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005

NUMERICAL APPROXIMATION FOR FUNCTIONALS OF REFLECTING DIFFUSION PROCESSES*

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Abstract. The aim of this paper is to approximate the expectation of a large class of functionals of the solution (X, ξ) of a stochastic differential equation with normal reflection in a piecewise smooth domain of $\mathbb{R}^d$. This also yields a Monte Carlo method for solving partial differential problems of parabolic type with mixed boundary conditions. The approximation is based on a modified Euler scheme for the stochastic differential equation. The scheme can be driven by a sequence of bounded independently and identically distributed (i.i.d.) random variables, or, when the domain is convex, by a sequence of Gaussian i.i.d. random variables. The order of (weak) convergence for both cases is given. In the former case the order of convergence is $1/2$, and it is shown to be exact by an example. In the last section numerical tests are presented. The behavior of the error as a function of the final time T, for fixed values of the discretization step, and as a function of the discretization step, for fixed values of the final time T, is analyzed.

Key words. stochastic differential equations with reflection, reflecting boundary conditions, Neumann boundary conditions, mixed boundary conditions, numerical schemes, weak convergence, Monte Carlo method

AMS subject classifications. 60F17, 60J50, 60H30, 65C05

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1. Introduction.

Numerical evaluation of expectations of functionals of diffusion processes is an important issue in physical, chemical and engineering problems (see, for example, the classical books [4] and [17]). It also provides an approach to the solution of boundary value problems using parallel computers (see, for example, [14]).

This paper proposes and analyzes an approximation for the expectation of a quite general functional of a reflecting diffusion process. In particular, this allows us to deal with pure Neumann and mixed boundary value problems.

Let D be a bounded domain in $\mathbb{R}^d$ with piecewise $\mathcal{C}^1$ boundary, and let $b : [0,T] \times \bar{D} \to \mathbb{R}^d$ and $\sigma : [0,T] \times \bar{D} \to \mathbb{R}^d \otimes \mathbb{R}^{d'}$ be continuous functions. Under suitable assumptions on D, b, and σ, specified in section 2 (in particular, if D is convex or has a $\mathcal{C}^2$ boundary and b, σ are uniformly Lipschitz continuous in space), it is well defined the diffusion process with coefficients b and σ and normal reflection in $\bar{D}$, starting at $X^0 \in \bar{D}$, i.e., a $\bar{D}$-valued continuous stochastic process $X = \{X_s\}_{t\le s\le T}$, $X_t = X^0$, for which there exist a (unique) continuous, increasing stochastic process $\xi = \{\xi_s\}_{t\le s\le T}$ (local time) and a (unique) stochastic process $n = \{n_s\}_{t\le s\le T}$, such that n_s is a suitably normalized inward normal vector at $X_s \in \partial D$ and the triple (X, ξ, n) satisfies

$$(1.1)\qquad \begin{cases} dX_s = b(s, X_s)ds + \sigma(s, X_s)dB_s + n_s d\xi_s, & t \le s \le T, \\ \xi_s = \int_t^s \mathbb{I}_{\partial D}(X_r)d\xi_r, & t \le s \le T, \end{cases}$$

where B is a standard Brownian motion.

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We consider a functional of X and ξ of the form

$$F_{t,T}(X,\xi) = \mathbb{I}_{T\leq\tau}\, f(X_T)\, \exp(Y_T + Z_T) + \mathbb{I}_{T>\tau}\, g_1(\tau, X_\tau)\, \exp(Y_\tau + Z_\tau) - \int_t^{T\wedge\tau} g_2(r, X_r)\, \exp(Y_r + Z_r)\, d\xi_r, \tag{1.2}$$

where $f : \bar{D} \to \mathbb{R}$, $g_1 : [0,T] \times \partial_1 D \to \mathbb{R}$, $g_2 : [0,T] \times (\partial D - \partial_1 D) \to \mathbb{R}$ are continuous functions and

(i) $\tau = \begin{cases} \inf\{s : t \leq s \leq T,\ X_s \in \partial_1 D\}, & \text{if } \{s : t \leq s \leq T,\ X_s \in \partial_1 D\} \neq \emptyset, \\ +\infty, & \text{otherwise,} \end{cases}$

$\partial_1 D$ being a (possibly empty) closed subset of ∂D,

(ii) $Y_s = \int_t^s c(r, X_r)dr$,

(iii) $Z_s = -\int_t^{s\wedge\tau} \lambda(r, X_r)d\xi_r$,

with $c : [0,T] \times \bar{D} \to \mathbb{R}$ and $\lambda : [0,T] \times (\partial D - \partial_1 D) \to \mathbb{R}$ continuous, and $\lambda(r,x) \geq 0$.

Important special cases of $\mathbb{E}\left[F_{t,T}(X,\xi)\right]$ are the expectation of a function f of X_T (corresponding to $\partial_1 D = \emptyset$, hence $\tau = +\infty$, $g_2 = c = \lambda = 0$), the expectation of the local time ξ_T (obtained by setting $\partial_1 D = \emptyset$, $f = c = \lambda = 0$, $g_2 = 1$), and the expectation of any compact support function g of the hitting time of $\partial_1 D$, (corresponding to T large enough, $g_1(t,x) = g(t)$, $f = g_2 = c = \lambda = 0$).

In addition, considering $X^0 = x$, $\mathbb{E}\left[F_{t,T}(X,\xi)\right] = \mathbb{E}_{t,x}\left[F_{t,T}(X,\xi)\right]$ can be viewed as a representation of the (classical) solution $u(t,x)$ of the backward partial differential equation with mixed boundary conditions

$$\begin{cases} u_t(t,x) + (L_t u + cu)(t,x) = 0, & \text{in } [0,T) \times D, \\ u(T,x) = f(x), & \text{in } \bar{D}, \\ u(t,x) = g_1(t,x), & \text{in } [0,T) \times \partial_1 D, \\ \left(\frac{\partial u}{\partial n} - \lambda u\right)(t,x) = g_2(t,x), & \text{in } [0,T) \times (\partial D - \partial_s D - \partial_1 D), \end{cases} \tag{1.3}$$

where

$$L_t u(t,x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(t,x) \frac{\partial^2 u}{\partial x_i \partial x_j}(t,x) + \sum_{i=1}^d b_i(t,x) \frac{\partial u}{\partial x_i}(t,x),$$

$$a_{ij}(t,x) = \sum_{k=1}^{d'} \sigma_{ik}(t,x)\sigma_{jk}(t,x).$$

$\partial_s D$ is the set of points $x \in \partial D$ for which there is not a unique unit inward normal vector $n(x)$ (see section 2 for a precise definition). This is a well-known result for smooth boundary and pure Dirichlet or pure Neumann boundary conditions, which we extend to piecewise smooth boundary and mixed boundary conditions in section 2.

The (classical) solution $v(t,x)$ of the forward partial differential equation with mixed boundary conditions

$$\begin{cases} v_t(t,x) = (L_t v + cv)(t,x), & \text{in } (0,T] \times D, \\ v(0,x) = f(x), & \text{in } \bar{D}, \\ v(t,x) = g_1(t,x), & \text{in } (0,T] \times \partial_1 D, \\ \left(\frac{\partial v}{\partial n} - \lambda v\right)(t,x) = g_2(t,x), & \text{in } (0,T] \times (\partial D - \partial_s D - \partial_1 D), \end{cases} \tag{1.4}$$

can also be represented as the expectation of a functional of the form (1.2), provided X is replaced by the diffusion process $\tilde{X} = \{\tilde{X}_s\}_{T-t\leq s\leq T}$, $\tilde{X}_{T-t} = x$, with coefficients

$\tilde{b}(s,x) = b(T-s,x)$, $\tilde{\sigma}(s,x) = \sigma(T-s,x)$; ξ is replaced by the corresponding local time $\tilde{\xi}$; t is replaced by $T-t$; and the functions $c(s,x), \lambda(s,x), g_i(s,x), i=1,2$, are replaced by $\tilde{c}(s,x) = c(T-s,x)$, $\tilde{\lambda}(s,x) = \lambda(T-s,x)$, $\tilde{g}_i(s,x) = g_i(T-s,x), i=1,2$, respectively.

In the time-homogeneous case ($b, \sigma, c, \lambda, g_i$ independent of the time variable) the expectation of (1.2) reduces to the expectation of $F_{0,T-t}(X,\xi) = F_{T-t}(X,\xi)$, X being the diffusion process starting from X^0 at time 0 and ξ being the corresponding local time, and we have $u(t,x) = \mathbb{E}_{0,x}[F_{T-t}(X,\xi)] = \mathbb{E}_x[F_{T-t}(X,\xi)]$, $v(t,x) = \mathbb{E}_{0,x}[F_t(X,\xi)] = \mathbb{E}_x[F_t(X,\xi)]$.

Therefore, any approximation of $F_{t,T}(X,\xi)$ yields a Monte Carlo method for solving (1.3) and (1.4). Our approximation of $F_{t,T}(X,\xi)$ is based on the following discretization of (1.1):

$$
\begin{aligned}
& s_0 = t,\ s_p = t + ph, \\
& X^h_{s_0} = X^0, \quad \xi^h_{s_0} = 0, \\
& \Delta_{p+1} W^h = hb(s_{p+1}, X^h_{s_p}) + \sqrt{h}\sigma(s_{p+1}, X^h_{s_p})\Delta_{p+1}\eta, \\
& \tilde{X}^h_{s_{p+1}} = X^h_{s_p} + \Delta_{p+1} W^h, \\
& X^h_{s_{p+1}} = \begin{cases} \tilde{X}^h_{s_{p+1}}, & \text{if } \tilde{X}^h_{s_{p+1}} \in \bar{D}, \\ \pi\left(\tilde{X}^h_{s_{p+1}}\right), & \text{if } \tilde{X}^h_{s_{p+1}} \notin \bar{D}, \end{cases} \\
& \xi^h_{s_{p+1}} = \begin{cases} \xi^h_{s_p}, & \text{if } \tilde{X}^h_{s_{p+1}} \in \bar{D}, \\ \xi^h_{s_p} + P\left(\tilde{X}^h_{s_{p+1}}\right), & \text{if } \tilde{X}^h_{s_{p+1}} \notin \bar{D}, \end{cases} \\
& X^h_s = X^h_{s_p}, \quad \xi^h_s = \xi^h_{s_p}, \quad \text{for} \quad s_p \le s < s_{p+1},
\end{aligned}
\tag{1.5}
$$

$p = 0, \ldots, N-1$, $Nh = T-t$, $N \ge 2$, where π denotes the normal projection on $\bar{D}$ and P is a function defined in section 2. Whenever there is a unique unit inward normal vector at $\pi(w)$, P is simply given by

$$P(w) = |\pi(w) - w|.$$

$\{\Delta_{p+1}\eta\}$ is a sequence of i.i.d. random variables, independent of X^0, verifying suitable assumptions on their first three moments (see section 3), bounded if D is not convex. The choice of evaluating b and σ at s_{p+1} allows us to require as little regularity of b and σ in the time variable as possible (see the proof of Lemma 3.1). (X^h, ξ^h) is the solution of the Skorohod problem (see section 2) for W^h,

$$
\begin{aligned}
& W^h_{s_{p+1}} = X^h_{s_0} + \textstyle\sum_{k=0}^{p} \Delta_{k+1} W^h,\ W^h_{s_0} = X^h_{s_0}, \\
& W^h_s = W^h_{s_p} \text{ for } s_p \le s < s_{p+1},\ p = 0, \ldots, N-1.
\end{aligned}
\tag{1.6}
$$

