Generalizing WDN simulation models to variable tank levels

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ABSTRACT

In water distribution network (WDN) steady-state modelling, tanks and reservoirs are modelled as nodes with known heads. As a result, the tank levels are upgraded after every steady-state simulation (snapshot) using external mass balance equations in extended period simulation (EPS). This approach can give rise to numerical instabilities, especially when tanks are in close proximity. In order to obtain a stable EPS model, an unsteady formulation of the WDN model has recently introduced. This work presents an extension of the steady-state WDN model, both for demand-driven and pressure-driven analyses, allowing the direct prediction of head variation of tank nodes with respect to an initial state. Head variations at those nodes are introduced as internal unknowns in the model, the variation of tank levels can be analyzed in the single steady-state simulation and EPS can be performed as a sequence of simulations without the need for external mass balances. The extension of mass balance at tank nodes allows the analysis of some technically relevant demand components. Furthermore, inlet and outlet head losses at tank nodes are introduced and large cross-sectional tank areas are allowed by the model and reservoirs become a special case of tanks. The solution algorithm is the generalized-global gradient algorithm (G-GGA), although the proposed WDN model generalization is universal.

Key words | extended period simulation, tank modelling, water distribution network, water network modelling, water network simulation

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NOMENCLATURE

$\bar{\mathbf{A}}_{pn}$	general topological matrix in the WDN model
$\mathbf{A}_{pn}$, $\mathbf{A}_{np}$, $\mathbf{A}_{p0}$	topological incidence sub-matrices in the WDN model
$\mathbf{A}_{pp}$	diagonal matrix in the WDN model
$\mathbf{B}_{pp}$	diagonal matrix used in the GGA or G-GGA
$\mathbf{d}_0^w$	vector of demands lumped in the nodes of tanks
$\mathbf{d}_0^{ext}$	vector of flow from external pipes to the WDN feeding tanks
$\mathbf{D}_{pp}$	derivative of head losses with respect to $\mathbf{Q}_p$
$\mathbf{D}_{nn}$	derivative of pressure-driven demands with respect to $\mathbf{H}_n$
$\mathbf{D}_0$	derivative of flow rate filling or emptying tanks with respect to $\Delta\mathbf{H}_0$
$\mathbf{F}_n$	temporary matrix used in the GGA or G-GGA
$\mathbf{H}_n$	vector of total network heads
$\mathbf{H}_0$	vector tank heads
$\mathbf{H}_0^{ini}$	vector of initial tank levels
$\mathbf{Q}_p$	vector of pipe flows/discharges
$\Delta\mathbf{H}_0$	vector of unknown levels of the tanks
Δt	time interval of the steady state of the G-GGA
$\mathbf{\Omega}_0$	column vector of cross-section area of the tanks

Operators and acronyms

EPS	extended period simulation
G-GGA	generalized-global gradient algorithm

doi: 10.2166/hydro.2011.224

GGA global gradient algorithm
WDN water distribution network

INTRODUCTION

Solutions to the steady state hydraulic network problem generally find their origin in the work of Hardy Cross who employed a local linearization method characterized by solution of the conservation equations one at a time (Cross 1936). Global linearization techniques, entailing the simultaneous solution of all the network equations, were developed about 40 years later and encompass work by several authors (Martin & Peters 1963; Shamir & Howard 1968; Epp & Fowler 1970; Hamam & Brammeller 1971; Kesavan & Chandrashekar 1972; Wood & Charles 1972; Collins *et al.* 1978; Isaacs & Mills 1980; Wood & Rayes 1981; Carpentier *et al.* 1987). Todini & Pilati (1988) developed the GGA which is used as the hydraulic solver in EPANET2 (Rossman 2000). Recently, GGA was enhanced by Giustolisi & Todini (2009) and Giustolisi (2010) in order to account for serial nodes without including them in the model network topology. All these algorithms apply to steady-state demand-driven simulation since withdrawals at nodes are assumed known *a priori*.

Relaxing the fixed demand assumption in network modelling was first undertaken by Bhave (1981) who considered the dependence of demand on system pressure. This was followed by Germanopoulos (1985) who combined a leakage term with pressure-dependent customer demands. Similar head-demand models were soon after proposed (Wagner *et al.* 1988; Reddy & Elango 1989; Chandapillai 1991). In particular, Wagner *et al.* (1988) suggested a generic pressure–demand model for controlled outlets that was indicated by Gupta & Bhave (1996) as the most feasible for predicting the performance of a water distribution system under pressure-deficient conditions.

As Wagner's model is hydraulically significant but not everywhere differentiable (Ackley *et al.* 2001), several methods to assure the differentiability of pressure–demand relationships were then developed (Tucciarelli *et al.* 1999; Tanyimboh *et al.* 2001; Tanyimboh & Templeman 2004).

For the same reason, Giustolisi *et al.* (2008a, b) introduced an adaptive over-relaxation parameter to pressure-driven analysis within GGA and Piller & van Zyl (2007, 2009) developed a pressure-driven WDN model using the 'content' and 'co-content' devised by Collins *et al.* (1978). Finally, Ang & Jowitt (2006) presented a different approach for analyzing water distribution systems under pressure-deficient conditions.

Despite generalizations to incorporate pressure-dependent demand, both the classical demand-driven and the recent head/pressure-driven steady-state WDN models assume that the nodal heads of tanks and reservoirs (i.e. the storage elements) are known. This assumption in EPS means the mass balance at the tank nodes is performed outside the steady-state runs in order to update the tank levels at each hydraulic snapshot. Such an approach, known as the Euler method, has been observed to generate model instability and convergence troubles in certain network configurations where the tanks are close to each other or when the hydraulic resistances of pipes connecting them are small. Various strategies were proposed to overcome the instability of the Euler method (Rao & Bree 1977; Rao *et al.* 1977; Bhave 1988; Brdys & Ulanicki 1994), including a recent proposal of van Zyl *et al.* (2006) to employ an 'explicit integration method'. As reported in Todini (2011), this decoupling of integration in time of the mass balance equations at the tank nodes from the steady-state network solutions is the cause of possible numerical instabilities. This corresponds to decoupling of the information in time (in the mass balance equations) from the information in space (in the energy balance equations) related to head variation of tanks. In other words, the topological positions of the tanks in the network, and the pipe hydraulic resistances of flow paths connecting them (spatial information), need to be coupled with the mass balance equations at the tank nodes. Todini (2011) discusses this issue. Furthermore, Avesani *et al.* (2011) report a case study in which such numerical instability occurred and describe their EPANET modification to deal with it.

