,T]\}$; then by Itô's formula we get

$$V_t^{\mathcal{F}} = g(t,X_t,S_t) = \int_0^t \left\{\frac{\partial g}{\partial x}\sigma_0(u,X_u)\mathrm{d}W_u^0 + \frac{\partial g}{\partial s}\sigma_1(u,S_u)S_u\mathrm{d}\widetilde{W}_u\right\}, \quad t \in [0,T],$$

and computing explicitly the sharp brackets ${}^{\mathbb{F}}\langle V^{\mathcal{F}}, S\rangle$ and ${}^{\mathbb{F}}\langle S\rangle$, we obtain

$$\beta_t^{\mathcal{F}} = \frac{\rho\sigma_0(t,X_{t^-})\frac{\partial g}{\partial x} + S_{t^-}\sigma_1(t,S_{t^-})\frac{\partial g}{\partial s}}{S_{t^-}\sigma_1(t,S_{t^-})}, \quad t \in [0,T].$$

Finally, taking (5.20) and the definition of the filter into account, we get that $\beta^{\mathcal{H}} = {}^{p,*}\beta^{\mathcal{F}}$, where ${}^{p,*}\beta^{\mathcal{F}}$ is the $(\mathbb{H},\mathbf{P}^*)$-predictable projection of the process $\beta^{\mathcal{F}}$.

Remark 5.7 (A pure jump market model)**.** Assume now that the stock price S is described by a pure jump process and consider the following partially observable system:

$$\begin{cases} \mathrm{d}X_t = \mu_0(t,X_t)\mathrm{d}t + \sigma_0(t,X_t)\mathrm{d}W_t^0 + \int_Z K_0(\zeta;t,X_{t^-})\mathcal{N}(\mathrm{d}t,\mathrm{d}\zeta), \quad X_0 = x \in \mathbb{R} \\ \mathrm{d}S_t = S_{t^-}\int_Z K_1(\zeta;t,X_{t^-},S_{t^-})\mathcal{N}(\mathrm{d}t,\mathrm{d}\zeta), \quad S_0 = s > 0, \end{cases} \tag{5.22}$$

with the same notations of the general jump diffusion case and analogous assumptions. Then taking the filter dynamics for the pure jump model given by (A.10) in Appendix A into account, the $\mathbb{H}$-pseudo optimal strategy is given by

$$\beta_t^{\mathcal{H}} = \widetilde{\beta}_t^{\mathcal{H}} + \phi_t^{\mathcal{H}}, \quad \mathbf{P}-a.s., \quad t \in [0,T],$$

where

$$\widetilde{\beta}_t^{\mathcal{H}} = \frac{\int_{\mathbb{R}} z\, w^g(t,z)\, \nu_t^{\mathbb{H},*}(\mathrm{d}z)}{\int_{\mathbb{R}} z^2\, \nu_t^{\mathbb{H},*}(\mathrm{d}z)}, \quad \phi_t^{\mathcal{H}} = \frac{\alpha_t^{\mathcal{H}} \int_{\mathbb{R}} z^2 \left\{ w^g(t,z) - \widetilde{\beta}_t^{\mathcal{H}} z \right\} \nu_t^{\mathbb{H}}(\mathrm{d}z)}{\int_{\mathbb{R}} z^2 \nu_t^{\mathbb{H}}(\mathrm{d}z)}, \quad t \in [0,T].$$

Here $g(t,x,s)$ is the solution to (5.3) with $\mathcal{L}^*_{X,S} = \mathcal{L}^2_{X,S}$ where $\mathcal{L}^2_{X,S}$ is the Markov generator of the pair (X,S) given by

$$\begin{aligned} \mathcal{L}^2_{X,S} f(t,x,s) = {} & \frac{\partial f}{\partial t} + \mu_0(t,x)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma_0^2(t,x)\frac{\partial^2 f}{\partial x^2} \\ & + \int_Z \Delta f(\zeta;t,x,s)\eta_t^*(\mathrm{d}\zeta) - \frac{\partial f}{\partial s} s \int_Z K_1(\zeta;t,x,s)\eta_t^*(\mathrm{d}\zeta), \end{aligned} \tag{5.23}$$

with

$$\Delta f(\zeta;t,x,s) := f\big(t, x + K_0(\zeta;t,x), s(1+K_1(\zeta;t,x,s))\big) - f(t,x,s),$$

and w^g is given by (A.5) replacing f with g.

5.1.3 A practical example: the $\mathbb{H}$-pseudo-optimal hedging strategy for a European put option

Now we consider the following simplified model for the discounted asset price

$$\mathrm{d}S_t = S_{t^-}\Big(\mu_1(X_t)\mathrm{d}t + \sigma_1 \mathrm{d}W_t^1 + \gamma(X_{t^-})(\mathrm{d}p_t - \mathrm{d}t)\Big), \quad S_0 = s > 0, \tag{5.24}$$

where W^1 is a standard Brownian motion and p is a standard Poisson process (with intensity 1) independent of W^1. Assume that the volatility σ_1 is a positive constant, and that $\mu_1(x)$ and $\gamma(x)$ are deterministic functions, with $\gamma(x) > -1$ for almost every $x \in \mathbb{R}$. Here the expected return rate and the jump sizes coefficient depend on an unobservable stochastic factor X that may represent the behaviour of a second risky asset price process, which is not negotiated on the market. We model the process X as

$$\mathrm{d}X_t = \gamma_0(X_{t^-})\mathrm{d}p_t^0, \quad X_0 = x \in \mathbb{R}, \tag{5.25}$$

where $\gamma_0(x)$ is a deterministic function and p^0 is a standard Poisson process (with intensity 1) independent of W^1 and p. This kind of equation for the dynamics of the unobservable process X is typically employed to model high frequency data.