As an approximation of $F_{t,T}(X,\xi)$ we take

$$
\begin{aligned}
F^h_{t,T}(X^h, \xi^h) = {} & \mathbb{I}_{N<\nu^h} f(X^h_{s_N}) \exp(Y^h_{s_N} + Z^h_{s_N}) \\
& + \mathbb{I}_{N \ge \nu^h} g_1(s_{\nu^h}, X^h_{s_{\nu^h}}) \exp(Y^h_{s_{\nu^h}} + Z^h_{s_{\nu^h}}) \\
& - \mathbb{I}_{2 \le \nu^h} \textstyle\sum_{p=0}^{(N\wedge\nu^h)-2} g_2(s_{p+1}, X^h_{s_{p+1}}) \exp(Y^h_{s_p} + Z^h_{s_p}) \Delta_{p+1}\xi^h \\
& - \mathbb{I}_{N<\nu^h} g_2(s_N, X^h_{s_N}) \exp(Y^h_{s_{N-1}} + Z^h_{s_{N-1}}) \Delta_N \xi^h,
\end{aligned}
\tag{1.7}
$$

where $\Delta_{p+1}\xi^h = \xi^h_{s_{p+1}} - \xi^h_{s_p}$ and

(i) $\nu^h = \begin{cases} \min\{p : 0 \le p \le N, X^h_{s_p} \in \partial_1 D\}, & \text{if } \{p : 0 \le p \le N, X^h_{s_p} \in \partial_1 D\} \neq \emptyset, \\ +\infty, & \text{otherwise,} \end{cases}$

(ii) $Y^h_{s_p} = \sum_{k=0}^{p-1} c(s_{k+1}, X^h_{s_k})h, \quad p = 1, \dots, N, \quad Y^h_{s_0} = 0,\ Y^h_s = Y^h_{s_p}$ for $s_p \le s < s_{p+1}$, $p = 0, \dots, N$,

(iii) $Z^h_{s_p} = -\sum_{k=0}^{p-1} \lambda(s_{k+1}, X^h_{s_{k+1}})\Delta_{k+1}\xi^h \mathbb{I}_{k<\nu^h-1}$, $p = 1, \dots, N$, $Z^h_{s_0} = 0$, $Z^h_s = Z^h_{s_p}$ for $s_p \le s < s_{p+1}$, $p = 0, \dots, N$.

In the case when $\partial_1 D = \emptyset$, and hence $\tau = \nu^h = +\infty$, by interpreting the integral in (1.2) as $\int_{t+}^{T+} g_2(r, X_r) \exp(Y_{r-} + Z_{r-}) d\xi_r$, one sees that

$$F^h_{t,T}(X^h, \xi^h) = F_{t,T}(X^h, \xi^h). \tag{1.8}$$

In section 3 we show that if either i) the random variables $\Delta_{p+1}\eta$ are bounded, or ii) D is convex, b and σ are time independent, and the random variables $\Delta_{p+1}\eta$ are Gaussian, then

$$\sup_{t \le T, x \in \bar{D}} \left| \mathbb{E}_{t,x}\left[F_{t,T}(X, \xi)\right] - \mathbb{E}_{t,x}\left[F^h_{t,T}(X^h, \xi^h)\right] \right| \longrightarrow 0 \text{ for } h \to 0. \tag{1.9}$$

Under suitable smoothness assumptions on $\partial D, b, \sigma, f, c, \lambda$, and g_i, $i = 1, 2$, we prove (Theorems 3.4 and 3.6) in the former case that there is a constant C such that

$$\sup_{t \le T, x \in \bar{D}} \left| \mathbb{E}_{t,x}\left[F_{t,T}(X, \xi)\right] - \mathbb{E}_{t,x}\left[F^h_{t,T}(X^h, \xi^h)\right] \right| \le C\, h^{1/2} \tag{1.10}$$

for h less than some h_1, and in the latter case that for every $\epsilon > 0$ there is a constant C^ϵ such that

$$\sup_{t \le T, x \in \bar{D}} \left| \mathbb{E}_{t,x}\left[F_{t,T}(X, \xi)\right] - \mathbb{E}_{t,x}\left[F^h_{t,T}(X^h, \xi^h)\right] \right| \le C^\epsilon\, h^{1/2-\epsilon}. \tag{1.11}$$

Finally, note that the operator L_t is not required to be uniformly elliptic or even nondegenerate and that the error bounds (1.10) and (1.11) are uniform in both time and space.

Moreover, we show that the estimate (1.10) is tight. In fact, for a standard reflecting Brownian motion in an interval and a suitable functional F_T, the left-hand side of (1.10) is bounded from below by a constant times $h^{1/2}$ (Example 3.1).

Discretization schemes for (1.1) have been recently considered in the literature. All the proposed schemes are modifications of the Euler scheme for a diffusion in $\mathbb{R}^d$. (For approximation methods of stochastic differential equations in $\mathbb{R}^d$, see [8] or [15].)

In [20] Słomiński considers two strong approximations of (X, ξ) in the time-homogeneous case: $(\widehat{X}^h, \widehat{\xi}^h)$ defined by scheme (5) in [20] and $(\bar{X}^h, \bar{\xi}^h)$ defined by scheme (4) in [20]. In the case of a convex set D, the author derives the strong convergence rate (in the sense of [8]) of both approximations; namely, he shows that for every $q \in \mathbb{N}$ there is a constant C_q such that

$$\mathbb{E}\left[\sup_{s \le T} |\widehat{X}^h_s - X_s|^{2q}\right] \le C_q\, h^q, \tag{1.12}$$

and that for every $\epsilon > 0$, $q \in \mathbb{N}$, there is a constant C^ϵ_q such that

$$\mathbb{E}\left[\sup_{s \le T} |\bar{X}^h_s - X_s|^{2q}\right] \le C^\epsilon_q\, h^{q/2-\epsilon}, \tag{1.13}$$

which is improved to

$$\mathbb{E}\left[\sup_{s\leq T}|\bar{X}^h_s - X_s|^{2q}\right] \leq C^\epsilon_q\, h^{q-\epsilon}, \tag{1.14}$$

if D is a convex polyhedron. For a half-space D, similar results were proved by Lepingle in [10] for $\widehat{X}^h$, and by Chitashvili and Lazrieva in [1] and Kinkladze in [7] for $\bar{X}^h$. $\widehat{X}^h$ is obtained by approximating X in each time step by a reflecting Brownian motion with linear drift, obtained by freezing b and σ. Therefore, this method cannot actually be implemented in a general convex domain, but in [11] Lepingle shows how it can be implemented in a half-space, an orthant, or a parallelepiped. The behavior of $\bar{X}^h$ is further analyzed in [21]. An approximation scheme for (X, ξ) based on a penalty method is proposed in [13].

$(\bar{X}^h, \bar{\xi}^h)$ coincides in law with (X^h, ξ^h) provided the $\Delta_{p+1}\eta$'s are taken to be Gaussian. (1.13) or (1.14), together with

$$\sup_{h\leq h_0} \mathbb{E}\left[\left(\xi^h_T\right)^q\right] < +\infty \quad \forall q \in \mathbb{N}, \tag{1.15}$$

which follows from the results of Słomiński in [20], imply (1.9). However, (1.13) (or (1.14)) and (1.15) can yield no estimate of the convergence rate of the left-hand side of (1.9): in fact, since (1.2) and (1.7) involve integrals in $d\xi$ and $d\xi^h$, such an estimate would require an upper bound on the total variation of $\xi - \bar{\xi}^h$.

In contrast, here we focus on weak convergence of the scheme (1.5), combining the technique used by Talay in [22] and by Talay and Tubaro in [23] for a diffusion process in $\mathbb{R}^d$, with results on the normal reflection Skorohod problem for càdlàg paths. Our approach enables us to obtain the error bounds (1.10) and (1.11) for a large class of functionals including integrals with respect to $d\xi$, and in a large class of domains. In addition, it allows us to use random variables $\Delta_{p+1}\eta$ with arbitrary law, which has advantages in the implementation of the scheme.

The fact that $h^{1/2}$ is the exact weak convergence rate of the scheme (1.5) with bounded $\Delta_{p+1}\eta$'s is, in our opinion, one of the most interesting points of this paper. In fact, for a diffusion in $\mathbb{R}^d$, the Euler scheme has strong convergence rate $h^{1/2}$ but weak convergence rate h (see, for instance, [8]).

In [16] two other weak discretization schemes of (1.1) are considered in domains with smooth boundary. These schemes differ from (1.5), in particular in the mechanism that approximates reflection on the boundary. One of them achieves the rate of convergence h, but, as pointed out by the author, is difficult to implement, while the other one is simpler, but has rate of convergence $h^{1/2}$.

In section 2 some definitions and theorems on the Skorohod problem and on reflecting diffusions are briefly recalled and the above-mentioned extension for the representation of the solutions of (1.3) and (1.4) is derived. In section 3 we prove our main results. In section 4 numerical tests are presented and their outcomes are analyzed. The scheme (1.5) is applied to an example of diffusion in $\mathbb{R}^2$, previously considered in [23]. We take $\bar{D} = [-L, L]\times[-L, L]$, with $L = 1, 1.2, 10$, and a functional $F_{0,T}(X, \xi)$ whose expectation can be computed exactly. $m = 10{,}000$ i.i.d. copies of (X^h, ξ^h) starting at $x = (1, 1)$ are simulated and the error $\mathbb{E}_{0,x}[F_{0,T}(X, \xi)] - \frac{1}{m}\sum_{i=1}^m F^h_{0,T}\left(X^{h,i}, \xi^{h,i}\right)$ is evaluated for different values of T and h. For each value of L and h, the evolution of the error with respect to T is shown. In particular, for $L = 10$, the simulated diffusion never hits the boundary and our results reproduce well those reported in [23]. For each value of L and some values of T the behavior of the error with respect to h is considered: it appears to be consistent with the theoretical analysis of section 3. These numerical results are obtained by using pseudorandom variables $\Delta_{p+1}\eta$ uniformly distributed on a finite number of values. In the last part of section 4 some of these results are compared with the corresponding ones obtained by using Gaussian pseudorandom variables $\Delta_{p+1}\eta$. The comparison shows that the CPU time consumed when using Gaussian pseudorandom variables is 1.2 times larger than the CPU time consumed when using discrete uniform pseudorandom variables, while the error raised using Gaussian variables is in general not smaller.

2. Preliminaries. The Skorohod problem with normal reflection has been studied chiefly by Tanaka in [24], for convex domains, and by Lions and Sznitman in [12] and Saisho in [18], for more general domains. These works deal with continuous paths. Extension of their results to càdlàg paths are contained in [2], as a special case of results on the Skorohod problem with oblique reflection, and in [19]. Here we follow [2] and [18].

Let D be a bounded domain in $\mathbb{R}^d$. Define the set $\mathcal{N}_x$ of inward normal vectors at $x \in \partial D$ by

$$\mathcal{N}_x = \{\alpha n : \alpha \geq 0, n \in \mathcal{N}_{x,\rho} \text{ for some } \rho > 0\},$$
$$\mathcal{N}_{x,\rho} = \left\{n \in \mathbb{R}^d : |n| = 1,\ B(x - \rho n, \rho) \cap D = \emptyset\right\},$$

where $B(x,\rho) = \{y \in \mathbb{R}^d : |y - x| < \rho\}$. We introduce two conditions on D.