Todini (2011) has recently proposed an unsteady WDN model formulation as a way to circumvent the instabilities that can emerge with the Euler method. In it, the steady parameters corresponding to the mass balance equations at time $t - \Delta t$ are used at time t, with an appropriate time averaging weight θ (set equal to 0.822), in order to compute the head variation of tanks.

This article presents a generalization of the WDN model based on the same premise of not decoupling time and space information in order to predict the head variation of tanks in the single steady-state snapshot. Thus, the G-GGA includes the unknown head variation of tanks with respect to their initial levels.

For this reason, the internal variable Δt is introduced as the time interval over which the boundary conditions of the steady-state snapshot can be kept constant, tank levels excluded. In this way, the proposed model allows accounting for the mass balances associated with tank nodes. As a result, the analysis of some demand components which are technically relevant, for example a fixed external supply, is permitted by the generalized WDN model.

The general formulation of the WDN model and of the solution algorithm (G-GGA) are developed and discussed from both mathematical/numerical and technical standpoints. The numerical stability of the G-GGA is then demonstrated using the same case study of Todini (2011), i.e. EPS condition. In addition, three tests involving the filling/emptying of the tanks at all of the nodes of the Apulian network (Giustolisi 2010) are reported as further evidence of the G-GGA's numerical stability.

GENERALIZATION OF THE WDN MODEL TO VARYING TANK LEVELS

Classical WDN modelling

The hydraulic model of a WDN is based on the following two equations which represent the energy and mass conservation laws for the network pipes and nodes, respectively,

$$\begin{aligned} & H_j - H_i + R_k Q_k |Q_k|^{n-1} + K_k^{ml} Q_k |Q_k| + \omega_k^{2-\gamma} r_k^p Q_k |Q_k|^{\gamma-1} \\ & \quad = \pm \omega_k^2 H_k^p \\ & \sum\nolimits_s \pm Q_{is} = d_i^w \end{aligned} \tag{1}$$

where H_i and H_j = unknown nodal heads at the upstream (ith) and downstream (jth) terminal nodes of the kth pipe according to the assumed positive direction for the unknown flow rate Q_k from node i to j; R_k = kth pipe hydraulic resistance; K_k^{ml} = minor loss (if any) along the kth pipe; ω_k and H_k^p, r_k^p, γ_k = speed factor, static head and parameters of internal head loss of the pump system (if any) along the kth pipe; Q_{is} = flow rates of the s pipes joining in the ith node and d_i^w = fixed demand at the ith node. The sign of Q_{is} depends on the assumed pipe positive direction for the sth pipe with respect to flow entering the node. The sign of ω_k depends on the installation direction of the pump with respect to the positive sign of the flow rate in the kth pipe. Note that the known terms in Equation (1) are moved to the right-hand side of the equations. Furthermore d_i^w can possibly be substituted by a head/pressure–demand relationship so that it becomes an unknown dependent on network status (i.e. $d_i^w = d_i^w(H)$), necessitating relocation to the left-hand side of mass balance equation. In the remainder of the text, $d_i^w(H)$ combines pressure–demand and leakage as, for example, in Giustolisi *et al.* (2008b).

When the ith node is a tank (i.e. the nodal head is assumed as known) the energy balance equation is modified by moving the known head $H_0 = H_i$ to the right-hand side,

$$\begin{aligned} & H_j + R_k Q_k |Q_k|^{n-1} + K_k^{ml} Q_k |Q_k| + \omega_k^{2-\gamma} r_k^p Q_k |Q_k|^{\gamma-1} \\ & \quad = H_0 \pm \omega_k^2 H_k^p \end{aligned} \tag{2}$$

while the mass balance equation disappears because it is not possible to fix both the head and demand in any node. This means that d_i^w (and eventually its pressure–demand and leakage components) disappears from the WDN model as the assumption of a fixed level means that any mass balance is preserved in that tank node, as further demonstrated later on in the text.

Consequently, the model of a hydraulic network of n_p pipes, n_n demand nodes (i.e. internal nodes) and n_0 tank or reservoir nodes (i.e. known heads) can be represented in a matrix form,

$$\begin{aligned} \mathbf{A}_{pp}\mathbf{Q}_p + \mathbf{A}_{pn}\mathbf{H}_n &= -\mathbf{A}_{p0}\mathbf{H}_0 + \mathbf{H}_p^p \\ \mathbf{A}_{np}\mathbf{Q}_p &= \mathbf{d}_n^w \end{aligned} \tag{3}$$

where $\mathbf{Q}_p = [n_p,1]$ column vector of unknown pipe flow rates; $\mathbf{H}_n = [n_n,1]$ column vector of unknown nodal heads; $\mathbf{H}_0 = [n_0,1]$ column vector of known nodal heads; $\mathbf{H}_p^p = [n_p,1]$ column vector of the static heads of pump systems installed along pipes (if any); $\mathbf{d}_n^w = [n_n,1]$ column vector of demands lumped in the nodes driving the simulation which are fixed *a priori* as a model assumption; $\mathbf{A}_{pn} = \mathbf{A}_{np}^T$ and $\mathbf{A}_{p0}$ = topological incidence sub-matrices of size $[n_p,n_n]$ and $[n_p,n_0]$, respectively. In the first equation of system (3)

$\mathbf{A}_{pp}\mathbf{Q}_p$ is the $[n_p,1]$ column vector of the evenly distributed pipe head losses eventually containing the terms related to internal head loss of pump systems and minor losses as in Equation (1).

When $\mathbf{d}_n^w$ is assumed dependent on the head/pressure status of the system and is substituted by a head–demand relationship (i.e. $\mathbf{d}_n^w = \mathbf{d}_n^w(\mathbf{H})$), the second matrix equation of system (3) becomes,

$$\mathbf{A}_{np}\mathbf{Q}_p - \mathbf{d}_n^w(\mathbf{H}) = \mathbf{0}_n \tag{4}$$

The simulation model modified with Equation (4) is named head/pressure-driven because of the vector $\mathbf{d}_n^w$ dependency on network head/pressure status while the classical model (with fixed $\mathbf{d}_n^w$) is named demand-driven.