Let us observe that when $\mu_1(x) = \mu_1 \in \mathbb{R}$ and $\gamma(x) = \gamma \in \mathbb{R}$, this model reduces to the example under full information considered in [12, Section 3.1].

To deduce the structure condition satisfied by the price process S note that its $(\mathbb{F}, \mathbf{P})$-martingale part M, the predictable quadratic variation $\langle M \rangle$ and $\alpha^{\mathcal{F}}$ are respectively given by

$$M_t = \int_0^t S_{u^-}\Big(\sigma_1 \mathrm{d}W^1_u + \gamma(X_u)(\mathrm{d}p_u - \mathrm{d}u)\Big), \quad \langle M \rangle_t = \int_0^t S^2_u(\sigma_1^2 + \gamma^2(X_u))\mathrm{d}u, \quad t \in [0,T],$$

$$\alpha^{\mathcal{F}}_t = \frac{\mu_1(X_{t^-})}{S_{t^-}(\sigma_1^2 + \gamma^2(X_{t^-}))}, \quad t \in [0,T].$$

Assume that $\mu_1(x)$ and $\gamma(x)$ are bounded for almost every $x \in \mathbb{R}$, and satisfy $\mu_1(x)\gamma(x) < \sigma_1^2 + \gamma^2(x)$; then the process $L = \{L_t,\ t \in [0,T]\}$ given by

$$L_t = \mathcal{E}\left(-\int \alpha^{\mathcal{F}}_u \mathrm{d}M_u\right)_t, \quad t \in [0,T],$$

is a square integrable $(\mathbb{F}, \mathbf{P})$-martingale and defines the density of the minimal martingale measure $\mathbf{P}^*$, that is $\frac{\mathrm{d}\mathbf{P}^*}{\mathrm{d}\mathbf{P}}\Big|_{\mathcal{F}_T} = L_T$. Moreover, the process S is an $(\mathbb{F}, \mathbf{P}^*)$-martingale satisfying

$$\mathrm{d}S_t = S_{t^-}\left\{\sigma_1 \mathrm{d}W^*_t + \gamma(X_{t^-})\Big(\mathrm{d}p_t - (1 + \gamma_1(X_{t^-})\ )\mathrm{d}t\Big)\right\}, \quad S_0 = s > 0, \qquad (5.26)$$

where the process $W^* = \{W^*_t,\ t \in [0,T]\}$ given by

$$W^*_t = W^1_t + \int_0^t \frac{\mu_1(X_s)\sigma_1}{\sigma_1^2 + \gamma^2(X_s)}\mathrm{d}s, \quad t \in [0,T],$$

is an $(\mathbb{F}, \mathbf{P}^*)$-Brownian motion and p is a point process with $(\mathbb{F}, \mathbf{P}^*)$-predictable intensity

$$1 + \gamma_1(X_{t^-}) = 1 - \frac{\mu_1(X_{t^-})\gamma(X_{t^-})}{\sigma_1^2 + \gamma^2(X_{t^-})}, \quad t \in [0,T].$$

Consider a European put option with strike price K, whose payoff is given by $\xi = H(T, S_T) = (K - S_T)^+$. By the Markov property and the stationarity of X, under $\mathbf{P}^*$ we get that

$$\begin{aligned} g(t,x,s) &= \mathbb{E}^{\mathbf{P}^*}\left[\left(K - s\frac{S_T}{S_t}\right)^+ \middle| X_t = x\right] \\ &= KF(T-t, x, \log(K/s)) - s\int_0^{\log\left(\frac{K}{s}\right)} e^y F_y(T-t,x,y)\mathrm{d}y \end{aligned}$$

where, for any fixed $x \in \mathbb{R}$, $F(t,x,y)$ denotes the following distribution function:

$$F(t,x,y) = \mathbf{P}^*\left(\sigma_1 W^*_t - \frac{1}{2}\sigma_1^2 t + \int_0^t \log(1 + \gamma(X^x_{s^-}))\mathrm{d}p_s \le y\right)$$

and X^x is the solution of (5.25) with initial condition $X_0 = x$.

To get the $\mathbb{H}$-pseudo optimal strategy for our model we notice that the integrand of the Galtchouk-Kunita-Watanabe decomposition of ξ under $\mathbf{P}^*$, is given by

$$\widetilde{\beta}^{\mathcal{H}}_t = \frac{S_{t^-}\sigma_1^2\pi_{t^-}\left(\frac{\partial g}{\partial s}\right) + \pi_{t^-}(g\gamma(1+\gamma_1)) - \pi_{t^-}(g)\pi_{t^-}(\gamma(1+\gamma_1)) + \pi_{t^-}(\Delta g\gamma(1+\gamma_1))}{S_{t^-}(\sigma_1^2 + \widetilde{\pi}_{t^-}(\gamma^2(1+\gamma_1)))},$$

for every $t \in [0,T]$. Here $\Delta g(t,x,s) := g(t,x,s(1+\gamma(x))) - g(t,x,s)$. Moreover,

$$\phi_t^{\mathcal{H}} = \frac{\widetilde{\pi}_{t^-}(\mu_1+\gamma)\left\{\widetilde{\pi}_{t^-}(\gamma^2)\Psi_{t^-} - \widetilde{\beta}_t^{\mathcal{H}}\widetilde{\pi}_{t^-}(\gamma^3)\pi_{t^-}(1+\gamma_1)S_{t^-}\right\}}{S_{t^-}(\sigma_1^2+\widetilde{\pi}_{t^-}(\gamma^2))^2\pi_{t^-}(1+\gamma_1)}, \quad t\in[0,T]$$

where $\Psi_t = \pi_t(g(1+\gamma_1)) - \pi_t(g)\pi_t((1+\gamma_1)) + \pi_t(\Delta g(1+\gamma_1))$ for every $t\in[0,T]$.