Condition A (uniform exterior sphere condition). There exists a constant $\rho_0 > 0$ such that $\mathcal{N}_{x,\rho_0} \neq \emptyset \ \forall x \in \partial D$.

Condition B. There exist constants $\delta > 0$ and $\beta \in [1, \infty)$ with the property that for every $x \in \partial D$ there is a unit vector e_x such that

$$e_x \cdot n \geq \frac{1}{\beta} \qquad \forall n \in \bigcup_{y \in B(x,\delta)\cap\partial D} \mathcal{N}_y, \quad |n| = 1.$$

If D is convex or has a $\mathcal{C}^2$ boundary then Conditions A and B are verified. In particular, if D is convex, Condition A holds with $\rho_0 = +\infty$.

Throughout this paper it will be assumed that D satisfies Conditions A and B.

Remark 2.1. If Condition A is verified, then for every $w \notin \bar{D}$ such that the distance of w from $\bar{D}$, $d(w,\bar{D})$, is strictly less than ρ_0, there exists a unique normal projection $\pi(w)$ on $\bar{D}$; i.e., there exists one and only one point $\pi(w) \in \bar{D}$ such that

$$|\pi(w) - w| = d(w, \bar{D}). \tag{2.1}$$

$\pi(w) \in \partial D$ and

$$\pi(w) - w \in \mathcal{N}_{\pi(w)}.$$

In addition to Conditions A and B it will always be assumed that D has a piecewise $\mathcal{C}^1$ boundary. More precisely, we will suppose that

$$D = \bigcap_{i=1}^{I} D_i, \tag{2.2}$$

where each D_i verifies the uniform exterior sphere condition, and

$$D_i = \{x : G_i(x) > 0\}, \quad \partial D_i = \{x : G_i(x) = 0\},$$
$$G_i \in \mathcal{C}^1(\mathbb{R}^d), \quad \inf_{\partial D_i} |\bigtriangledown G_i(x)| > 0, \quad i = 1, \ldots, I.$$

Let $\partial_s D$ be the set of points in ∂D that belong to ∂D_i for more than one value of i, and, for $x \in \partial_s D$, let $\{i_1(x), \ldots, i_k(x)\} = \{i_1, \ldots, i_k\}$, $2 \leq k \leq I$, be the unique set of indexes such that

$$x \in \left(\bigcap_{j=1}^{k} \partial D_{i_j} \right) \cap \left(\bigcap_{i \neq i_1, \ldots, i_k} D_i \right).$$

Denoting by $n^{i_j}(x)$ the unit inward normal at x with respect to ∂D_{i_j}, it will be assumed that for every $n \in \mathcal{N}_x$ there exists a unique set of nonnegative measurable functions $\{\alpha_1(n), \ldots, \alpha_k(n)\}$ such that

$$n = \sum_{j=1}^{k} \alpha_j(n) n^{i_j}(x). \tag{2.3}$$

Then there exists a positive constant α such that

$$|n| \leq \sum_{j=1}^{k} \alpha_j(n) \leq \alpha |n| \quad \forall n \in \mathcal{N}_x,\ x \in \partial_s D. \tag{2.4}$$

Define

$$\begin{aligned} \mathcal{N}_x^1 &= \left\{ n \in \mathcal{N}_x : \textstyle\sum_{j=1}^{k} \alpha_j(n) = 1 \right\} && \text{for } x \in \partial_s D, \\ \mathcal{N}_x^1 &= \{ n \in \mathcal{N}_x : |n| = 1 \} && \text{for } x \in \partial D - \partial_s D. \end{aligned} \tag{2.5}$$

We now introduce the normal reflection Skorohod problem in a formulation devised for application in this work. Denote by $\mathcal{D}([t,T], \mathbb{R}^d)$ ($\mathcal{D}([t,T], \bar{D})$) the space of càdlàg paths with values in $\mathbb{R}^d$ (respectively, $\bar{D}$) and by $\mathcal{I}([t,T], \mathbb{R}_+)$ the space of càdlàg increasing paths with values in $\mathbb{R}_+$.

DEFINITION 2.1. *A solution of the normal reflection Skorohod problem in $\bar{D}$ for $w \in \mathcal{D}([t,T], \mathbb{R}^d)$, $w_t \in \bar{D}$, is a pair (x, ξ), $x \in \mathcal{D}([t,T], \bar{D})$, $\xi \in \mathcal{I}([t,T], \mathbb{R}_+)$, for which there exists a measurable function n such that $n_r \in \mathcal{N}_{x_r}^1$, $d\xi_r$-almost everywhere and the triple (x, ξ, n) satisfies*

$$\begin{cases} x_s = w_s + \int_t^s n_r d\xi_r, & t \leq s \leq T, \\ \xi_s = \int_t^s \mathbb{I}_{\partial D}(x_r) d\xi_r, & t \leq s \leq T. \end{cases}$$

Remark 2.2. The usual definition of the normal reflection Skorohod problem requires $n_r \in \mathcal{N}_{x_r}$, $|n_r| = 1$, rather than $n_r \in \mathcal{N}_{x_r}^1$. The reason why we take $n_r \in \mathcal{N}_{x_r}^1$ is that with this normalization the Neumann boundary condition in (1.3) extends automatically to $\partial_s D$. In fact, if the last equation of (1.3) is verified, then, by (2.3) and (2.5),

$$\sum_{i=1}^{d} \frac{\partial u}{\partial x_i}(t,x) n_i - \lambda(t,x) u(t,x) = g_2(t,x) \quad \forall t \in [0,T],\ n \in \mathcal{N}_x^1,\ x \in \partial D - \partial_1 D.$$

However, there is a one to one correspondence between solutions of the normal reflection Skorohod problem in the usual sense and in the sense of definition (2.2). Therefore all existence and uniqueness results carry over. Moreover, by (2.4) the estimates of Theorem 2.2 below, proved in [2] with the usual definition of the Skorohod problem, carry over too.

Under the Conditions A and B, in [18] it is proved that there exists a unique solution to the normal reflection Skorohod problem for every $w \in \mathcal{C}([t,T], \mathbb{R}^d)$ (the space of continuous paths with values in $\mathbb{R}^d$). For a smaller class of domains, in [2] it is proved that there exists a solution for every $w \in \mathcal{D}([t,T],\mathbb{R}^d)$ such that $\sup_{t\le s\le T}|w_s - w_{s-}|$ is bounded by a constant determined by D; in addition, for every $c > 0$ there exists $c' > 0$ such that for every $w \in \mathcal{D}([t,T],\mathbb{R}^d)$ verifying $\sup_{t\le s\le T}|w_s - w_{s-}| < c'$ there exists a solution (x,ξ) verifying $\sup_{t\le s\le T}|x_s - x_{s-}| \le c$.

THEOREM 2.2 (see [2] and Remark 2.2). *For every $w \in \mathcal{D}([t,T],\mathbb{R}^d)$, $w_t \in \bar{D}$, there exist constants $K(w,t,T)$, $K'(w,t,T)$ such that for any solution (x,ξ) of the normal reflection Skorohod problem in $\bar{D}$ for w such that $\sup_{t\le s\le T}|x_s - x_{s-}| \le c_0$, c_0 being determined by D ($c_0 = +\infty$ if D is convex), it holds that*

$$\begin{aligned}(2.6)\qquad &\sup_{t_1\le r_1<r_2\le t_2}|x(r_1)-x(r_2)| \\ &\le K(w,t,T)\sup_{t_1\le r_1<r_2\le t_2}|w(r_1)-w(r_2)|, \quad t\le t_1\le t_2\le T,\end{aligned}$$

$$(2.7)\qquad \xi_{t_2}-\xi_{t_1}\le K'(w,t,T)\sup_{t_1\le r_1<r_2\le t_2}|w(r_1)-w(r_2)|, \quad t\le t_1\le t_2\le T.$$

$K(w,t,T)$ and $K'(w,t,T)$ are given by

$$(2.8)\qquad K(w,t,T)=\frac{(T-t)C}{\delta_w(M,t,T)}\,,\quad K'(w,t,T)=\frac{(T-t)C'}{\delta_w(M,t,T)}\,,$$

where C, C', and M are constants depending only upon D and

$$\delta_w(M,t,T)=\sup\{\delta:\omega'_w(\delta,t,T)<M\}\,,$$

$$\omega'_w(\delta,t,T)=\inf_{\{r_i\}\in\mathcal{P}}\max_i\sup_{u,r\in[r_i,r_{i+1})}|w(u)-w(r)|,$$

$\mathcal{P}$ being the set of the partitions $\{r_i\}$ of $[t,T]$ such that $\min_i|r_i-r_{i+1}|>\delta$. $\delta_w(M,t,T)$ is bounded away from zero as w varies in a relatively compact subset of $\mathcal{D}([t,T],\mathbb{R}^d)$.

Note that (2.6) and (2.7) imply that for a continuous datum w, any solution (x,ξ) such that $\sup_{t\le s\le T}|x_s - x_{s-}|\le c_0$ is necessarily continuous.

Remark 2.3. If w is a step function on $[t,T]$, $w_s = w_{s_p}$ for $s_p \le s < s_{p+1}$, $p = 0,\dots,N-1$, $t = s_o < s_1 < \cdots < s_N = T$, such that $\max_p|w_{s_{p+1}} - w_{s_p}| < \rho_0$, then there is one and only one solution of the Skorohod problem among the step functions on $[t,T]$ that are constant on $[s_p,s_{p+1})$, $p = 0,\dots,N-1$. The solution is given by

$$x_{s_{p+1}}=\begin{cases} x_{s_p}+\Delta_{p+1}w, & \text{if}\quad x_{s_p}+\Delta_{p+1}w\in\bar{D},\\ \pi(x_{s_p}+\Delta_{p+1}w), & \text{if}\quad x_{s_p}+\Delta_{p+1}w\notin\bar{D},\end{cases}$$

$$\xi_{s_{p+1}}=\begin{cases} \xi_{s_p}, & \text{if}\quad x_{s_p}+\Delta_{p+1}w\in\bar{D},\\ \xi_{s_p}+P\left(\pi(x_{s_p}+\Delta_{p+1}w),x_{s_p}+\Delta_{p+1}w\right), & \text{if}\quad x_{s_p}+\Delta_{p+1}w\notin\bar{D},\end{cases}$$

$$x_s=x_{s_p},\ \xi_s=\xi_{s_p}\quad\text{for}\quad s_p\le s<s_{p+1},\ p=0,\dots,N-1,$$

where $\Delta_{p+1}w = w_{s_{p+1}} - w_{s_p}$ and, for $w\notin\bar{D}$,

$$(2.9)\qquad P(w)=\begin{cases}|\pi(w)-w| & \text{for}\quad \pi(w)\in\partial D-\partial_s D,\\ \sum_j\alpha_j\left(\pi(w)-w\right) & \text{for}\quad \pi(w)\in\partial_s D.\end{cases}$$

Therefore, (X^h,ξ^h), defined recursively by (1.5), is a solution of the normal reflection Skorohod problem for W^h, defined by (1.6), almost surely. Note that, by (2.9), (2.1), and (2.4),

$$(2.10)\qquad \Delta_{p+1}\xi^h\le\alpha|\Delta_{p+1}W^h|.$$

Let $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{t\le s\le T}, P)$ be a filtered probability space, B be an $\mathbb{R}^{d'}$-valued $(\mathcal{F}_s)_{t\le s\le T}$-Brownian motion, and $b : [t,T]\times \bar{D} \to \mathbb{R}^d$, $\sigma : [t,T]\times\bar{D}\to\mathbb{R}^d\otimes\mathbb{R}^{d'}$ be continuous functions.