The GGA solution of system (3), with or without implementing Equation (4), can be obtained by iteratively solving the following equations in (5) (Piller *et al.* 2003; Todini 2003; Cheung *et al.* 2005; Giustolisi *et al.* 2008a),

$$\begin{aligned}
\mathbf{B}_{pp}^{iter} &= \left(\mathbf{D}_{pp}^{iter}\right)^{-1}\mathbf{A}_{pp}^{iter} \\
\mathbf{F}_n^{iter} &= \mathbf{A}_{np}\left(\mathbf{Q}_p^{iter} - \mathbf{B}_{pp}^{iter}\mathbf{Q}_p^{iter}\right) - \mathbf{A}_{np}\left(\mathbf{D}_{pp}^{iter}\right)^{-1} \\
&\quad \times \left(\mathbf{A}_{p0}\mathbf{H}_0 + \mathbf{H}_p^p\right) - \mathbf{C}_n \\
\mathbf{H}_n^{iter+1} &= \left[\mathbf{A}_{np}\left(\mathbf{D}_{pp}^{iter}\right)^{-1}\mathbf{A}_{pn} + \mathbf{D}_{nn}^{iter}\right]^{-1}\mathbf{F}_n^{iter} \\
\mathbf{Q}_p^{iter+1} &= \left(\mathbf{Q}_p^{iter} - \mathbf{B}_{pp}^{iter}\mathbf{Q}_p^{iter}\right) - \left(\mathbf{D}_{pp}^{iter}\right)^{-1} \\
&\quad \times \left(\mathbf{A}_{p0}\mathbf{H}_0 + \mathbf{H}_p^p + \mathbf{A}_{pn}\mathbf{H}_n^{iter+1}\right)
\end{aligned} \tag{5}$$

$$\begin{aligned}
&\mathbf{C}_n = \mathbf{d}_n^w \text{ and } \mathbf{D}_{nn}^{iter} = \mathbf{0}_{nn} && \text{for demand-driven analysis} \\
&\mathbf{C}_n = \left(\mathbf{d}_n^w(\mathbf{H})\right)^{iter} - \mathbf{D}_{nn}^{iter}\mathbf{H}_n^{iter} && \text{for pressure-driven analysis}
\end{aligned}$$

where *iter* is a counter of the iterative solving algorithm, $\mathbf{D}_{pp}$ = diagonal matrix whose elements are the derivatives of the head loss components (minor losses, losses in pump systems and evenly distributed losses) with respect to $\mathbf{Q}_p$, and $\mathbf{D}_{nn}$ = diagonal matrix whose elements are the derivatives of $\mathbf{d}_n^w$ with respect to nodal heads. $\mathbf{D}_{nn}$ is different from zero in pressure-driven simulation only.

For example, in Giustolisi *et al.* (2008b), $\mathbf{d}_n^w(\mathbf{H}_n) = \mathbf{d}_n^{act}(\mathbf{H}_n) + \mathbf{d}_n^l(\mathbf{H}_n)$ where $\mathbf{d}_n^{act}$ and $\mathbf{d}_n^l = [n_n,1]$ column vectors of demands related to customers (Wagner *et al.* 1988) and to background leakage (Germanopoulos 1985), respectively.

Generalization of classical WDN models

Let us start from Equation (1) for a tank node adding to both sides the initial head H_i^{ini} of the steady-state simulation in order to consider its level (head) variability with volume:

$$\begin{aligned}
&H_j - \left(H_i - H_i^{ini}\right) + R_k Q_k|Q_k|^{n-1} + K_k^{ml} Q_k|Q_k| \\
&\quad + \omega_k^{2-\gamma} r_k^p Q_k|Q_k|^{\gamma-1} = \pm\omega_k^2 H_k^p + H_i^{ini} \\
&\sum_s \pm Q_{is} = d_i^w(H) + d_i^{ext} + d_i^v \\
&\qquad = d_i^w(H) + d_i^{ext} + \frac{\Omega_i(H_i)}{\Delta t}\left(H_i - H_i^{ini}\right)
\end{aligned} \tag{6}$$

where d_i^{ext} = flow from an external pipe generally supplying water to the tank in the *i*th node from outside the network; Ω_i = cross-sectional area of the tank in the *i*th node (possibly dependent on head level – Ω_i (H_i)); Δt = time interval of the steady-state simulation and H_i^{ini} = initial head of the steady-state condition for the tank in the *i*th node.

The first equation in (6) differs with respect to (1) because the term for the unknown variation in tank level (head) $\Delta H_i = H_i - H_i^{ini}$ appears. Thus, unlike in Equation (2), the mass balance equation for the *i*th node does not disappear and can accommodate the term d_i^v related to the water volume/level change during the time interval Δt due to the global adjustments in the network through the flow rates Q_{is}. Furthermore, the possibility of representing the actual mass balance in the *i*th tank node allows the introduction of an important component in the mass balance from a hydraulic standpoint; that is d_i^{ext}. In fact, a tank could be fed by an external pipe and, in that case, d_i^{ext} is relevant for the prediction of the level variation.

It is worth noting that d_i^v depends on nodal head H_i through the level variation ΔH_i, the cross-sectional area of the tank (Ω_i) and Δt. d_i^{ext} does not depend on the head status of the network as it is a known discharge generally filling the tank during a fixed time interval.

The WDN model in Equation (3), extended or not to the pressure-driven case by means of Equation (4), can be arranged to have one mass balance equation for each tank node along with the new unknowns, $\Delta\mathbf{H} = \mathbf{H}_0 - \mathbf{H}_0^{ini}$, representing the variation of tank heads (levels) during Δt.

Thus, the matrix form of the new WDN model is,

$$\begin{aligned} \mathbf{A}_{pp}\mathbf{Q}_p + \bar{\mathbf{A}}_{pn}\begin{bmatrix}\mathbf{H}_n \\ \cdots \\ \Delta\mathbf{H}_0\end{bmatrix} &= -\mathbf{A}_{p0}\mathbf{H}_0^{ini} + \mathbf{H}_p^p \\ \bar{\mathbf{A}}_{np}\mathbf{Q}_p - \begin{bmatrix}\mathbf{d}_n^w(\mathbf{H}) \\ \cdots \\ \mathbf{d}_0^w(\mathbf{H}_0) + \dfrac{\mathbf{\Omega}_0\Delta\mathbf{H}_0}{\Delta t}\end{bmatrix} &= \begin{bmatrix}\mathbf{0}_n \\ \cdots \\ \mathbf{d}_0^{ext}\end{bmatrix} \end{aligned} \tag{7}$$

where $\Delta\mathbf{H}_0 = [n_0, 1]$ column vector of unknown tank level variations; $\mathbf{H}_0^{ini} = [n_0,1]$ column vector of initial tank heads; $\mathbf{d}_0^w = [n_0,1]$ column vector of demands lumped at tank nodes; $\mathbf{d}_0^{ext} = [n_0,1]$ column vector of flows feeding tanks from external pipes to the WDN; $\mathbf{\Omega}_0 = [n_0,1]$ column vector of cross-sectional area of tanks; $\Delta t =$ time interval of the steady-state simulation model.