Then $\mathbb{H}$-pseudo optimal strategy is given by $\beta^{\mathcal{H}} = \widetilde{\beta}^{\mathcal{H}} + \phi^{\mathcal{H}}$, $\mathbf{P}-a.s.$.

Note that when $\mu_1(x)=\mu_1\in\mathbb{R}$ and $\gamma(x)=\gamma\in\mathbb{R}$, the expressions for integrands in the Galtchouk-Kunita-Watanabe decomposition under $\mathbf{P}^*$, $\widetilde{\beta}^{\mathcal{H}}$, and in the Föllmer-Schweizer decomposition under $\mathbf{P}$, $\beta^{\mathcal{H}}$, reduce to those obtained in [12, Section 3.1] under full information.

Finally we show how to compute the filter under $\mathbf{P}^*$ and under $\mathbf{P}$, given by π and $\widetilde{\pi}$ respectively, for this example.

We assume that X takes values in a finite space $\mathcal{S}=\{x_i,...,x_d\}$ with $x_i\in\mathbb{R}$ for every $i=1,...,d$. Hence for any measurable function $f(t,x,s)$ such that $\mathbb{E}^{\mathbf{P}^*}\left[|f(t,X_t,S_t)|\right]<\infty$, $\forall t\in[0,T]$, we can write

$$\pi_t(f) = \sum_{i=1}^d f(t,x_i,S_t)\pi_t(f_i), \quad f_i(x)=\mathbf{1}_{\{x=x_i\}},\ i=1,...,d.$$

Then, in order to characterize the filter, it is sufficient to compute the dynamics of $\{\pi(f_i)\}_{i=1,\dots,d}$. Denoting by $\{T_n\}_{n\geq 0}$ the increasing sequence of jump times of S, we get that the Kushner-Stratonovich equation, see (A.2) below, for $t\in[T_n,T_{n+1})$ reduces to the following system of equations

$$\begin{aligned}\pi_t(f_i) =&\pi_{T_n}(f_i) + \int_{T_n}^t \left[\sum_{j=1}^d \pi_s(f_j)\mathbf{1}_{\{\gamma_0(x_j)=x_i-x_j\}} - \pi_s(f_i)\right]\mathrm{d}s\\ &- \int_{T_n}^t \left[\pi_s(f_i)\mathbf{1}_{\{\gamma(x_i)\neq 0\}}(1+\gamma_1(x_i)) - \pi_s(f_i)\sum_{j=1}^d \mathbf{1}_{\{\gamma(x_j)\neq 0\}}(1+\gamma_1(x_j))\pi_s(f_j)\right]\mathrm{d}s,\end{aligned}$$

for $i=1,...,d$, where

$$\pi_{T_n}(f_i) = \frac{\pi_{T_n^-}(f_i)\mathbf{1}_{\{\gamma(x_i)\neq 0\}}(1+\gamma_1(x_i))}{\sum_{j=1}^d \pi_{T_n^-}(f_j)\mathbf{1}_{\{\gamma(x_j)\neq 0\}}(1+\gamma_1(x_j))}, \quad i=1,...,d.$$

Note that $\{\pi_{T_n}(f_i)\}_{i=1,\dots,d}$ is completely determined by the knowledge of $\{\pi_t(f_i)\}_{i=1,\dots,d}$, for $t\in[T_{n-1},T_n)$, since $\pi_{T_n^-}(f_i)=\lim_{t\to T_n^-}\pi_t(f_i)$.

Analogously, we get that $\widetilde{\pi}_t(f) = \sum_{i=1}^d f(t,x_i,S_t)\widetilde{\pi}_t(f_i)$ with $f_i(x)=\mathbf{1}_{\{x=x_i\}}$, and the equation satisfied by $\{\widetilde{\pi}(f_i)\}_{i=1,\dots,d}$ (see equation (A.11) below) becomes

$$\begin{aligned}\widetilde{\pi}_t(f_i) &= \widetilde{\pi}_{T_n}(f_i) + \frac{1}{\sigma_1}\int_{T_n}^t \left[\widetilde{\pi}_s(f_i)(\mu_1(x_i)-\gamma(x_i)) - \widetilde{\pi}_s(f_i)\sum_{j=1}^d \widetilde{\pi}_s(f_j)(\mu_1(x_j)-\gamma(x_j))\right]\mathrm{d}I_s\\ &+ \int_{T_n}^t \left[\sum_{j=1}^d \widetilde{\pi}_s(f_j)\mathbf{1}_{\{\gamma_0(x_j)=x_i-x_j\}} - \widetilde{\pi}_s(f_i)\left(1+\mathbf{1}_{\{\gamma(x_i)\neq 0\}} - \sum_{j=1}^d \mathbf{1}_{\{\gamma(x_j)\neq 0\}}\widetilde{\pi}_s(f_j)\right)\right]\mathrm{d}s\end{aligned}$$

for $i=1,...,d$, and for every $t\in[T_n,T_{n+1})$, where the process $I=\{I_t,\ t\in[0,T]\}$ is the $(\mathbb{H},\mathbf{P})$-Brownian motion defined by

$$I_t = W_t^1 + \frac{1}{\sigma_1}\int_0^t \left[(\mu_1(X_s)-\gamma(X_s)) - \widetilde{\pi}_s(\mu_1-\gamma)\right]\mathrm{d}s,$$

and

$$\widetilde{\pi}_{T_n}(f_i) = \frac{\widetilde{\pi}_{T_n^-}(f_i)\mathbf{1}_{\{\gamma(x_i)\neq 0\}}}{\sum_{j=1}^d \widetilde{\pi}_{T_n^-}(f_j)\mathbf{1}_{\{\gamma(x_j)\neq 0\}}}, \quad i = 1, ..., d.$$