DEFINITION 2.3. *A pair (X,ξ) of $(\mathcal{F}_s)_{t\le s\le T}$-adapted stochastic processes, X with paths in $\mathcal{C}([t,T],\bar{D})$, ξ with paths in $\mathcal{I}([t,T],\mathbb{R}_+)$, is a (strong) solution of the stochastic differential equation (SDE) of coefficients b and σ with normal reflection in $\bar{D}$ if it satisfies* (1.1), *for some adapted stochastic process n such that $n_s\in\mathcal{N}^1_{X_s}$ $d\xi_s$-almost everywhere, almost surely. X is called a diffusion process with coefficients b and σ and normal reflection in $\bar{D}$, and ξ is called local time.*

Remark 2.4. Of course, if (X,ξ) is a solution of the SDE of coefficients b and σ with normal reflection in $\bar{D}$, then (X,ξ) is a solution of the normal reflection Skorohod problem for $W_s=\int_t^s b(r,X_r)dr+\int_t^s\sigma(r,X_r)dB_r$, almost surely.

The following theorem is proved in [18] for b and σ time-independent, but it extends to time-dependent b and σ as well.

THEOREM 2.4 (see [18]). *Assume Conditions A and B hold and suppose there exists a constant $L>0$ such that*

$$\begin{aligned}|\sigma(s,x)-\sigma(s,y)|&\le L|x-y|, \quad s\in[t,T], \quad x,y\in\bar{D},\\ |b(s,x)-b(s,y)|&\le L|x-y|, \quad s\in[t,T], \quad x,y\in\bar{D}.\end{aligned}\tag{2.11}$$

Then there exists one and only one solution of the SDE of coefficients b and σ with normal reflection in $\bar{D}$.

In the sequel we will always assume that b and σ satisfy (2.11). We conclude this section with some results about the solutions u and v of problems (1.3) and (1.4). By a classical solution of (1.3) or (1.4) we mean a solution in $\mathcal{C}^{1,2}([0,T]\times\bar{D})$.

THEOREM 2.5. *Let u and v be classical solutions of* (1.3) *and* (1.4), *respectively, and suppose either that D is convex or that u and v can be extended to functions in $C^{1,2}([0,T]\times\mathbb{R}^d)$ (see, for instance, Remark* 2.5 *below). Then, for every $(t,x)\in[0,T]\times\bar{D}$, the following representations hold:*

- (i)

$$u(t,x)=\mathbb{E}_{t,x}\left[F_{t,T}(X,\xi)\right],\tag{2.12}$$

where (X,ξ) is the solution of the SDE, of coefficients b and σ with normal reflection in $\bar{D}$, $X_t=x$, and $F_{t,T}(X,\xi)$ is the functional defined by (1.2);

- (ii)

$$v(t,x)=\mathbb{E}_{T-t,x}\left[\tilde{F}_{T-t,T}(\tilde{X},\tilde{\xi})\right],\tag{2.13}$$

where $(\tilde{X},\tilde{\xi})$ is the solution of the SDE of coefficients $\tilde{b}(s,x)=b(T-s,x)$ and $\tilde{\sigma}(s,x)=\sigma(T-s,x)$ and normal reflection in $\bar{D}$, $\tilde{X}_{T-t}=x$. $\tilde{F}_{T-t,T}(\tilde{X},\tilde{\xi})$ is the functional defined by (1.2) *with t replaced by $T-t$ and the functions $c(s,x)$, $\lambda(s,x)$, $g_i(s,x), i=1,2$, replaced by $\tilde{c}(s,x)=c(T-s,x)$, $\tilde{\lambda}(s,x)=\lambda(T-s,x)$, $\tilde{g}_i(s,x)=g_i(T-s,x)$, $i=1,2$, respectively.*

Proof. Consider the assertion (i). When D has sufficiently smooth boundary, and pure Dirichlet or pure Neumann boundary conditions are considered, this is a well-known result, which can be found, for instance, in [3]. For the mixed boundary conditions considered here, it is enough to apply the proof for Dirichlet conditions to the reflecting diffusion X, i.e., to apply Ito's formula to the function $\varphi(s,x,y,z)=u(s,x)\exp(y+z)$, and to the stochastic process (X,Y,Z) with Y and Z defined as in (1.2). When D has a piecewise $\mathcal{C}^1$ boundary, the assertion follows from Remark 2.2 and the fact that, under the present assumptions, Ito's formula still holds. Finally,

(ii) follows from the fact that $\tilde{u}(t,x) = v(T-t,x)$ is a solution of a problem of the form (1.3) with coefficients $\tilde{b}$, $\tilde{\sigma}$, $\tilde{c}$, λ and data $\tilde{g}_1$, $\tilde{g}_2$, f. □

When D has a C^2 boundary, i.e.,

$$D = \{x : G(x) > 0\}, \quad G \in C^2, \qquad \partial D = \{x : G(x) = 0\}, \quad \inf_{x\in\partial D} |\nabla G(x)| > 0, \tag{2.14}$$

the following theorems, given in [9], discuss existence, uniqueness, and regularity of the solutions of (1.3) and (1.4), at least for the pure Dirichlet and pure Neumann cases. We consider only (1.3), since (1.4) reduces to (1.3) by a time change. Let $l > 0$ be a noninteger number and denote by $H^l(\bar{D})$ the space of functions $f \in C^{[l]}(\bar{D})$ such that $D_x^{[l]} f$ (with this notation we mean any derivative of order $[l]$ with respect to the variables $x_1, \ldots, x_d$) is a Hölder continuous function of exponent $l - [l]$ (see [9]). Analogously, let $H^{l/2,l}([0,T] \times \bar{D})$ denote the space of functions u with continuous derivatives $D_t^i D_x^j u(t,x)$ for $2i + j < l$, such that $D_t^i D_x^j u(t,x)$ for $2i + j = [l]$ are Hölder continuous of exponent $l - [l]$. By the phrase "∂D belongs to H^l" we mean that the function G in (2.14) belongs to $H^l = H^l(\mathbb{R}^d)$. For a function ψ defined on ∂D ($[0,T] \times \partial D$), $\psi \in H^l(\partial D)$ ($\psi \in H^{l/2,l}([0,T] \times \partial D)$) means that ψ is the restriction to ∂D of a function in $H^l(\{x : d(x,\partial D) \le \rho\})$ ($H^{l/2,l}([0,T] \times \{x : d(x,\partial D) \le \rho\})$) for some $\rho > 0$.

We say that the compatibility conditions of order $m \ge 0$ are fulfilled for the pure Dirichlet problem ($\partial_1 D = \partial D$) if

$$\gamma_k|_{\partial D} = \left.\frac{\partial^k g_1}{\partial t^k}\right|_{t=T}, \quad k = 0, \ldots, m, \tag{2.15}$$

where γ_k are the functions defined recursively by

$$\begin{cases} \gamma_0 = f, \\ \gamma_{k+1} = \displaystyle\sum_{j=0}^{k} \binom{k}{j} \mathcal{L}^{(j)} \gamma_{k-j}, \end{cases} \tag{2.16}$$

with

$$\mathcal{L}^{(j)}\gamma(x) = -\frac{1}{2}\sum_{i,l=1}^{d} \frac{\partial^j a_{il}}{\partial t^j}(T,x)\frac{\partial^2 \gamma}{\partial x_i \partial x_l}(x) - \sum_{i=1}^{d} \frac{\partial^j b_i}{\partial t^j}(T,x)\frac{\partial \gamma}{\partial x_i}(x) - \frac{\partial^j c}{\partial t^j}(T,x)\gamma(x).$$

Analogously, we say that the compatibility conditions of order $m \ge 0$ are fulfilled for the pure Neumann problem ($\partial_1 D = \emptyset$) if

$$\left.\left[\frac{\partial \gamma_k}{\partial n} - \sum_{j=0}^{k} \binom{k}{j} \left.\frac{\partial^j \lambda}{\partial t^j}\right|_{t=T} \gamma_{k-j}\right]\right|_{\partial D} = \left.\frac{\partial^k g_2}{\partial t^k}\right|_{t=T}, \ k = 0, \ldots, m, \tag{2.17}$$

where γ_k are the functions defined in (2.16).

THEOREM 2.6 (see [9]). *Let $\partial_1 D = \partial D$. Suppose the coefficients b, σ and the function c belong to $H^{l/2,l}([0,T] \times \bar{D})$ and the boundary ∂D belongs to H^{l+2} for some $l > 0$. Then for every $f \in H^{l+2}(\bar{D})$, $g_1 \in H^{l/2+1,l+2}([0,T] \times \partial D)$ satisfying the compatibility conditions* (2.15) *of order* $[l/2] + 1$, *the problem* (1.3) *has one and only one solution in $H^{l/2+1,l+2}([0,T] \times \bar{D})$.*

THEOREM 2.7 (see [9]). *Let $\partial_1 D = \emptyset$. Suppose ∂D belongs to H^{l+2} and the coefficients b, σ and the function c are in the class $H^{l/2,l}([0,T]\times\bar{D})$, $\lambda \in H^{(l+1)/2,l+1}([0,T]\times$*

∂D), for some $l > 0$. Then, for every $f \in H^{l+2}(\bar{D})$, $g_2 \in H^{(l+1)/2,l+1}([0,T] \times \partial D)$ satisfying the compatibility conditions (2.17) *of order* $[(l+1)/2]$, *the problem* (1.3) *has one and only one solution in $H^{l/2+1,l+2}([0,T] \times \bar{D})$.*

Remark 2.5. If D is not convex, in Theorem 2.5 above and in section 3 we need to assume that the solutions u and v of (1.3) and (1.4) can be extended to functions in $\mathcal{C}^{1,2}([0,T] \times \mathbb{R}^d)$. This is always true if ∂D and the coefficients and data are sufficiently smooth. In particular, if ∂D belongs to H^{l+2}, for some $l > 0$, any function in $H^{l/2+1,l+2}([0,T] \times \bar{D})$ can be extended to a function in $H^{l/2+1,l+2}([0,T] \times \mathbb{R}^d)$ (Proposition 1.17 in [5]).

3. Main results. Let X be the diffusion process with coefficients b and σ and normal reflection in $\bar{D}$ (see Definition 2.3 and Theorem 2.4) and ξ be the corresponding local time, and let (X^h, ξ^h), $X_t^h = X^0 = x$, be defined by (1.5) for h small enough ($h < h_0$, where h_0 is defined by (3.6) below). Let $\{\Delta_{p+1}\eta\}$ be a sequence of d'-dimensional i.i.d. random variables, on some probability space $(\Omega, \mathcal{F}, P)$, such that

$$\mathbb{E}\left[\Delta_{p+1}\eta\right] = 0, \tag{3.1}$$

$$\mathbb{E}\left[(\Delta_{p+1}\eta_i)(\Delta_{p+1}\eta_j)\right] = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \tag{3.2}$$

$$\mathbb{E}\left[|\Delta_{p+1}\eta|^3\right] < +\infty. \tag{3.3}$$

In addition, if D is not convex, take $\Delta_{p+1}\eta$ bounded. Set

$$H_N = \max_{0 \le p \le N-1} |\Delta_{p+1}\eta| \tag{3.4}$$

and

$$H = \sup_{\omega \in \Omega} |\Delta_{p+1}\eta(\omega)| \tag{3.5}$$

if $\Delta_{p+1}\eta$ is bounded.