The model in Equation (7) represents a non-linear system of equations based on energy and mass balance conservation, as those in Equations (2) and (3), having as many as n_0 new unknowns for head variation of tanks $\Delta\mathbf{H}_0$ and n_0 new mass balance equations written for those tank nodes. For this reason, the general topological matrix $\bar{\mathbf{A}}_{pn} = [\mathbf{A}_{pn} \mid \mathbf{A}_{p0}]$ of size $[n_p, n_n + n_0]$ and its transpose ($\bar{\mathbf{A}}_{np}$) are used instead of $\mathbf{A}_{pn}$ and $\mathbf{A}_{np}$.

As a consequence, the new GGA formulation for the model in Equation (7), G-GGA, is derived from Equation (5) as follows,

$$\begin{aligned} \mathbf{B}_{pp}^{iter} &= \left(\mathbf{D}_{pp}^{iter}\right)^{-1}\mathbf{A}_{pp}^{iter} \\ \mathbf{F}_n^{iter} &= \bar{\mathbf{A}}_{np}\left(\mathbf{Q}_p^{iter} - \mathbf{B}_{pp}^{iter}\mathbf{Q}_p^{iter}\right) - \bar{\mathbf{A}}_{np}\left(\mathbf{D}_{pp}^{iter}\right)^{-1} \\ &\quad \times \left(\mathbf{A}_{p0}\mathbf{H}_0^{ini} + \mathbf{H}_p^p\right) - \mathbf{C}_n \\ \mathbf{H}_n^{iter+1} &= \left[\bar{\mathbf{A}}_{np}\left(\mathbf{D}_{pp}^{iter}\right)^{-1}\bar{\mathbf{A}}_{pn} + \begin{pmatrix}\mathbf{D}_{nn}^{iter} & \mathbf{0}_{nt} \\ \mathbf{0}_{nt}^T & \mathbf{D}_0^{iter}\end{pmatrix}\right]^{-1}\mathbf{F}_n^{iter} \\ \mathbf{Q}_p^{iter+1} &= \left(\mathbf{Q}_p^{iter} - \mathbf{B}_{pp}^{iter}\mathbf{Q}_p^{iter}\right) - \left(\mathbf{D}_{pp}^{iter}\right)^{-1} \\ &\quad \times \left(\mathbf{A}_{p0}\mathbf{H}_0^{ini} + \mathbf{H}_p^p + \bar{\mathbf{A}}_{pn}\begin{bmatrix}\mathbf{H}_n \\ \cdots \\ \Delta\mathbf{H}_0\end{bmatrix}^{iter+1}\right) \end{aligned} \tag{8}$$

$$\mathbf{C}_n = \begin{bmatrix}\mathbf{d}_n^w \\ \cdots \\ \mathbf{d}_0^w + \mathbf{d}_0^{ext} + \left(\dfrac{\mathbf{\Omega}_0\Delta\mathbf{H}_0}{\Delta t}\right)^{iter}\end{bmatrix} - \begin{bmatrix}\mathbf{0}_{nn} \\ \cdots \\ \mathbf{D}_0^{iter}\end{bmatrix}\begin{bmatrix}\mathbf{0}_n \\ \cdots \\ \Delta\mathbf{H}_0\end{bmatrix}$$

for demand-driven analysis

$$\mathbf{C}_n = \begin{bmatrix}\left(\mathbf{d}_n^w(\mathbf{H})\right)^{iter} \\ \cdots \\ \left(\mathbf{d}_n^w(\mathbf{H})\right)^{iter} + \mathbf{d}_0^{ext} + \left(\dfrac{\mathbf{\Omega}_0\Delta\mathbf{H}_0}{\Delta t}\right)^{iter}\end{bmatrix} - \begin{bmatrix}\mathbf{D}_{nn}^{iter} \\ \cdots \\ \mathbf{D}_0^{iter}\end{bmatrix}\begin{bmatrix}\mathbf{H}_n \\ \cdots \\ \Delta\mathbf{H}_0\end{bmatrix}$$

for pressure-driven analysis

where $\mathbf{0}_{nt} = [n_n, n_0]$ null matrix.

Both expressions of $\mathbf{C}_n$ contain the component of the mass balance at tanks ($\mathbf{\Omega}_0\Delta\mathbf{H}_0/\Delta t$). In particular, $\mathbf{D}_0$ ($=\mathbf{\Omega}_0/\Delta t$ provided that $\mathbf{\Omega}_0$ is constant, as for cylindrical tanks) needs to be iteratively computed in both the analyses (demand and pressure-driven).

Regarding the third equation of Equation (8), since $\bar{\mathbf{A}}_{pn}$ is the incidence matrix of the graph representing the WDN topology, the adjacency matrix of the nodes $\bar{\mathbf{A}}_{np} \times \bar{\mathbf{A}}_{pn}$ is not full rank because of its structure, independently on the term $\mathbf{D}_{pp}$. The rank is n_n–1 provided that the graph is composed of one component. Then, $\bar{\mathbf{A}}_{np}(\mathbf{D}_{pp})^{-1}\bar{\mathbf{A}}_{pn}$ in Equation (8) is not positive definite, but the addition of the information about just one tank generates a $\mathbf{D}_0$ (a scalar value in the specific case) that renders positive definite the following matrix,

$$\bar{\mathbf{A}}_{np}\left(\mathbf{D}_{pp}^{iter}\right)^{-1}\bar{\mathbf{A}}_{pn} + \begin{pmatrix}\mathbf{0}_{nn} & \mathbf{0}_{nt} \\ \mathbf{0}_{nt}^T & \mathbf{D}_0^{iter}\end{pmatrix} \tag{9}$$

This observation holds independently on the matrix $\mathbf{D}_{nn}$ of the pressure-driven case. In other words, in a classical demand-driven model (i.e. without accounting for variable tank levels) it is necessary to have one fixed head to guarantee that $\mathbf{A}_{np}(\mathbf{D}_{pp})^{-1}\mathbf{A}_{pn}$ ($\mathbf{A}_{np}$ is a sub-matrix of $\bar{\mathbf{A}}_{np}$) is positive definite. In G-GGA, the information about level variation of at least one tank is a scalar value added to the diagonal element of $\bar{\mathbf{A}}_{np}(\mathbf{D}_{pp})^{-1}\bar{\mathbf{A}}_{pn}$ corresponding to that tank node, which guarantees positive definiteness.

Also in the proposed WDN model, tanks with constant head (reservoirs) can be considered by removing the associated mass balance and unknown ΔH, i.e. removing the corresponding columns in $\bar{\mathbf{A}}_{pn}$. Actually, reservoirs can be considered as a special case of tanks with a large cross-sectional area as clarified subsequently.