A more tractable equation can be obtained by considering the unnormalized version of the filter $\{v_t(f_i)\}_{i=1,\dots,d}$ ($\{\widetilde{v}_t(f_i)\}_{i=1,\dots,d}$, respectively) which can be characterized as the unique solution of linear systems between two consecutive jump times and satisfies $\pi_t(f_i) = \frac{v_t(f_i)}{v_t(1)}$ ($\widetilde{\pi}_t(f_i) = \frac{\widetilde{v}_t(f_i)}{\widetilde{v}_t(1)}$, respectively), see for instance [5, 18] and references therein.

A The filtering equation

We recall that the filter with respect to the minimal martingale measure $\mathbf{P}^*$ is given by

$$\pi_t(f) = \mathbb{E}^{\mathbf{P}^*}[f(t, X_t, S_t)|\mathcal{F}_t^S], \quad t \in [0, T]$$

for any measurable function $f(t, x, s)$ such that $\mathbb{E}^{\mathbf{P}^*}[|f(t, X_t, S_t)|] < \infty$, for every $t \in [0, T]$.

Here, we derive the filter dynamics for the jump-diffusion model and deduce the equations for the continuous model and for the pure jump one, as particular cases. Hence, using the same notations of Section 5.1, we assume that the dynamics of the pair signal-observation under $\mathbf{P}^*$ is given by

$$\begin{cases} \mathrm{d}X_t = \mu_0(t, X_t)\mathrm{d}t + \sigma_0(t, X_t)\mathrm{d}W_t^0 + \displaystyle\int_Z K_0(\zeta; t, X_{t^-})\mathcal{N}(\mathrm{d}t, \mathrm{d}\zeta), \quad X_0 = x \in \mathbb{R} \\ \mathrm{d}S_t = S_{t^-}\left\{\sigma_1(t, S_t)\mathrm{d}W_t^* + \displaystyle\int_Z K_1(\zeta; t, X_{t^-}, S_{t^-})\widetilde{\mathcal{N}}^*(\mathrm{d}t, \mathrm{d}\zeta)\right\}, \quad S_0 = s > 0. \end{cases}$$

We assume (5.13) and (5.14), in addition to (5.8), which in particular imply that the processes X and S have finite first moment under $\mathbf{P}^*$. We recall that the jump part of the process S can be described by the integer-valued random measure $m(\mathrm{d}t, \mathrm{d}z)$ defined in (5.1). We denote by $\nu_t^{\mathbb{F},*}(\mathrm{d}z)\mathrm{d}t$ its $(\mathbb{F}, \mathbf{P}^*)$-predictable dual projection and by $\nu_t^{\mathbb{H},*}(\mathrm{d}z)\mathrm{d}t$ its $(\mathbb{H}, \mathbf{P}^*)$-predictable dual projection and the following relation holds

$$\nu_t^{\mathbb{H},*}(\mathrm{d}z)\mathrm{d}t = \pi_{t^-}(\nu^{\mathbb{F},*}(\mathrm{d}z))\mathrm{d}t$$

thanks to [2, Proposition 2.2].

Remark A.1. An essential tool to derive the filtering equation is represented by the Martingale Representation Theorem (see [4, Proposition 2.6]). In particular, it states that every $(\mathbb{H}, \mathbf{P}^*)$-local martingale admits the following representation

$$M_t = M_0 + \int_0^t h_u \mathrm{d}I_t^* + \int_0^t \int_{\mathbb{R}} w(u, z)\left(m(\mathrm{d}u, \mathrm{d}z) - \nu_u^{\mathbb{H},*}(\mathrm{d}z)\mathrm{d}t\right), \quad t \in [0, T],$$

for suitable $\mathbb{H}$-adapted and $\mathbb{H}$-predictable processes $h = \{h_t,\ t \in [0, T]\}$ and $w(\cdot, z) = \{w(t, z),\ t \in [0, T]\}$ for every $z \in \mathbb{R}$, satisfying

$$\int_0^T \left(h_t^2 + \int_{\mathbb{R}} |w(t, z)|\nu_t^{\mathbb{H},*}(\mathrm{d}z)\right)\mathrm{d}t < \infty \quad \mathbf{P}^* - \text{a.s.}$$

where $I^* = \{I_t^*,\ t \in [0, T]\}$ is the $(\mathbb{H}, \mathbf{P}^*)$-Brownian motion given by

$$I_t^* = W_t^* + \int_0^t \left\{\frac{b(u, X_u, S_u)}{\sigma_1(u, S_u)} - \pi_u\left(\frac{b}{\sigma_1}\right)\right\}\mathrm{d}u, \quad t \in [0, T] \tag{A.1}$$

with $b(t, X_t, S_t) = \displaystyle\int_Z K_1(\zeta; t, X_t, S_t)\ \eta_t^*(\mathrm{d}\zeta)$ for every $t \in [0, T]$.

The following result provides the filter dynamics.