In this section, we will always assume that there exists a classical solution, u, of (1.3) (see, for instance, Theorems 2.6, 2.7) and, if D is not convex, that u can be extended to a function in $\mathcal{C}^{1,2}([0,T] \times \mathbb{R}^d)$ (see Remark 2.5). The extended function will still be denoted by u, and any assumptions on u will be referred to the extension.

Let

$$h_0 = \left(\frac{\rho_0}{\sup_{[0,T]\times\bar{D}} |b(s,x)| + H \sup_{[0,T]\times\bar{D}} \|\sigma(s,x)\|} \right)^2 \wedge 1 \tag{3.6}$$

(where ρ_0 is the constant of Condition A in section 2), if D is not convex, and $h_0 = 1$, if D is convex. In all statements and computations, C and $C(T)$ will denote constants depending only on D, the coefficients, and the data of the problem. Consider the functionals $F_{t,T}$ and $F_{t,T}^h$ defined by (1.2) and (1.7), respectively.

LEMMA 3.1. *For $h < h_0$, for every $x \in \bar{D}$, $t \in [0,T]$, it holds that*

$$\begin{aligned}
&\left|\mathbb{E}_{t,x}\left[F_{t,T}(X,\xi)\right] - \mathbb{E}_{t,x}\left[F_{t,T}^h(X^h,\xi^h)\right]\right| \\
&\le C(T)\mathbb{E}_{t,x}\left[\mathbb{I}_{N \ge \nu^h} H_N\right]\sqrt{h} \\
&+C(T)\mathbb{E}_{t,x}\left[H_N \xi_T^h\right]\sqrt{h} \\
&+C(T)\left\{1 + \mathbb{E}_{t,x}\left[\xi_T^h\right] + \mathbb{E}_{t,x}\left[H_N \xi_T^h\right] + \mathbb{E}_{t,x}\left[H_N^2 \xi_T^h\right]\right\} h \\
&+\left|\mathbb{E}_{t,x}\left[r(t,T,h)\right]\right| \\
&+\left|\mathbb{E}_{t,x}\left[R(t,T,h)\right]\right|,
\end{aligned} \tag{3.7}$$

where

$$r(t,T,h) \tag{3.8}$$
$$= \mathbb{I}_{\nu^h \geq 1} \sum_{p=0}^{(N\wedge\nu^h)-1} \int_{s_p}^{s_{p+1}} \left[u_s(s,X_s^h) - u_s(s_{p+1},X_s^h)\right] \exp\left(Y_s^h + Z_s^h\right) ds,$$

$$R(t,T,h)$$
$$= \frac{1}{2}\, \mathbb{I}_{\nu^h \geq 1} \sum_{p=0}^{(N\wedge\nu^h)-1} \sum_{i,j=1}^{d+2} \Big[\varphi_{v_i v_j}(s_{p+1}, V_{s_p}^h + \theta_p^h \Delta_{p+1} V^h) \tag{3.9}$$
$$- \varphi_{v_i v_j}(s_{p+1}, V_{s_p}^h)\Big] \Delta_{p+1} V_i^h \Delta_{p+1} V_j^h,$$

$$v_i = x_i, (V_{s_p}^h)_i = (X_{s_p}^h)_i, \Delta_{p+1}V_i^h = \Delta_{p+1}X_i^h \quad \textit{for} \quad i = 1,\ldots,d,$$

$$v_{d+1} = y, (V_{s_p}^h)_{d+1} = (Y_{s_p}^h), \Delta_{p+1}V_{d+1}^h = \Delta_{p+1}Y^h,$$

$$v_{d+2} = z, (V_{s_p}^h)_{d+2} = (Z_{s_p}^h), \Delta_{p+1}V_{d+2}^h = \Delta_{p+1}Z^h,$$

$$\varphi(t,v) = \varphi(t,x,y,z) = u(t,x)\exp(y+z), \quad 0 < \theta_p^h < 1,$$

and $C(T)$ *is a constant depending only on* T, D *the coefficients of* (1.1), *and the data of* (1.2).

Proof. By applying Ito's formula for semimartingales to the function $\varphi(t,x,y,z) = u(t,x)\exp(y+z)$, we have

$$\varphi(T\wedge s_{\nu^h}, X_{T\wedge s_{\nu^h}}^h, Y_{T\wedge s_{\nu^h}}^h, Z_{T\wedge s_{\nu^h}}^h) - \varphi(t,x,0,0)$$
$$= \int_t^{T\wedge s_{\nu^h}} \varphi_s(s, X_s^h, Y_s^h, Z_s^h) ds \tag{3.10}$$
$$+ \mathbb{I}_{\nu^h \geq 1} \sum_{p=0}^{(N\wedge\nu^h)-1} \left[\varphi(s_{p+1}, X_{s_{p+1}}^h, Y_{s_{p+1}}^h, Z_{s_{p+1}}^h) - \varphi(s_{p+1}, X_{s_p}^h, Y_{s_p}^h, Z_{s_p}^h)\right].$$

By expanding the summands in the right-hand side of (3.10) by Taylor's formula, introducing the notation $u^p = u(s_{p+1}, X_{s_p}^h)$, $u_s^p = u_s(s_{p+1}, X_{s_p}^h)$, $u_{x_i}^p = u_{x_i}(s_{p+1}, X_{s_p}^h)$, $u_{x_ix_j}^p = u_{x_ix_j}(s_{p+1}, X_{s_p}^h)$, $b^p = b(s_{p+1}, X_{s_p}^h)$, $\sigma^p = \sigma(s_{p+1}, X_{s_p}^h)$, $c^p = c(s_{p+1}, X_{s_p}^h)$, $\lambda^p = \lambda(s_{p+1}, X_{s_p}^h)$, $n_i^p = n_i(X_{s_p}^h)$, $e^p = \exp(Y_{s_p}^h + Z_{s_p}^h)$, $u^{p+1} = u(s_{p+1}, X_{s_{p+1}}^h)$, $u_s^{p+1} = u_s(s_{p+1}, X_{s_{p+1}}^h)$, $u_{x_i}^{p+1} = u_{x_i}(s_{p+1}, X_{s_{p+1}}^h)$, $u_{x_ix_j}^{p+1} = u_{x_ix_j}(s_{p+1}, X_{s_{p+1}}^h)$, $b^{p+1} = b(s_{p+1}, X_{s_{p+1}}^h)$, $\sigma^{p+1} = \sigma(s_{p+1}, X_{s_{p+1}}^h)$, $c^{p+1} = c(s_{p+1}, X_{s_{p+1}}^h)$, $\lambda^{p+1} = \lambda(s_{p+1}, X_{s_{p+1}}^h)$, $n_i^{p+1} = n_i(X_{s_{p+1}}^h)$, $e^{p+1} = \exp(Y_{s_{p+1}}^h + Z_{s_{p+1}}^h)$, we get

$$\varphi(T\wedge s_{\nu^h}, X_{T\wedge s_{\nu^h}}^h, Y_{T\wedge s_{\nu^h}}^h, Z_{T\wedge s_{\nu^h}}^h) - \varphi(t,x,0,0)$$

$$= \int_t^{T\wedge s_{\nu^h}} u_s(s, X_s^h)\exp(Y_s^h + Z_s^h) ds$$
$$+ \mathbb{I}_{\nu^h \geq 1} \sum_{p=0}^{(N\wedge\nu^h)-1} e^p \left\{ \sum_{i=1}^{d} u_{x_i}^p \Delta_{p+1} X_i^h + u^p c^p h - u^p \lambda^{p+1} \Delta_{p+1} \xi^h \mathbb{I}_{p<\nu^h-1} \right\}$$
$$+ \frac{1}{2} \mathbb{I}_{\nu^h \geq 1} \sum_{p=0}^{(N\wedge\nu^h)-1} e^p \sum_{i,j=1}^{d} u_{x_ix_j}^p \Delta_{p+1} X_i^h \Delta_{p+1} X_j^h$$
$$+ \mathbb{I}_{\nu^h \geq 1} \sum_{p=0}^{(N\wedge\nu^h)-1} e^p \sum_{i=1}^{d} \left\{ u_{x_i}^p \Delta_{p+1} X_i^h c^p h - u_{x_i}^p \Delta_{p+1} X_i^h \lambda^{p+1} \Delta_{p+1} \xi^h \mathbb{I}_{p<\nu^h-1} \right\}$$

$$+\frac{1}{2}\mathbb{I}_{\nu^h\geq 1}\sum_{p=0}^{(N\wedge\nu^h)-1} e^p\left\{u^p(c^p)^2h^2 - 2u^pc^p\lambda^{p+1}h\Delta_{p+1}\xi^h\mathbb{I}_{p<\nu^h-1}\right.$$
$$\left.+u^p(\lambda^{p+1})^2(\Delta_{p+1}\xi^h)^2\mathbb{I}_{p<\nu^h-1}\right\} + R(t,T,h),$$

where $R(t,T,h)$ is given by (3.9). Recalling that we have set

$$\Delta_{p+1}W^h = hb(s_{p+1},X^h_{s_p}) + \sqrt{h}\sigma(s_{p+1},X^h_{s_p})\Delta_{p+1}\eta, \tag{3.11}$$

so that

$$\sum_{i,j=1}^{d} u^p_{x_ix_j}\Delta_{p+1}X^h_i\Delta_{p+1}X^h_j$$
$$= \sum_{i,j=1}^{d} u^p_{x_ix_j}\left\{\Delta_{p+1}W^h_i\Delta_{p+1}W^h_j + 2\Delta_{p+1}X^h_in^{p+1}_j\Delta_{p+1}\xi^h\right.$$
$$\left.-n^{p+1}_in^{p+1}_j(\Delta_{p+1}\xi^h)^2\right\},$$

we obtain, rearranging the terms,

$$\varphi(T\wedge s_{\nu^h}, X^h_{T\wedge s_{\nu^h}}, Y^h_{T\wedge s_{\nu^h}}, Z^h_{T\wedge s_{\nu^h}}) - \varphi(t,x,0,0)$$
$$= \int_t^{T\wedge s_{\nu^h}} u_s(s,X^h_s)\exp\left(Y^h_s + Z^h_s\right)ds$$
$$+\mathbb{I}_{\nu^h\geq 1}\sum_{p=0}^{(N\wedge\nu^h)-1} e^p\left\{\sum_{i=1}^{d} u^p_{x_i}\Delta_{p+1}W^h_i + \frac{1}{2}\sum_{i,j=1}^{d} u^p_{x_ix_j}\Delta_{p+1}W^h_i\Delta_{p+1}W^h_j + u^pc^ph\right\}$$
$$+\mathbb{I}_{\nu^h\geq 1}\sum_{p=0}^{(N\wedge\nu^h)-1} e^p\left\{\sum_{i=1}^{d} u^p_{x_i}n^{p+1}_i - u^p\lambda^{p+1}\mathbb{I}_{p<\nu^h-1}\right\}\Delta_{p+1}\xi^h$$
$$+\mathbb{I}_{\nu^h\geq 1}\sum_{p=0}^{(N\wedge\nu^h)-1} e^p\sum_{j=1}^{d}\left\{\sum_{i=1}^{d} u^p_{x_ix_j}n^{p+1}_i - u^p_{x_j}\lambda^{p+1}\mathbb{I}_{p<\nu^h-1}\right\}\Delta_{p+1}X^h_j\Delta_{p+1}\xi^h$$
$$+\frac{1}{2}\mathbb{I}_{\nu^h\geq 1}\sum_{p=0}^{(N\wedge\nu^h)-1} e^p\left\{u^p(\lambda^{p+1})^2\mathbb{I}_{p<\nu^h-1} - \sum_{i,j=1}^{d} u^p_{x_ix_j}n^{p+1}_in^{p+1}_j\right\}(\Delta_{p+1}\xi^h)^2$$
$$+\mathbb{I}_{\nu^h\geq 1}\sum_{p=0}^{(N\wedge\nu^h)-1} e^pc^p\sum_{i=1}^{d} u^p_{x_i}\Delta_{p+1}W^h_ih$$
$$+\frac{1}{2}\mathbb{I}_{\nu^h\geq 1}\sum_{p=0}^{(N\wedge\nu^h)-1} e^pu^p(c^p)^2h^2$$
$$+\mathbb{I}_{\nu^h\geq 1}\sum_{p=0}^{(N\wedge\nu^h)-1} e^pc^p\left\{\sum_{i=1}^{d} u^p_{x_i}n^{p+1}_i - u^p\lambda^{p+1}\mathbb{I}_{p<\nu^h-1}\right\}\Delta_{p+1}\xi^hh$$
$$+R(t,T,h).$$