Some remarks on Δt with respect to modelling assumptions

A steady-state analysis of a hydraulic network is generally performed under the assumption of slow time-varying boundary conditions such as demands and water tank levels. Then, the related inertial and dynamic effects are considered negligible. For this reason, the meaning of demands in a WDN model, which are the most variable boundary condition components of the hydraulic system, need to be further clarified with respect to the underlying steady-state assumption.

Demands are actually pulses having a stochastic behaviour which are reported to a larger scale of the steady-state WDN model by means of their spatial and temporal aggregation. Therefore, demands need to be considered as the average values of any statistical model describing the stochastic behaviour of the pulses in order to preserve the actual mass balance in the system. In other words, demands in a WDN model are assumed to be stationary parameters over the time window Δt of the single snapshot.

For this reason Δt, which represents the time interval during which the steady-state condition holds, is generally determined by the need for stationary of the average values of pulse intensity representing the model demands. The technical lower bound of Δt is a few minutes and depends on the spatial and time aggregation scales of demands. The selection of a very small Δt might contradict the assumption of slow time-varying boundary conditions. Obviously, this practical advice holds independently on numerical tests designed to ascertain algorithm stability.

In order to clarify some aspects of the proposed model, we consider an energy balance equation for a pipe having one tank as a terminal node and which ignores pumps and minor losses for simplicity. With these specifications, it is possible to write,

$$\begin{aligned} &\int_{\Delta t} H_j - \Delta H_i + RQ|Q|^{n-1}\mathrm{d}t = \int_{\Delta t} H_i^{ini}\,\mathrm{d}t \\ &\Rightarrow (\bar{H}_j - \Delta\bar{H}_i)_{\Delta t} + R\frac{\int_{\Delta t} Q|Q|^{n-1}\mathrm{d}t}{\Delta t} = H_i^{ini} \\ &\bar{Q} = \frac{\int_{\Delta t} Q\mathrm{d}t}{\Delta t} \Rightarrow \frac{\int_{\Delta t} Q|Q|^{n-1}\mathrm{d}t}{\Delta t} \neq \bar{Q}|\bar{Q}|^{n-1} \end{aligned} \tag{10}$$

Integration of the energy balance in time, similar to that for the mass balance, indicates that the state variables are actually average values in Δt. For this reason, the head loss computed using $\bar{Q}$ differs from the integration in time as in the second part of Equation (10). This discrepancy is not related to the generalized WDN model or G-GGA formulation but to extension in time of the single steady-state snapshot. Equation (10) illustrates how Δt cannot be too large if the constants for tank filling/emptying are to remain viable.

Some remarks on mass balance equation for tank nodes

Considering the mass balance equation for a tank node as in Equation (6), or its equivalent matrix form, it is possible to write,

$$\left(H_i - H_i^{ini}\right) = \left(\sum\nolimits_s Q_{is} - d_i^w(H) - d_i^{ext}\right)\frac{\Delta t}{\Omega_i}$$

or

$$\begin{aligned} \Delta\mathbf{H}_0 &= \frac{\Delta t}{\mathbf{\Omega}_0}\left[\left(\mathbf{A}_{p0}\right)^T\mathbf{Q}_p - \mathbf{d}_0^{ext} - \mathbf{d}_0^w\right] \\ &= f\left\{\mathbf{\Omega}_0, \Delta t, \left[\left(\mathbf{A}_{p0}\right)^T\mathbf{Q}_p - \mathbf{d}_0^{ext} - \mathbf{d}_0^w\right]\right\} \end{aligned} \tag{11}$$

Equation (11) states that the tank level variation ($\Delta\mathbf{H}_0$) depends on the cross-sectional area, the time interval of the steady-state condition and the mass balance of: (i) the network flow rates ($\mathbf{A}_{0p}\mathbf{Q}_p$); (ii) the nodal withdrawals ($\mathbf{d}_0^w$) generated by outflow along pipes (customer demands and leakages) lumped in the model (Giustolisi 2010) and (iii) flow rates ($\mathbf{d}_0^{ext}$) feeding the tanks by means of external pipes to the network.

Equation (11) helps to identify the conditions for the constant tank level assumption in the classical WDN models. The following two cases hold:

1. A very small Δt and/or very large $\mathbf{\Omega}_0$. In this circumstance the right-hand side of Equation (11) tends to a null value and $\Delta\mathbf{H}_0 \approx \mathbf{0}$ holds. This means that for large cross-sectional areas with respect to the time interval Δt of the steady-state condition it is possible

to assume $\mathbf{H}_0 \approx \mathbf{H}_0^{ini}$, as is typically assumed for reservoirs. It is noteworthy that the proposed WDN model (and G-GGA) allows selecting Δt in order to analyze a tank level variation inside a single steady-state condition of the system with respect to other boundary conditions. Thus, for example, the effect of the component $\mathbf{d}_0^{ext}$ with respect to the demand required by the network ($\mathbf{A}_{0p}\mathbf{Q}_p$ and $\mathbf{d}_0^{w}$) can be analyzed.

2. $(\mathbf{A}_{0p}\mathbf{Q}_p - \mathbf{d}_0^{w} - \mathbf{d}_0^{ext}) \approx \mathbf{0}$. In this case network demand is in equilibrium with that supplied by external feeding, irrespective of the tank cross-sectional area and time interval of the analysis. Such equilibrium implies $\Delta\mathbf{H}_0 \approx \mathbf{0}$; that is, a constant $\mathbf{H}_0^{ini}$.

Some remarks on head losses close to tanks

The preceding section showed that the G-GGA is a generalization of GGA for the proposed WDN model since reservoirs (i.e. fixed heads) are an approximation of tanks having large cross-sectional area $\mathbf{\Omega}_0$ with respect to the steady-state condition time step Δt.

In improving the modelling of tanks, it is also important to account for the inlet and outlet head losses close to the tanks (related to the kinetic component of Bernoulli's equation). In fact, although these contributions to head losses are negligible in standard conditions, they are not when $\mathbf{\Omega}_0\Delta\mathbf{H}_0/\Delta t$ increases. Ignoring inlet and outlet head losses produces lack of convergence when $\mathbf{\Omega}_0/\Delta t$ increases and the G-GGA cannot then be used for large $\mathbf{\Omega}_0/\Delta t$.

From a numerical standpoint, this is caused by growth of the derivative terms in $\mathbf{D}_0$. From a physical standpoint, high velocities are limited by the existence of these losses. Therefore, it is necessary to obtain the terms of $\mathbf{D}_0$ (i.e. the derivatives of the demand at a tank node with respect to the head variation) accounting for inlet or outlet head losses.