Proposition A.2 (The filtering equation). *Under* (5.8)*,* (5.13) *and* (5.14) *the filter solves the Kushner-Stratonovich equation for every function* $f(t,x,s) \in \mathcal{C}_b^{1,2,2}([0,T]\times\mathbb{R}\times\mathbb{R}^+)$*, given by*

$$\pi_t(f) = f(0,x_0,s_0) + \int_0^t \pi_s(\mathcal{L}^*_{X,S} f)\mathrm{d}s + \int_0^t h_s(f)\mathrm{d}I^*_s + \int_0^t \int_{\mathbb{R}} w^f(s,z)(m(\mathrm{d}s,\mathrm{d}z) - \nu_s^{\mathbb{H},*}(\mathrm{d}z)\mathrm{d}s), \tag{A.2}$$

for every $t \in [0,T]$*, where*

$$h_t(f) = \rho\pi_t\left(\sigma_0 \frac{\partial f}{\partial x}\right) + S_t\sigma_1(t,S_t)\pi_t\left(\frac{\partial f}{\partial s}\right), \tag{A.3}$$

$$w^f(t,z) = \frac{\mathrm{d}\pi_{t^-}(f\nu^{\mathbb{F},*})}{\mathrm{d}\nu_t^{\mathbb{H},*}}(z) - \pi_{t^-}(f) + \frac{\mathrm{d}\pi_{t^-}(\mathcal{R}f)}{\mathrm{d}\nu_t^{\mathbb{H},*}}(z), \tag{A.4}$$

$\mathcal{L}^*_{X,S}$ *is given in* (5.15) *and* $\mathcal{R}f(t,x,s,\mathcal{A}) := \int_{d^{\mathcal{A}}(t,x,s)}\{f(t,x+K_0(\zeta;t,x),s(1+K_1(\zeta;t,x,s))) - f(t,x,s)\}\eta_t^*(\mathrm{d}\zeta)$*, for every* $\mathcal{A} \in \mathcal{B}(\mathbb{R})$*, where* $d^{\mathcal{A}}(t,x,s) = \{\zeta \in Z : K_1(\zeta;t,x,s) \in \mathcal{A}\setminus\{0\}\}$*.*

Proof. We consider the semimartingale $Z = \{Z_t = f(t,X_t,S_t),\ t\in[0,T]\}$ whose decomposition is given by

$$Z_t = f(0,X_0,S_0) + \int_0^t \mathcal{L}^*_{X,S} f(u,X_u,S_u)\mathrm{d}u + M_t^{3,f}, \quad t\in[0,T], \tag{A.5}$$

where $\mathcal{L}^*_{X,S}$ and $M^{3,f}$ are defined in (5.15) and (5.16), respectively. By taking the conditional expectation with respect to $\mathcal{H}_t$ in (A.5), we get

$$\pi_t(f) = \pi_0(f) + \int_0^t \pi_u(\mathcal{L}^*_{X,S} f)\mathrm{d}u + \widetilde{M}_t^f, \quad t\in[0,T],$$

where

$$\widetilde{M}_t^f = {}^{o,*}M_t^{3,f} + {}^{o,*}\left(\int_0^t \mathcal{L}^*_{X,S} f(u,X_u,S_u)\mathrm{d}u\right) - \int {}^{o,*}(\mathcal{L}^*_{X,S} f(u,X_u,S_u))\mathrm{d}u, \quad t\in[0,T],$$

and ${}^{o,*}Y$ is the $(\mathbb{H},\mathbf{P}^*)$-optional projection of a given process Y, and $\widetilde{M}^f$ is an $(\mathbb{H},\mathbf{P}^*)$-martingale. Thanks to Remark A.1, there exist an $\mathbb{H}$-adapted process $h(f)$ and an $\mathbb{H}$-predictable process w^f such that

$$\mathbb{E}^{\mathbf{P}^*}\left[\int_0^T \left(h_s^2(f) + \int_{\mathbb{R}} |w^f(s,z)|\nu_s^{\mathbb{H},*}(\mathrm{d}z)\right)\mathrm{d}s\right] < \infty$$

and

$$\begin{aligned}\widetilde{M}_t^f &= \pi_t(f) - \pi_0(f) - \int_0^t \pi_u(\mathcal{L}^*_{X,S} f)\mathrm{d}u \\ &= \int_0^t h_u(f)\mathrm{d}I^*_u + \int_0^t \int_{\mathbb{R}} w^f(u,z)\left(m(\mathrm{d}u,\mathrm{d}z) - \nu_u^{\mathbb{H},*}(\mathrm{d}z)\mathrm{d}u\right), \quad t\in[0,T].\end{aligned}$$

To identify the process $h(f)$ we define the process $\widetilde{W}^* = \{\widetilde{W}^*_t,\ t\in[0,T]\}$ by

$$\widetilde{W}^*_t := I^*_t + \int_0^t \frac{\pi_u(b)}{\sigma_1(u,S_u)}\mathrm{d}u, \quad t\in[0,T].$$

Then we compute ${}^{o,*}\left(Z\widetilde{W}^*\right)$ and ${}^{o,*}Z\widetilde{W}^*$ separately and since $\widetilde{W}^*$ is $\mathbb{H}$-adapted, the equality ${}^{o,*}\left(Z\widetilde{W}^*\right) = {}^{o,*}Z\widetilde{W}^*$ holds. By Itô's product rule, we have

$$\mathrm{d}(Z_t\widetilde{W}_t^*) = Z_t\mathrm{d}\widetilde{W}_t^* + \widetilde{W}_t^*\,\mathcal{L}_{X,S}^* f(t,X_t,S_t)\mathrm{d}t + \frac{\partial f}{\partial x}\sigma_0(t,X_t)\rho\mathrm{d}t + \frac{\partial f}{\partial s}S_t\sigma_1(t,S_t)\,\mathrm{d}t + \mathrm{d}M_t^1,$$

where $M^1 := \int \widetilde{W}_s^*\mathrm{d}M_s^{3,f}$ is an $(\mathbb{F},\mathbf{P}^*)$-local martingale. We now introduce an $\mathbb{H}$-localizing sequence for M^1:

$$\widetilde{\tau}_n = T\wedge\inf\left\{t : |\widetilde{W}_t^*| \geq n\right\}, \quad n\geq 1.$$