Then, by adding and subtracting

$$\sum_{p=0}^{(N\wedge\nu^h)-1} e^pu^p_sh$$

and

$$\mathbb{I}_{\nu^h \geq 1} \sum_{p=0}^{(N\wedge\nu^h)-1} e^p \left\{ \sum_{i=1}^d u_{x_i}^{p+1} n_i^{p+1} - u^{p+1}\lambda^{p+1}\mathbb{I}_{p<\nu^h-1} \right\} \Delta_{p+1}\xi^h,$$

and by taking expectations, we have

$$\mathbb{E}_{t,x}\left[\varphi(T\wedge s_{\nu^h}, X^h_{T\wedge s_{\nu^h}}, Y^h_{T\wedge s_{\nu^h}}, Z^h_{T\wedge s_{\nu^h}})\right] - \varphi(t,x,0,0)$$

$$= \mathbb{E}_{t,x}\left[\mathbb{I}_{\nu^h\geq 1}\sum_{p=0}^{(N\wedge\nu^h)-1}\int_{s_p}^{s_{p+1}}\left[u_s(s,X_s^h) - u_s(s_{p+1},X_s^h)\right]\exp\left(Y_s^h + Z_s^h\right)ds\right]$$

$$+\mathbb{E}_{t,x}\left[\mathbb{I}_{\nu^h\geq 1}\sum_{p=0}^{(N\wedge\nu^h)-1} e^p \quad \left\{u_s(s_{p+1},X_{s_p}^h) + L_{s_{p+1}}u(s_{p+1},X_{s_p}^h)\right.\right.$$

$$\left.\left. +c(s_{p+1},X_{s_p}^h)u(s_{p+1},X_{s_p}^h)\right\} h\right]$$

$$+\mathbb{E}_{t,x}\left[\mathbb{I}_{\nu^h\geq 2}\sum_{p=0}^{(N\wedge\nu^h)-2} e^p \quad \left\{\sum_{i=1}^d u_{x_i}(s_{p+1},X_{s_{p+1}}^h)n_i(X_{s_{p+1}}^h)\right.\right.$$

$$\left.\left. -\lambda(s_{p+1},X_{s_{p+1}}^h)u(s_{p+1},X_{s_{p+1}}^h) \right\}\Delta_{p+1}\xi^h\right]$$

$$+\mathbb{E}_{t,x}\left[\mathbb{I}_{N<\nu^h}\, e^{N-1}\left\{\sum_{i=1}^d u_{x_i}(s_N,X_{s_N}^h)n_i(X_{s_N}^h) - \lambda(s_N,X_{s_N}^h)u(s_N,X_{s_N}^h)\right\}\Delta_N\xi^h\right]$$

$$+\mathbb{E}_{t,x}\left[\mathbb{I}_{N\geq\nu^h\geq 1}\, e^{\nu^h-1}\sum_{i=1}^d u_{x_i}(s_{\nu^h},X_{s_{\nu^h}}^h)n_i(X_{s_{\nu^h}}^h)\Delta_{\nu^h}\xi^h\right]$$

$$+\mathbb{E}_{t,x}\left[\mathbb{I}_{\nu^h\geq 1}\sum_{p=0}^{(N\wedge\nu^h)-1} e^p\sum_{i=1}^d n_i^{p+1}\left\{\left(u_{x_i}^p - u_{x_i}^{p+1}\right) + \sum_{j=1}^d u_{x_ix_j}^p\Delta_{p+1}X_j^h\right\}\Delta_{p+1}\xi^h\right]$$

$$+\mathbb{E}_{t,x}\left[\mathbb{I}_{\nu^h\geq 1}\sum_{p=0}^{(N\wedge\nu^h)-1} e^p\lambda^{p+1}\mathbb{I}_{p<\nu^h-1}\left\{\left(u^{p+1}-u^p\right) - \sum_{j=1}^d u_{x_j}^p\Delta_{p+1}X_j^h\right\}\Delta_{p+1}\xi^h\right]$$

$$+\frac{1}{2}\mathbb{E}_{t,x}\left[\mathbb{I}_{\nu^h\geq 1}\sum_{p=0}^{(N\wedge\nu^h)-1} e^p\left\{u^p(\lambda^{p+1})^2\mathbb{I}_{p<\nu^h-1} - \sum_{i,j=1}^d u_{x_ix_j}^p n_i^{p+1}n_j^{p+1}\right\}(\Delta_{p+1}\xi^h)^2\right]$$

$$+\frac{1}{2}\mathbb{E}_{t,x}\left[\mathbb{I}_{\nu^h\geq 1}\sum_{p=0}^{(N\wedge\nu^h)-1} e^p\left\{\sum_{i.j=1}^d u_{x_ix_j}^p b_i^p b_j^p + 2c^p\ \sum_{i=1}^d u_{x_i}^p b_i^p + u^p(c^p)^2\right\}h^2\right]$$

$$+\mathbb{E}_{t,x}\left[\mathbb{I}_{\nu^h\geq 1}\sum_{p=0}^{(N\wedge\nu^h)-1} e^p c^p\left\{\sum_{i=1}^d u_{x_i}^p n_i^{p+1} - u^p\lambda^{p+1}\mathbb{I}_{p<\nu^h-1}\right\}\Delta_{p+1}\xi^h h\right]$$

Therefore, taking into account that u is a solution of problem (1.3), and Remark 2.2, we get, by moving the appropriate terms to the left-hand side and rearranging the other ones,

$$
\begin{aligned}
&\mathbb{E}_{t,x}\left[F_{t,T}(X,\xi)\right] - \mathbb{E}_{t,x}\left[F^h_{t,T}(X^h,\xi^h)\right] \\
&= -\mathbb{E}_{t,x}\left[\mathbb{I}_{N\geq\nu^h\geq 1}\, e^{\nu^h-1}\sum_{i=1}^d u_{x_i}(s_{N\wedge\nu^h}, X^h_{s_{N\wedge\nu^h}})n_i(X^h_{s_{N\wedge\nu^h}})\Delta_{N\wedge\nu^h}\xi^h\right] && \text{(a)}\\
&+\tfrac{1}{2}\mathbb{E}_{t,x}\left[\mathbb{I}_{\nu^h\geq 1}\sum_{p=0}^{(N\wedge\nu^h)-1} e^p\{\sum_{i,j=1}^d u^p_{x_ix_j}n_i^{p+1}n_j^{p+1} - u^p(\lambda^{p+1})^2\mathbb{I}_{p<\nu^h-1}\}\left(\Delta_{p+1}\xi^h\right)^2\right] && \text{(b)}\\
&+\mathbb{E}_{t,x}\left[\mathbb{I}_{\nu^h\geq 1}\sum_{p=0}^{(N\wedge\nu^h)-1} e^p \sum_{i=1}^d n_i^{p+1}\sum_{j=1}^d\left\{u_{x_ix_j}(s_{p+1}, X^h_{s_p}+\vartheta^h_{i,p}\Delta_{p+1}X^h)\right.\right.\\
&\qquad\left.\left.-u_{x_ix_j}(s_{p+1},X^h_{s_p})\right\}\Delta_{p+1}X^h_j\Delta_{p+1}\xi^h\right] && \text{(c)}\\
&-\tfrac{1}{2}\mathbb{E}_{t,x}\left[\mathbb{I}_{\nu^h\geq 1}\sum_{p=0}^{(N\wedge\nu^h)-1} e^p\left\{\sum_{i,j=1}^d u^p_{x_ix_j}b_i^pb_j^p +2\,c^p\sum_{i=1}^d u^p_{x_i}b_i^p + u^p(c^p)^2\right\}h^2\right] && \text{(d)}\\
&+\mathbb{E}_{t,x}\left[\mathbb{I}_{\nu^h\geq 1}\sum_{p=0}^{(N\wedge\nu^h)-1} e^pc^p\left\{u^p\lambda^{p+1}\mathbb{I}_{p<\nu^h-1} - \sum_{i=1}^d u^p_{x_i}n_i^{p+1}\right\}\Delta_{p+1}\xi^h h\right] && \text{(e)}\\
&+\mathbb{E}_{t,x}\left[\mathbb{I}_{\nu^h\geq 1}\sum_{p=0}^{(N\wedge\nu^h)-1} e^p\lambda^{p+1}\,\mathbb{I}_{p<\nu^h-1}\sum_{j=1}^d\left\{u_{x_j}(s_{p+1},X^h_{s_p})\right.\right.\\
&\qquad\left.\left.-u_{x_j}(s_{p+1},X^h_{s_p}+\tilde{\vartheta}^h_p\Delta_{p+1}X^h)\right\}\Delta_{p+1}X^h_j\Delta_{p+1}\xi^h\right] && \text{(f)}\\
&-\mathbb{E}_{t,x}\left[r(t,T,h)\right]\\
&-\mathbb{E}_{t,x}\left[R(t,T,h)\right],
\end{aligned}
\tag{3.12}
$$

where $0<\vartheta^h_{i,p}<1$, $\quad 0<\tilde{\vartheta}^h_p<1$ and $r(t,T,h)$ is defined by (3.8).

We can now estimate the terms (a) through (f).

(a) By (2.10), we have almost surely, for every p,

$$|\Delta_{p+1}\xi^h| \leq \alpha\left|hb(s_{p+1},X^h_{s_p}) + \sqrt{h}\sigma(s_{p+1},X^h_{s_p})\Delta_{p+1}\eta\right|, \tag{3.13}$$

and hence

$$
\begin{aligned}
|(\mathrm{a})| &\leq C(T)\mathbb{E}_{t,x}\left[\mathbb{I}_{N\geq\nu^h}|hb(s_{\nu^h},X^h_{s_{\nu^h-1}}) + \sqrt{h}\sigma(s_{\nu^h},X^h_{\nu^h-1})\Delta_{\nu^h}\eta|\right]\\
&\leq C(T)\left\{P_{t,x}(N\geq\nu^h)h + \mathbb{E}_{t,x}\left[\mathbb{I}_{N\geq\nu^h}H_N\right]\sqrt{h}\right\},
\end{aligned}
$$

where H_N is defined by (3.4).