Starting from the following two equations,

$$\begin{cases} H_{model} - H_{tank} = \dfrac{\alpha V|V|}{2g} = \dfrac{\alpha}{2gA^2}|d^v|d^v = K_t d^v|d^v| \\ d^v = \dfrac{\Omega}{\Delta t}(H_{tank} - H_{ini}) \end{cases} \tag{12}$$

it is possible to obtain,

$$\begin{aligned} d^v &= \frac{-1 + \sqrt{1 + \mathrm{sign}(d^v)4K_t(\mathbf{\Omega}/\Delta t)^2(H_{model} - H_{ini})}}{\mathrm{sign}(d^v)(K_t\mathbf{\Omega}/\Delta t)} \\ &= \frac{\mathrm{sign}(\Delta H)\Delta t}{2K_t\mathbf{\Omega}}\left(-1 + \sqrt{1 + 4K_t\left(\frac{\boldsymbol{\Omega}}{\Delta t}\right)^2|\Delta H|}\right) \\ \frac{d}{d\Delta H}(d^v) &= \frac{\Delta t}{4K_t\boldsymbol{\Omega}}\frac{1}{\sqrt{1 + 4K_t(\mathbf{\Omega}/\Delta t)^2|\Delta H|}}4K_t\left(\frac{\boldsymbol{\Omega}}{\Delta t}\right)^2 \\ &= \frac{\boldsymbol{\Omega}}{\Delta t}\frac{1}{\sqrt{1 + 4K_t(\boldsymbol{\Omega}/\Delta t)^2|\Delta H|}} \end{aligned} \tag{13}$$

where V = mean velocity in the pipe connecting the network with a tank; d^v = flow rate of the tank; K_t = coefficient of the inlet/outlet head losses; g = gravity acceleration and α = coefficient accounting for the actual velocity distribution in the cross-sectional area of the pipe joining the tank.

Equation (13) allows updating of the G-GGA as for a tank flow rate and its derivative with respect to the level variation differently from the previously defined $\boldsymbol{\Omega}\Delta H/\Delta t$ and $\boldsymbol{\Omega}/\Delta t$, respectively. This way G-GGA can be used for a large $\boldsymbol{\Omega}/\Delta t$ in order to approximate fixed head levels (see Equation (11)).

During implementation of G-GGA in the software package WDNetXL (2011), it was observed that $\boldsymbol{\Omega}/\Delta t$ needs to be upper bounded to 10,000 m^2/s for numerical reasons. This means that if Δt = 15 min = 900 s, $\boldsymbol{\Omega} \leq$ 9,000,000 m^2, i.e. a circular tank with a diameter greater than 1,000 m). Decreasing Δt does not require prediction of tank level variation within a few digits considering model accuracy and then either $\boldsymbol{\Omega}$ can be decreased or the head kept constant. Increasing Δt raises the maximum allowed $\boldsymbol{\Omega}$. In this circumstance, a large value of Δt is useful for assessing tank level variation in a single snapshot with respect to mass balances in the system, bearing in mind that the other boundary conditions (e.g. demands) need to be kept constant throughout Δt. Finally, in order to constrain the maximum head of a tank it is possible to use a large $\boldsymbol{\Omega}/\Delta t$. This is easily done within G-GGA because it allows modelling reservoirs as a special case. In essence, tanks become reservoirs beyond a maximum level without any particular code modification.

Simple system

In order to test the algorithm's stability the system in Figure 1, composed of three tanks and three pipes, was examined (Todini 2011). The two interconnected cylindrical tanks, 1 and 2, each having 3.56 m diameter and initial heads of 20 and 30 m, respectively, begin filling and emptying at time $t = 0$. The third tank has diameter $d = 1{,}000$ m and an initial level of zero. The pipe lengths are 100 m and the diameters of pipes 2 and 3 are 0.1 m while for pipe 1 is double. The Hazen–Williams equation was used to represent the head losses with $C = 130$.

As in Todini (2011), it is expected that a flow from tank 2 to tank 1 will initially occur in pipe 1 until the two tanks reach the same water level. At this point, the flow in pipe 1 will stop and both tanks will empty at the same rate.

The EPS was performed on the system in Figure 1 using time steps $\Delta t = \{5, 15, 60, 120\}$ min. Figure 2 presents eight diagrams: (i) four diagrams of varying levels of tanks 1 and 2 for each time step Δt; (ii) four diagrams of varying flow rate in the pipes 1, 2 and 3 for each time step.

In the EPS, G-GGA allows simple updating from the tanks' initial level at time t with that at time $t - \Delta t$ and, then, the diagrams of flow rate in the pipes 1, 2 and 3 appear to finish one-Δt backward. For example, in the case of $\Delta t = 120$ min, the initial ($t = 0$) levels of tanks 1 and 2 (20 and 30 m, respectively) both become about 9 m at $t = 120$ min and the corresponding flow rate in pipe 1 (about equal to 7 L/s) at $t = 0$ min represents the hydraulic status in pipe 1 from 0 to 2 h, then, the last ($t = 6$ h) represents that from 6 to 8 h.

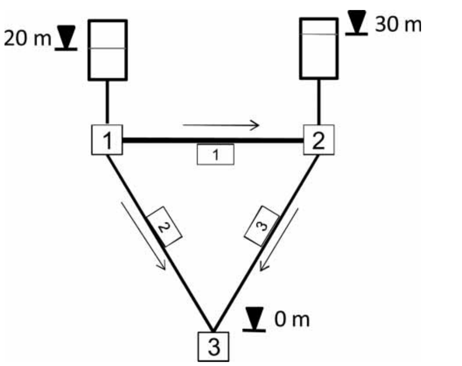

Figure 1 | Two emptying interconnected tanks starting at time zero with different water levels (from Todini 2011).

However, the results show that G-GGA are consistent with those reported in Todini (2011) and slightly superior as flow rate oscillations do not occur in any case. In particular, the flow rate in pipe 1 demonstrates that G-GGA is extremely stable as it reaches the equilibrium (both tanks 1 and 2 reach the same level corresponding to the flow rate in pipe 1 diminishing to zero after a few minutes) without any pipe 1 flow rate oscillation, also for a very large time step ($\Delta t = 120$ min), which means approximately half of the time for emptying the tanks 1 and 2.

Finally, it should be noted that the duration of the emptying process, especially with increasing Δt, varies significantly for reasons explained by Equation (10).

The filling/emptying process in a larger network

In order to further prove the stability of the proposed G-GGA, the Apulian network in Figure 3 is considered. Network details can be found in Giustolisi (2010).

Note that the nodal demands of the network are all assumed to be zero, while its topology and pipe hydraulic resistances are left unchanged. Furthermore, cylindrical tanks (with constant cross-sectional area equal to 10 m^2) are assumed at all 23 nodes of the network while the original reservoir (i.e. a tank of fixed head equal to 36.4 m) remains at node 24. A cross-sectional area of 100,000 m^2 has been assumed in order to fix the tank's level.