If we take the conditional expectation with respect to $\mathcal{H}_t$, on $\{t\leq\widetilde{\tau}_n\}$ we get

$$\mathrm{d}^{o,*}(Z_t\widetilde{W}_t^*) \quad = \quad {}^{o,*}\left(\widetilde{W}_t^*\,\mathcal{L}_{X,S}^* f(t,X_t,S_t) + \frac{\partial f}{\partial x}(t)\sigma_0(t,X_t)\rho + \frac{\partial f}{\partial s}S_t\sigma_1(t,S_t)\right)\mathrm{d}t + \mathrm{d}\widetilde{M}_t^1,$$

where $\widetilde{M}^1$ is an $(\mathbb{H},\mathbf{P}^*)$-local martingale. On the other hand

$$\mathrm{d}({}^{o,*}Z_t\widetilde{W}_t^*) = \left(\widetilde{W}_t^*\;{}^{o,*}\mathcal{L}_{X,S}^* f(t,X_t,S_t) + h_t(f)\right)\mathrm{d}t + \mathrm{d}M_t^2,$$

where M^2 is an $(\mathbb{H},\mathbf{P}^*)$-local martingale. By the equality ${}^{o,*}\left(Z\widetilde{W}^*\right) = {}^{o,*}Z\widetilde{W}^*$, the bounded variation terms must be equal, which means that

$$h_t(f) = \rho\pi_t\left(\sigma_0\frac{\partial f}{\partial x}\right) + S_t\sigma_1(t,S_t)\pi_t\left(\frac{\partial f}{\partial s}\right)$$

on $\{t\leq\widetilde{\tau}_n\}$. Now, when $n\to\infty$, $\widetilde{\tau}_n$ goes to T $\mathbf{P}$-a.s. and so the process $h_t(f)$ is completely defined for every $t\in[0,T]$. Following the same arguments of the proof of Theorem 3.2 in [4] we obtain the expression of $w^f(t,z)$. □

Remark A.3. Strong uniqueness for the solution of the filtering equation is analyzed in [4] and [5] for the pair signal-observation given by the system (5.7). These results can be applied to deduce suitable conditions which ensure strong uniqueness of the solution to the filtering equation (A.2) under the minimal martingale measure $\mathbf{P}^*$. In [5] the authors analyzed strong uniqueness for the Zakai equation solved by the unnormalized version of the filter, and the relation with pathwise uniqueness for the Kushner-Stratonovich equation. In particular, whenever the signal process X is a pure jump process taking values in a countable space, the Zakai equation can be solved recursively (see Section 5.3 in [5] and [9]) and pathwise uniqueness holds under the hypothesis that X takes values in a finite space or when X and S have only common jump times.

Remark A.4 (The diffusion market model)**.** The dynamics of the pair (X,S) in system (5.19) under the minimal martingale measure is given by

$$\begin{cases} \mathrm{d}X_t = \mu_0(t,X_t)\mathrm{d}t + \sigma_0(t,X_t)\mathrm{d}W_t^0, \quad X_0 = x\in\mathbb{R}, \\ \mathrm{d}S_t = S_t\sigma_1(t,S_t)\mathrm{d}\widetilde{W}_t, \quad S_0 = s > 0. \end{cases} \tag{A.6}$$

Now we observe that

$$W_t^* = W_t^1 + \int_0^t \frac{\mu_1(u,X_u,S_u)}{\sigma_1(u,S_u)}\mathrm{d}u = \widetilde{W}_t, \quad t\in[0,T],$$

then $I_t^* = I_t$, for every $t \in [0,T]$, with I given in (5.11). Therefore, the filtering equation becomes

$$\pi_t(f) = f(0, x_0, s_0) + \int_0^t \pi_s(\mathcal{L}^1_{X,S} f)\mathrm{d}s + \int_0^t h_s(f)\mathrm{d}I_s \tag{A.7}$$

for every $t \in [0,T]$ and for every function $f \in \mathcal{C}^{1,2,2}([0,T] \times \mathbb{R} \times \mathbb{R}^+)$, where

$$h_t(f) = \rho\pi_t\left(\sigma_0 \frac{\partial f}{\partial x}\right) + S_t\sigma_1(t, S_t)\pi_t\left(\frac{\partial f}{\partial s}\right), \quad t \in [0,T]. \tag{A.8}$$

Remark A.5 (The pure jump market model)**.** Under the minimal martingale measure the dynamics of the pair (X, S) in system (5.22) becomes

$$\begin{cases} \mathrm{d}X_t = \mu_0(t, X_t)\mathrm{d}t + \sigma_0(t, X_t)\mathrm{d}W_t^0 + \int_Z K_0(\zeta; t, X_{t^-})\mathcal{N}(\mathrm{d}t, \mathrm{d}\zeta), \quad X_0 = x \in \mathbb{R} \\ \mathrm{d}S_t = S_{t^-}\int_Z K_1(\zeta; t, X_{t^-}, S_{t^-})\widetilde{\mathcal{N}}^*(\mathrm{d}t, \mathrm{d}\zeta), \quad S_0 = s > 0. \end{cases} \tag{A.9}$$