(b) By (3.13),

$$\begin{aligned}|(\mathrm{b})| &\leq C(T)\mathbb{E}_{t,x}\left[\sum_{p=0}^{(N\wedge\nu^h)-1}\left(\Delta_{p+1}\xi^h\right)^2\right]\\ &\leq C(T)\mathbb{E}_{t,x}\left[\sum_{p=0}^{(N\wedge\nu^h)-1}|hb(s_{p+1},X^h_{s_p})+\sqrt{h}\sigma(s_{p+1},X^h_{s_p})\Delta_{p+1}\eta|\Delta_{p+1}\xi^h\right]\\ &\leq C(T)\left\{\mathbb{E}_{t,x}\left[\xi^h_T\right]h+\mathbb{E}_{t,x}\left[H_N\xi^h_T\right]\sqrt{h}\right\}.\end{aligned}$$

(c) Taking into account that

$$|\Delta_{p+1}X^h| \leq 2|hb(s_{p+1},X^h_{s_p})+\sqrt{h}\sigma(s_{p+1},X^h_{s_p})\Delta_{p+1}\eta|, \tag{3.14}$$

we have

$$\begin{aligned}|(\mathrm{c})| &\leq C(T)\mathbb{E}_{t,x}\left[\sum_{p=0}^{N-1}|hb(s_{p+1},X^h_{s_p})+\sqrt{h}\sigma(s_{p+1},X^h_{s_p})\Delta_{p+1}\eta|\Delta_{p+1}\xi^h\right]\\ &\leq C(T)\left\{\mathbb{E}_{t,x}\left[\xi^h_T\right]h+\mathbb{E}_{t,x}\left[H_N\xi^h_T\right]\sqrt{h}\right\}.\end{aligned}$$

(d) $|(\mathrm{d})| \leq C(T)h$.
(e) $|(\mathrm{e})| \leq C(T)\mathbb{E}_{t,x}\left[\xi^h_T\right]h$.
(f) By (3.14):

$$\begin{aligned}|(\mathrm{f})| &\leq C(T)\mathbb{E}_{t,x}\left[\sum_{p=0}^{N-1}|\Delta_{p+1}X^h|^2\Delta_{p+1}\xi^h\right]\\ &\leq C(T)\left\{\mathbb{E}_{t,x}\left[\xi^h_T\right]h^2+\mathbb{E}_{t,x}\left[H_N\xi^h_T\right]h\sqrt{h}+\mathbb{E}_{t,x}\left[H_N^2\xi^h_T\right]h\right\}.\end{aligned}$$

This concludes the proof. □

Remark 3.1.

(i) If $\partial_1 D = \emptyset$, that is (1.3) is a pure Neumann problem, then $(a) = 0$ since $\nu^h = +\infty$.

(ii) If $u_{x_ix_j}$, $i,j = 1,\ldots,d$, is a Lipschitz continuous function of x we have

$$|u_{x_ix_j}(s_{p+1},X^h_{s_{p+1}}) - u_{x_ix_j}(s_{p+1},X^h_{s_{p+1}}+\vartheta^h_{i,p}\Delta_{p+1}X^h)| \leq C(T)|\Delta_{p+1}X^h|,$$

and therefore,

$$\begin{aligned}|(\mathrm{c})| &\leq C(T)\mathbb{E}_{t,x}\left[\sum_{p=0}^{(N\wedge\nu^h)-1}|\Delta_{p+1}X^h|^2\Delta_{p+1}\xi^h\right]\\ &\leq C(T)\left\{\mathbb{E}_{t,x}\left[\xi^h_T\right]h^2+\mathbb{E}_{t,x}\left[H_N\xi^h_T\right]h\sqrt{h}+\mathbb{E}_{t,x}\left[H_N^2\xi^h_T\right]h\right\}.\end{aligned}$$

LEMMA 3.2.

- (i)

$$\sup_{t\leq T,x\in\bar{D}} |\mathbb{E}_{t,x}\left[r(t,T,h)\right]| \longrightarrow 0 \quad \textit{for} \quad h\to 0.$$

If u_s is a Hölder continuous function of s of exponent $1/2$ then, for $h<h_0$,

$$\sup_{t\leq T,x\in\bar{D}} |\mathbb{E}_{t,x}\left[r(t,T,h)\right]| \leq C(T)\sqrt{h}.$$

If u_s is a Lipschitz continuous function of s then, for $h < h_0$,

$$\sup_{t\le T, x\in\bar D} |\mathbb{E}_{t,x}[r(t,T,h)]| \le C(T)h.$$

- (ii)

$$\sup_{t\le T, x\in\bar D} |\mathbb{E}_{t,x}[R(t,T,h)]| \longrightarrow 0 \quad \textit{for} \quad h\to 0.$$

If $u_{x_ix_j}$, $i,j = 1,\dots,d$, is a Lipschitz continuous function of x, then, for $h < h_0$,

$$\sup_{t\le T, x\in\bar D} |\mathbb{E}_{t,x}[R(t,T,h)]| \le C(T)\sqrt{h}.$$

Proof. (i) It is enough to observe that

$$|\mathbb{E}_{t,x}[r(t,T,h)]| \le C(T) \sup_{\substack{t\le s_1\le s_2\le T\\ |s_1-s_2|\le h\\ y\in\bar D}} |u_s(s_1,y)-u_s(s_2,y)|.$$

(ii) Setting

$$\Phi_{i,j}^{h,p} = |\varphi_{v_iv_j}(s_{p+1}, V_{s_p}^h) - \varphi_{v_iv_j}(s_{p+1}, V_{s_p}^h + \theta_p^h\Delta_{p+1}V^h)|,$$

we have, by (3.13) and (3.14),

$$\begin{aligned} &|\mathbb{E}_{t,x}[R(t,T,h)]| \\ \le\ & \tfrac{1}{2}\textstyle\sum_{p=0}^{N-1}\sum_{i,j=1}^{d+2}\mathbb{E}_{t,x}\left[\Phi_{i,j}^{h,p}|\Delta_{p+1}V^h|^2\right] \\ \le\ & C\textstyle\sum_{p=0}^{N-1}\sum_{i,j=1}^{d+2}\mathbb{E}_{t,x}\left[\Phi_{i,j}^{h,p}\left\{|\Delta_{p+1}X^h|^2 + h^2 + \left(\Delta_{p+1}\xi^h\right)^2\right\}\right] \\ \le\ & C(T)\textstyle\sum_{p=0}^{N-1}\sum_{i,j=1}^{d+2}\mathbb{E}_{t,x}\left[\left(\Phi_{i,j}^{h,p}\right)^3\right]^{1/3}\left\{h + \mathbb{E}_{t,x}\left[|\Delta_{p+1}\eta|^3\right]^{2/3}\right\}h \\ \le\ & C(T)\max_{0\le p\le N-1}\textstyle\sum_{i,j=1}^{d+2}\mathbb{E}_{t,x}\left[\left(\Phi_{i,j}^{h,p}\right)^3\right]^{1/3}. \end{aligned}$$

We have, by (1.7), (3.12), and (3.13),

$$\begin{aligned} &\mathbb{E}_{t,x}\left[\left(\Phi_{i,j}^{h,p}\right)^3\right] \\ &= \mathbb{E}_{t,x}\left[\left(\Phi_{i,j}^{h,p}\right)^3\mathbb{I}_{|\Delta_{p+1}V^h|<\delta}\right] + \mathbb{E}_{t,x}\left[\left(\Phi_{i,j}^{h,p}\right)^3\mathbb{I}_{|\Delta_{p+1}V^h|\ge\delta}\right] \\ &\le C(T)\left\{\left(\omega''_{\varphi_{vv}}(\delta,T,\bar D\times F)\right)^3 + P_{t,x}\left(|\Delta_{p+1}V^h|\ge\delta\right)\right\}, \end{aligned}$$

where $F = \left[0, \sup_{[0,T]\times\bar D}|c(s,x)|T\right] \times (-\infty, 0]$ is the range of (v_{d+1}, v_{d+2}) and

$$\omega''_{\varphi_{vv}}(\delta,T,\bar D\times F) = \max_{i,j}\sup_{s\in[0,T]}\sup_{\substack{v,\tilde v\in\bar D\times F\\ |v-\tilde v|\le\delta}} |\varphi_{v_iv_j}(s,v) - \varphi_{v_iv_j}(s,\tilde v)|.$$

Therefore, by (3.13), (3.14), and (3.3),

$$\limsup_{h\to 0} \sup_{t\le T, x\in\bar D} \max_p \sum_{i,j=1}^{d+2} \mathbb{E}_{t,x}\left[\left(\Phi_{i,j}^{h,p}\right)^3\right]$$

$$\le C(T) \limsup_{\delta\to 0} \limsup_{h\to 0} \left\{\left(\omega''_{\varphi_{vv}}(\delta, T, \bar D\times F)\right)^3\right.$$

$$\left. + P\left(C(T)\left(h+\sqrt{h}|\Delta_{p+1}\eta|\right)\ge\delta\right)\right\}$$

$$= 0.$$

If $u_{x_i x_j}$ is a Lipschitz continuous function we have, again by (3.13) and (3.14),

$$\mathbb{E}_{t,x}\left[\left(\Phi_{i,j}^{h,p}\right)^3\right] \le C(T)\mathbb{E}_{t,x}|\Delta_{p+1}V^h|^3 \le C(T)\left[h^{3/2}+\mathbb{E}|\Delta_{p+1}\eta|^3\right]h^{3/2},$$

so that, by (3.3),

$$\sup_{t\le T, x\in\bar D} |\mathbb{E}_{t,x}\left[R(t,T,h)\right]| \le C(T)\sqrt{h}. \qquad \square$$

Let

$$h_1 = \frac{1}{4}\left(\frac{\rho_0\wedge c_0}{\sup_{[0,T]\times\bar D}|b(s,x)| + H\sup_{[0,T]\times\bar D}||\sigma(s,x)||}\right)^2 \wedge 1, \tag{3.15}$$

where c_0 is the constant in Theorem 2.2 if D is not convex, and $h_1 = 1$ if D is convex.

LEMMA 3.3. *If $\Delta_{p+1}\eta$ is bounded, then for any $q\in\mathbb{N}$*

$$\sup_{h<h_1}\sup_{t\le T, x\in\bar D}\mathbb{E}_{t,x}\left[\left(\xi_T^h\right)^q\right] = C(T) < +\infty,$$

where $C(T)$ is a constant depending only on T, D, H, q, and the coefficients of (1.1).