Three numerical experiments were then performed in EPS:

1. All tanks were assumed to have zero head at $t = 0$ and were then filled by the reservoir at node 24.
2. All tanks were assigned a head of 72.8 m at $t = 0$ and were then emptied as water flowed to node 24.
3. The tanks in selected nodes close to node 24 {1–6, 16–19, 21 and 23} were assumed to have a level of 0 m at $t = 0$ while the others, farther from node 24, were given a level of 72.8 m. This initial configuration permitted simultaneous examination of the filling process at certain nodes {1–6, 16–19, 21 and 23} and the emptying at others. The reservoir/tank at node 24 initially fills nodes {1–6, 16–19, 21 and 23} before receiving water from the network as other nodes empty.

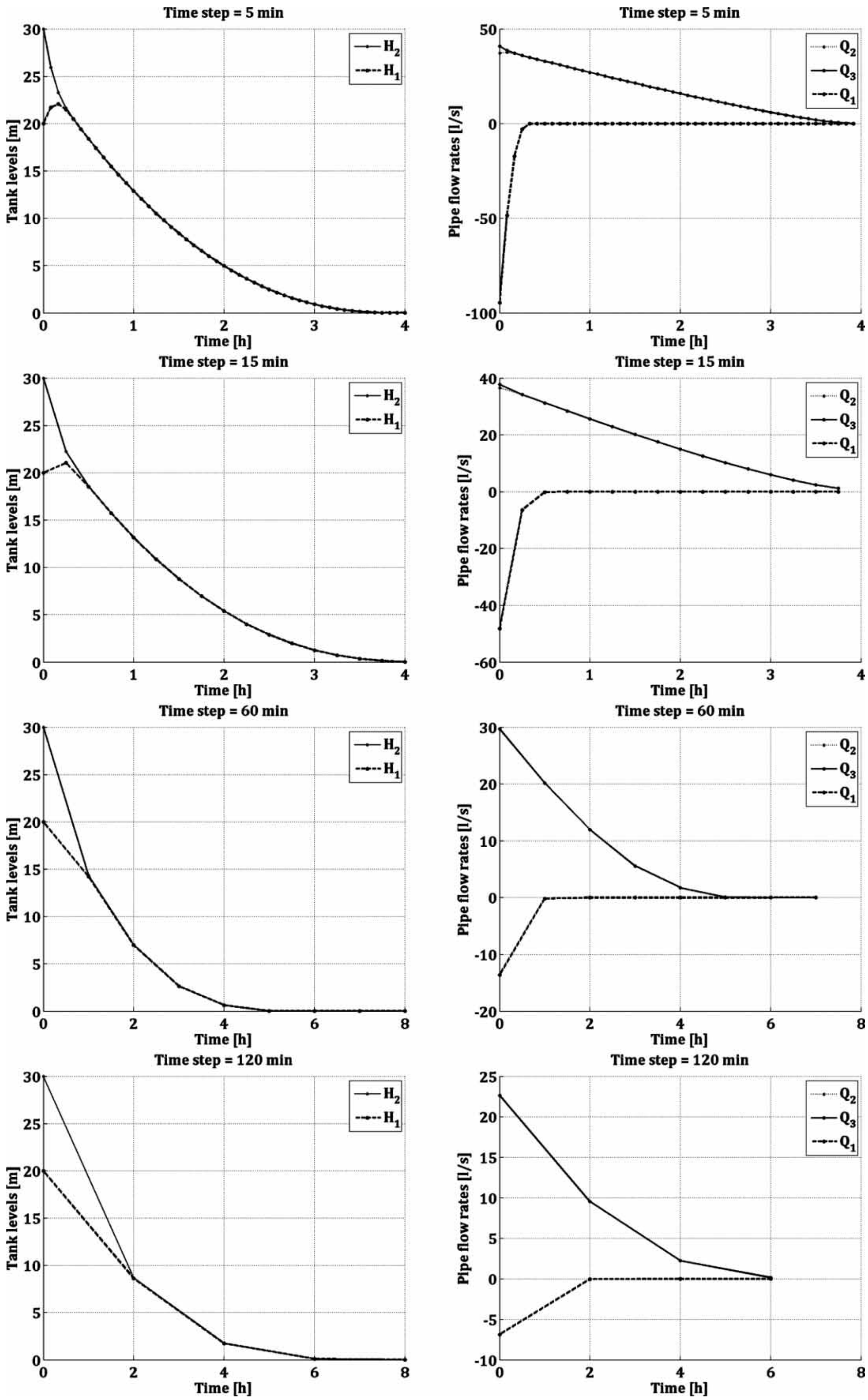

Figure 2 | G-GGA tank levels and flow rates computed for the system in Figure 1 in EPS.

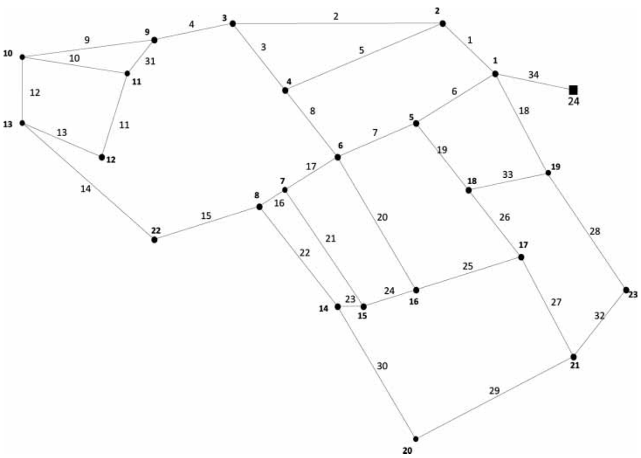

Figure 3 | Apulian network layout.

The EPS was performed for 12 h using $\Delta t = 10$ min. Figures 4–6 report the flow rate in all pipes and the heads at all nodes for each time step of the simulation.