Then the filter can be characterized by

$$\pi_t(f) = f(0, x_0, s_0) + \int_0^t \pi_s(\mathcal{L}^2_{X,S} f)\mathrm{d}s + \int_0^t \int_{\mathbb{R}} w^f(s,z)(m(\mathrm{d}s, \mathrm{d}z) - \nu_s^{\mathbb{H},*}(\mathrm{d}z)\mathrm{d}s), \tag{A.10}$$

where

$$w^f(t,z) = \frac{\mathrm{d}\pi_{t^-}(f\nu^{\mathbb{F},*})}{\mathrm{d}\nu_t^{\mathbb{H},*}}(z) - \pi_{t^-}(f) + \frac{\mathrm{d}\pi_{t^-}(\mathcal{R}f)}{\mathrm{d}\nu_t^{\mathbb{H},*}}(z).$$

In [10] an explicit representation of the filter is obtained by the Feynman-Kac formula using a linearization method. This representation allows one to provide a recursive algorithm for the computation of the filter.

A.1 The filtering equation under the real-world probability measure

As pointed out in Section 5.1, to derive the $\mathbb{H}$-pseudo optimal strategy we also need to compute $\nu_t^{\mathbb{H}}(\mathrm{d}z)\mathrm{d}t$ which is the $(\mathbb{H}, \mathbf{P})$-predictable dual projection of the integer valued random measure $m(\mathrm{d}t, \mathrm{d}z)$. We observed that $\nu_t^{\mathbb{H}}(\mathrm{d}z)\mathrm{d}t$ has a representation in terms of $\widetilde{\pi}$ which is the filter under the real-world probability measure $\mathbf{P}$, given by $\nu_t^{\mathbb{H}}(\mathrm{d}z) = \widetilde{\pi}_{t^-}(\nu^{\mathbb{F}}(\mathrm{d}z))$.

Under (5.8) and (5.13), (5.14), formulated under $\mathbf{P}$, by extending the results in [4], the filter $\widetilde{\pi}$ solves the following Kushner-Stratonovich equation

$$\begin{aligned} \widetilde{\pi}_t(f) = f(0, x_0, S_0) + \int_0^t \widetilde{\pi}_s(\widetilde{\mathcal{L}}_{X,S} f)\mathrm{d}s + \int_0^t \widetilde{h}_s(f)\mathrm{d}I_s \\ \int_0^t \int_{\mathbb{R}} \widetilde{w}^f(s,z)(m(\mathrm{d}s, \mathrm{d}z) - \widetilde{\pi}_{s^-}(\nu^{\mathbb{F}}(\mathrm{d}z))\mathrm{d}s), \end{aligned} \tag{A.11}$$

for every function $f \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R} \times \mathbb{R}^+)$ and for every $t \in [0,T]$, where

$$\widetilde{w}^f(t,z) = \frac{\mathrm{d}\widetilde{\pi}_{t^-}(\nu^{\mathbb{F}} f)}{\mathrm{d}\widetilde{\pi}_{t^-}(\nu^{\mathbb{F}})}(z) - \widetilde{\pi}_{t^-}(f) + \frac{\mathrm{d}\widetilde{\pi}_{t^-}(\widetilde{\mathcal{R}} f)}{\mathrm{d}\widetilde{\pi}_{t^-}(\nu^{\mathbb{F}})}(z), \quad t \in [0,T]$$

$$\widetilde{h}_t(f) = \frac{\widetilde{\pi}_t(\mu_1 f) - \widetilde{\pi}_t(\mu_1)\widetilde{\pi}_t(f)}{\sigma_1(t, S_t)} + \rho\widetilde{\pi}_t\left(\sigma_0\frac{\partial f}{\partial x}\right) + S_t\sigma_1(t, S_t)\widetilde{\pi}_t\left(\frac{\partial f}{\partial s}\right), \quad t \in [0,T],$$

and I is the innovation process defined by (5.11). The operator $\widetilde{\mathcal{L}}_{X,S}$ denotes the generator of (X,S) under $\mathbf{P}$, which is given by

$$\widetilde{\mathcal{L}}_{X,S}f(t,x,s) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\mu_0(t,x) + \frac{\partial f}{\partial s}s\mu_1(t,x,s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma_0^2(t,x) + \frac{1}{2}\frac{\partial^2 f}{\partial s^2}s^2\sigma_1^2(t,s)$$
$$+ \frac{\partial^2 f}{\partial x \partial s}s\sigma_0(t,x)\sigma_1(t,s)\rho + \int_Z \Delta f(\zeta;t,x,s)\eta(\mathrm{d}\zeta)$$

with $\Delta f(\zeta;t,x,s) = f\big(t, x + K_0(\zeta;t,x), s(1 + K_1(\zeta;t,x,s))\big) - f(t,x,s)$, and for every $\mathcal{A} \in \mathcal{B}(\mathbb{R})$, the operator $\widetilde{\mathcal{R}}$, defined by

$$\widetilde{\mathcal{R}}f(t,x,s,\mathcal{A}) := \int_{d^{\mathcal{A}}(t,x,s)} \Delta f(\zeta;t,x,s)\eta(\mathrm{d}\zeta),$$

where $d^{\mathcal{A}}(t,x,s) = \{\zeta \in Z : K_1(\zeta;t,x,s) \in \mathcal{A} \setminus \{0\}\}$, takes common jump times between the signal X and the observation S into account.

Remark A.6 (The pure jump market model)**.** Clearly, we can deduce the filtering equation for the pure jump model as a particular case of equation (A.11), with $\widetilde{h}(f) = 0$ and

$$\widetilde{\mathcal{L}}_{X,S}f(t,x,s) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\mu_0(t,x) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma_0^2(t,x) + \int_Z \Delta f(\zeta;t,x,s)\eta(\mathrm{d}\zeta).$$

B Some proofs

Proof of Lemma 4.3. We denote by $(B^{\mathbb{F}}, C^{\mathbb{F}}, \nu^{\mathbb{F}})$ the $(\mathbb{F}, \mathbf{P})$-predictable characteristics of S (see [22] for more details) and by $(B^{\mathbb{H}}, C^{\mathbb{H}}, \nu^{\mathbb{H}})$ the $(\mathbb{H}, \mathbf{P})$-predictable characteristics of S.

Assume now that S has continuous trajectories. Then $\nu^{\mathbb{F}} = \nu^{\mathbb{H}} = 0$ and the $(\mathbb{F}, \mathbf{P})$-predictable characteristics of S are given by

$$B_t^{\mathbb{F}} = \int_0^t \alpha_u^{\mathcal{F}} \mathrm{d}\langle M \rangle_u \qquad C_t^{\mathbb{F}} = \langle S \rangle_t = \langle M \rangle_t, \quad t \in [0,T],$$

and the $(\mathbb{H}, \mathbf{P})$-predictable characteristics of S are

$$B_t^{\mathbb{H}} = \int_0^t \alpha_u^{\mathcal{H}} \mathrm{d}\langle N \rangle_u \qquad C_t^{\mathbb{H}} = \langle S \rangle_t = \langle N \rangle_t, \quad t \in [0,T].$$

We also recall that in the continuous trajectories case we get that $\alpha^{\mathcal{H}} = {}^p(\alpha^{\mathcal{F}})$ and $\langle S \rangle = \langle M \rangle = \langle N \rangle$.

This means that the $(\mathbb{H}, \mathbf{P})$-predictable characteristics of S can also be written as

$$B_t^{\mathbb{H}} = \int_0^t {}^p\alpha_u^{\mathcal{F}} \mathrm{d}\langle M \rangle_u \qquad C_t^{\mathbb{H}} = \langle S \rangle_t = \langle M \rangle_t, \quad t \in [0,T].$$

Using the definition of S we get that

$$S_t - S_0 = M_t + \int_0^t \alpha_u^{\mathcal{F}} \mathrm{d}\langle M \rangle_u = N_t + \int_0^t \alpha_u^{\mathcal{H}} \mathrm{d}\langle N \rangle_u, \quad t \in [0,T].$$

Hence, by the Girsanov theorem we get that S has $(\mathbb{F}, \mathbf{P}^*)$-predictable characteristics $(0, \langle M \rangle, 0)$ and since $\langle M \rangle = \langle N \rangle$, these are also the $(\mathbb{H}, \mathbf{P}^*)$-predictable characteristics of S.

Again, by the Girsanov theorem S has $(\mathbb{H}, \mathbf{P}^0)$-predictable characteristics given by $(0, \langle N \rangle, 0)$.

Therefore since the $(\mathbb{H}, \mathbf{P}^*)$-predictable characteristics of S coincide with its $(\mathbb{H}, \mathbf{P}^0)$-predictable characteristics and $\mathbf{P}^0|_{\mathcal{H}_0} = \mathbf{P}^*|_{\mathcal{H}_0}$, by [22, Chapter 3, Corollary 4.31] we can conclude that $\mathbf{P}^0$ is the restriction of $\mathbf{P}^*$ over $\mathbb{H}$. □

Proof of Proposition 5.5. Observe that the change of probability measure $\left.\frac{\mathrm{d}\mathbf{P}^*}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_T}$ is Markovian since $\alpha_t^{\mathcal{F}} = \alpha^{\mathcal{F}}(t, X_{t^-}, S_{t^-})$, for each $t \in [0,T]$ (see [11, Proposition 3.4]). Then the pair (X,S) is still an $(\mathbb{F}, \mathbf{P}^*)$-Markov process. To compute the generator $\mathcal{L}^*_{X,S}$, we apply Itô's formula to the function $f(t, X_t, S_t)$, and we get

$$f(t, X_t, S_t) = f(0, x_0, s_0) + \int_0^t \mathcal{L}^*_{X,S} f(r, X_r, S_r)\mathrm{d}r + M_t^{3,f},$$

where $\mathcal{L}^*_{X,S}$ is the operator in (5.15) and $M^{3,f}$ is given in (5.16). Moreover, under conditions (5.13), (5.14), the process $M^{3,f}$ is an $(\mathbb{F}, \mathbf{P}^*)$-martingale; indeed

$$\mathbb{E}^{\mathbf{P}^*}\left[\int_0^T \sigma_0^2(t, X_t)\left(\frac{\partial f}{\partial x}\right)^2 \mathrm{d}t\right] < \infty, \qquad \mathbb{E}^{\mathbf{P}^*}\left[\int_0^T \sigma_1^2(t, S_t) S_t^2 \left(\frac{\partial f}{\partial s}\right)^2 \mathrm{d}t\right] < \infty$$

and

$$\mathbb{E}^{\mathbf{P}^*}\left[\int_0^T \int_Z |\Delta f(\zeta; t, X_{t^-}, S_{t^-})| \eta_t^*(\mathrm{d}\zeta)\, \mathrm{d}t\right] \leq 2\|f\| \mathbb{E}^{\mathbf{P}^*}\left[\int_0^T \{\eta_t^*(D_t^0) + \eta_t^*(D_t)\}\mathrm{d}t\right] < \infty,$$

where $\|f\| = \sup\{f(t,x,s) | (t,x,s) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+\}$. □

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