The case study reveals that G-GGA was quite stable as the flow in all pipes did not oscillate and the filling process was consistently simulated. In particular, in the first case (see Figure 4) the uppermost curve represents the flow rate in pipe 34 which is that connecting the tank in node 24 to the rest of the network. Such a flow rate is positive (i.e. towards the network) and is very high in the first hours (about 800 L/s corresponding to high head losses in the network). The diagram on the left clearly shows the filling of other nodes. The reverse situation occurs in the second

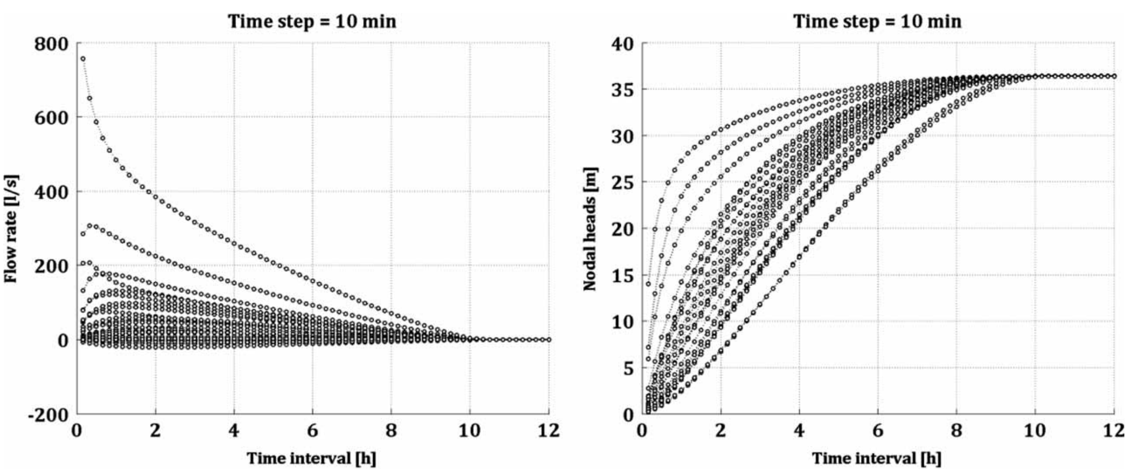

Figure 4 | All the flow rates and nodal heads in the system during the filling process for case 1.

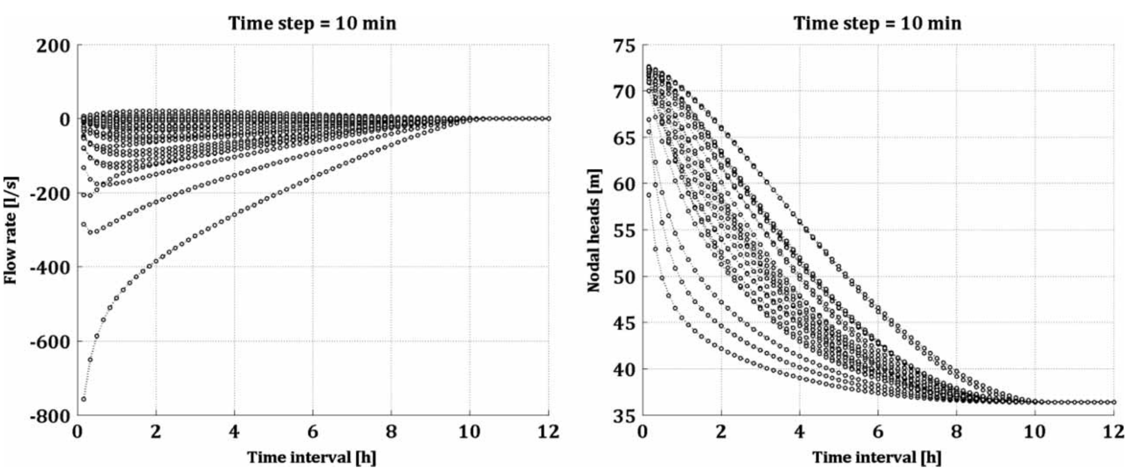

Figure 5 | All flow rates and nodal heads in the system during the emptying process for case 2.

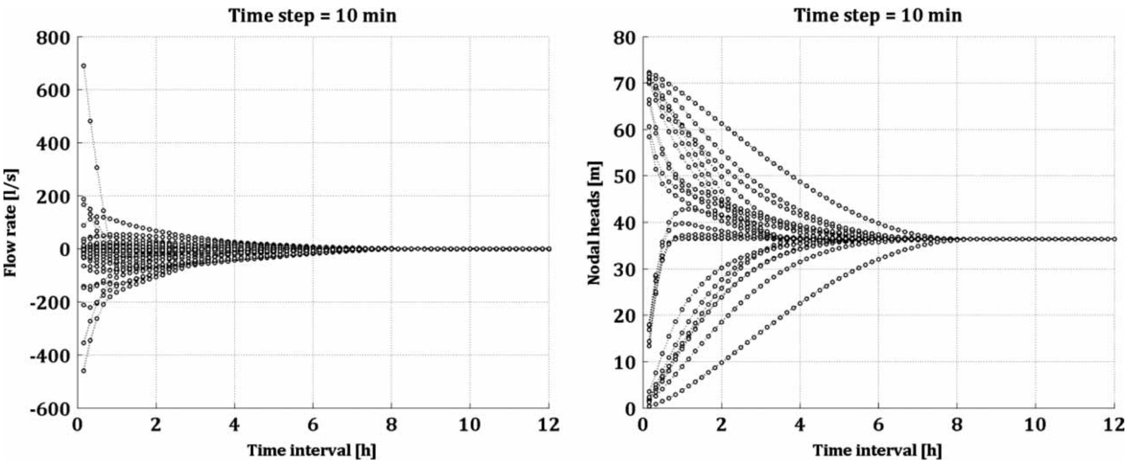

Figure 6 | All the flow rates and nodal heads in the system during the filling/emptying process for case 3.

case (Figure 5) as there is an emptying of the tanks inside the network starting from a reverse initial condition. In the third case (Figure 6), some tanks filled while others emptied during EPS, with flows reversing direction in the network pipes during the first hour. This is shown, for example, by the pipe 34 flow rate in Figure 6. It is worth noting that, due to the complex filling/emptying process of different tanks, some (e.g. the tank at node 24) might experience a sudden filling followed by a slower emptying phase. This is actually a consequence of different pipe resistances connecting tank nodes.

CONCLUSION

An expanded generalized framework for WDN modelling has been presented which is applicable to both demand-driven and pressure-driven analyses and allows consideration of varying tank levels and mass balances at tank nodes. This is in contrast to the classical network hydraulic models which assume a constant head level at tank nodes and thus neglect the relevant mass balance components.

The stability of the resulting G-GGA was tested using the simple case study from Todini (2011). Subsequently, the G-GGA was applied to a larger network by modelling the filling/emptying processes of tanks situated at all the nodes. G-GGA is a promising enhancement to the hydraulic network model in WDN analysis because it can account for the volume balance in tanks (by recovering the related mass balance equation) during EPS and also for single steady-state simulation.

ACKNOWLEDGEMENTS

The authors wish to thank Dr Andrew F. Colombo for his assistance in reviewing the English of this manuscript.

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First received 26 February 2011; accepted in revised form 22 August 2011. Available online 29 November 2011