Proof. By Theorem 2.2 and Remark 2.3,

$$\xi_T^h \le \frac{C(T)}{\delta_{W^h}(M,t,T)}\sup_{r_1,r_2\in[t,T]}|W_{r_2}^h - W_{r_1}^h|,$$

where W^h is defined by (1.5) and (1.6). Therefore

$$\sup_{t\le T, x\in\bar D}\mathbb{E}_{t,x}\left[\left(\xi_T^h\right)^q\right]$$

$$\le C(T)\sup_{t\le T, x\in\bar D}\mathbb{E}_{t,x}\left[\frac{1}{\delta_{W^h}(M,t,T)^{2q}}\right]^{1/2}$$

$$\times\left\{T^q + \sup_{t\le T, x\in\bar D}\mathbb{E}_{t,x}\left[\max_{0\le p\le p'\le N-1}\left|\sum_{k=p}^{p'}\sqrt{h}\sigma(s_{k+1},X_{s_k}^h)\Delta_{k+1}\eta\right|^{2q}\right]^{1/2}\right\}.$$

Let us prove

(i)

$$\sup_{h<h_1}\sup_{t\leq T,x\in\bar{D}}\mathbb{E}_{t,x}\left[\frac{1}{\delta_{W^h}(M,t,T)^{2q}}\right]<+\infty,$$

(ii)

$$\sup_{h<h_1}\sup_{t\leq T,x\in\bar{D}}\mathbb{E}_{t,x}\left[\max_{0\leq p\leq p'\leq N-1}\left|\sum_{k=p}^{p'}\sqrt{h}\sigma(s_{k+1},X^h_{s_k})\Delta_{k+1}\eta\right|^{2q}\right]<+\infty.$$

Concerning (ii) we have, by (3.2) and the Burkholder–Davis–Gundy inequality, for any $q\in\mathbb{N}$,

$$\begin{aligned}&\sup\nolimits_{t\leq T,x\in\bar{D}}\mathbb{E}_{t,x}\left[\max\nolimits_{0\leq p\leq p'\leq N-1}\left|\sum\nolimits_{k=p}^{p'}\sqrt{h}\sigma(s_{k+1},X^h_{s_k})\Delta_{k+1}\eta\right|^{2q}\right]\\ \leq\;& C\sup\nolimits_{t\leq T,x\in\bar{D}}\mathbb{E}_{t,x}\left[\max\nolimits_{0\leq p\leq N-1}\left|\sum\nolimits_{k=0}^{p}\sqrt{h}\sigma(s_{k+1},X^h_{s_k})\Delta_{k+1}\eta\right|^{2q}\right]\\ \leq\;& C\mathbb{E}\left[\left(\sum\nolimits_{k=0}^{N-1}h|\Delta_{k+1}\eta|^2\right)^q\right]\\ \leq\;& C(TH^2)^q.\end{aligned}$$

For (i) we will show that

$$\sup_{h<h_1}\sup_{t\leq T,x\in\bar{D}}\int_{0+}^{\infty}P_{t,x}\left(\frac{1}{\delta_{W^h}(M,t,T)^{2q}}>u\right)du<\infty.$$

We have

$$\begin{aligned}P_{t,x}\left(\frac{1}{\delta_{W^h}(M,t,T)^{2q}}>u\right)&=P_{t,x}\left(\delta_{W^h}(M,t,T)<\left(\frac{1}{u}\right)^{1/(2q)}\right)\\&=1-P_{t,x}\left(\delta_{W^h}(M,t,T)\geq\left(\frac{1}{u}\right)^{1/(2q)}\right).\end{aligned}$$

By definition (see Theorem 2.2)

$$\begin{aligned}P_{t,x}\left(\delta_{W^h}(M,t,T)\geq\left(\frac{1}{u}\right)^{1/(2q)}\right)&\geq P_{t,x}\left(\omega'_{W^h}\left(\left(\frac{1}{u}\right)^{1/(2q)},t,T\right)<M\right)\\&\geq P_{t,x}\left(\max_i\sup_{r,r'\in[r_i,r_{i+1})}|W^h_r-W^h_{r'}|<M\right)\end{aligned}$$

for any partition $\{r_i\}$ of $[t,T]$ such that $\min_i(r_{i+1}-r_i)>\left(\frac{1}{u}\right)^{1/(2q)}$. We choose a suitable partition; namely, we take $m-1=\left[\frac{1}{hu^{1/(2q)}}\right]$, where $[\cdot]$ denotes the integer part, and $r_i=t+imh$, $i=0,\dots,[N/m]-1$, $r_{[N/m]}=T$. In this way, W^h has at most $m-1$ jumps in $[r_i,r_{i+1})$ for $i=0,\dots,[N/m]-2$ and at most $2m-2$ jumps in $[r_{[N/m]-1},r_{[N/m]})$. Moreover,

$$
\begin{aligned}
&\max_{0\le i\le [N/m]-1} \sup_{r,r'\in[r_i,r_{i+1})} |W^h_r - W^h_{r'}| \\
&\le 2 \max_{0\le i\le [N/m]-1} \sup_{r\in[r_i,r_{i+1})} |W^h_r - W^h_{r_i}| \\
&\le C(T) \max\left\{ \max_{0\le i\le [N/m]-2} \max_{1\le p\le m-1} \left| \sum_{k=0}^{p-1} \sqrt{h}\sigma(s_{im+k+1}, X^h_{s_{im+k}})\Delta_{im+k+1}\eta \right| + \left(\frac{1}{u}\right)^{1/(2q)}, \right. \\
&\max_{1\le p\le N-[N/m]m+m-1} \left| \sum_{k=0}^{p-1} \sqrt{h}\sigma(s_{([N/m]-1)m+k+1}, X^h_{s_{([N/m]-1)m+k}})\Delta_{([N/m]-1)m+k+1}\eta \right| \\
&\left. +2\left(\frac{1}{u}\right)^{1/(2q)} \right\},
\end{aligned}
$$

Let $\mathcal{F}_p$ denote the σ-algebra generated by $\{\Delta_{k+1}\eta\}_{0\le k\le p-1}$, for every $p = 1,\ldots,N-1$. By the Burkholder–Davis–Gundy inequality, for any $q' \in \mathbb{N}$ and for u larger than some $u_{q,q'}$ independent of t, x, and h, we have

$$
\begin{aligned}
&P_{t,x}\left(\max_{1\le p\le m-1} \left| \sum_{k=0}^{p-1} \sigma(s_{im+k+1}, X^h_{s_{im+k}})\Delta_{im+k+1}\eta \right| \right. \\
&\left. \ge \frac{Mu^{1/(4q)} - (1/u)^{1/(4q)}C(T)}{C(T)}(m-1)^{1/2} \,\middle|\, \mathcal{F}_{im} \right) \\
&\le \frac{C(T)H^{2q'}}{\left[Mu^{1/(4q)} - (1/u)^{1/(4q)}C(T)\right]^{2q'}}
\end{aligned}
$$

and

$$
\begin{aligned}
&P_{t,x}\left(\max_{1\le p\le N-[N/m]m+m-1} \left| \sum_{k=0}^{p-1} \sigma(s_{([N/m]-1)m+k+1}, X^h_{s_{([N/m]-1)m+k}})\Delta_{([N/m]-1)m+k+1}\eta \right| \right. \\
&\left. \ge \frac{Mu^{1/(4q)} - (1/u)^{1/(4q)}C(T)}{C(T)}(m-1)^{1/2} \,\middle|\, \mathcal{F}_{([N/m]-1)m} \right) \\
&\le \frac{C(T)H^{2q'}}{\left[Mu^{1/(4q)} - (1/u)^{1/(4q)}C(T)\right]^{2q'}}.
\end{aligned}
$$

Therefore, for $u_{q,q'}$ sufficiently large,

$$
\begin{aligned}
&\int_{0+}^{\infty} P_{t,x}\left(\frac{1}{\delta_{W^h}(M,t,T)^{2q}} > u \right) du \\
&\le u_{q,q'} + \int_{u_{q,q'}}^{\infty} \left[1 - \left(1 - \frac{C(T)H^{2q'}}{u^{q'/(2q)}} \right)^{Tu^{1/(2q)}} \right] du.
\end{aligned}
$$

It is easy to show that the function

$$f(u) = \left[1 - \left(1 - \frac{C(T)H^{2q'}}{u^{q'/(2q)}} \right)^{Tu^{1/(2q)}} \right] u^{q'/(2q)}$$

is bounded for $u \geq u_{q,q'}$, so that for $q' > 2q$, the integral in the right-hand side is finite and independent of t, x, and h. □

THEOREM 3.4. *If $\Delta_{p+1}\eta$ is bounded, then*

$$\sup_{t \leq T, x \in \bar{D}} \left| \mathbb{E}_{t,x} \left[F_{t,T}(X, \xi) \right] - \mathbb{E}_{t,x} \left[F_{t,T}^h(X^h, \xi^h) \right] \right| \longrightarrow 0 \quad \textit{for } h \to 0.$$

If, furthermore, u_s is a Hölder continuous function of s of exponent $1/2$ and $u_{x_i x_j}$, $i, j = 1, \ldots, d$, is a Lipschitz continuous function of x, then, for $h < h_1$,

$$\sup_{t \leq T, x \in \bar{D}} \left| \mathbb{E}_{t,x} \left[F_{t,T}(X, \xi) \right] - \mathbb{E}_{t,x} \left[F_{t,T}^h(X^h, \xi^h) \right] \right| \leq C(T) h^{1/2},$$

where $C(T)$ is a constant depending only on T, D, H, the coefficients of (1.1), *and the data of* (1.2).

Proof. The assertions follow immediately from Lemmas 3.1, 3.2, and 3.3. □

Example 3.1. In this example we prove that in general the estimate of Theorem 3.4 cannot be improved. Let X be the reflecting Brownian motion in $D = [-1, 1]$ (i.e., $b = 0$ and $\sigma = 1$), ξ be the corresponding local time, and consider the functional (1.2) with $t = 0$, $T = 1$, $\partial_1 D = \emptyset$, $f(x) = 1 - x^2$, $c = \lambda = 0$, $g_2 = 2$, that is,

$$F_1(X, \xi) = 1 - X_1^2 - 2\xi_1.$$

In this case (1.3) takes the form

$$\begin{cases} u_t(t,x) + \frac{1}{2} u_{xx}(t,x) = 0, \\ u(1,x) = 1 - x^2, \\ \frac{\partial u}{\partial n}\Big|_{\partial D} = 2. \end{cases} \tag{3.16}$$

Since the assumptions of Theorem 2.7 are verified for every $l > 0$, (3.16) has one and only one classical solution, which is given by

$$u(t,x) = t - x^2.$$

Let X^h and ξ^h be defined by (1.5) with $\Delta_{p+1}\eta$ uniformly distributed on $\{-1, 1\}$. Then, by (3.12) we have

$$\begin{aligned} \mathbb{E}_x \left[F_1(X, \xi) \right] - \mathbb{E}_x \left[F_1^h(X^h, \xi^h) \right] &= \frac{1}{2} \mathbb{E}_x \left[\sum_{p=0}^{N-1} u_{xx}^p (n^{p+1})^2 (\Delta_{p+1}\xi^h)^2 \right] \\ &= -\mathbb{E}_x \left[\sum_{p=0}^{N-1} (\Delta_{p+1}\xi^h)^2 \right]. \end{aligned}$$

Let $h = 1/i^2$ for some $i \in \mathbb{N}$, $i \geq 2$. Then, if the starting point is $x = 0$,

$$(\Delta_{p+1}\xi^h)^2 = \sqrt{h}\Delta_{p+1}\xi^h;$$

hence

$$\mathbb{E}_0\left[F_1(X,\xi)\right]-\mathbb{E}_0\left[F_1^h(X^h,\xi^h)\right]=-\sqrt{h}\mathbb{E}_0\left[\sum_{p=0}^{N-1}(\Delta_{p+1}\xi^h)\right]$$
$$=-\sqrt{h}\mathbb{E}_0\left[\xi_1^h\right].$$

By the continuity properties of the Skorohod problem (see, for instance, [2, Corollary 3.3]), ξ_1^h converges in law to ξ_1, for $h\to 0$. Therefore, by Lemma 3.3,

$$\mathbb{E}_0[\xi_1^h]\longrightarrow\mathbb{E}_0[\xi_1]\quad\text{for } h\to 0$$

and

$$\mathbb{E}_0[F_1(X,\xi)]-\mathbb{E}_0[F_1[X^h,\xi^h]=-\sqrt{h}\mathbb{E}_0[\xi_1]+o(\sqrt{h}).$$

Since $\mathbb{E}_0[\xi_1]\neq 0$, this shows that the estimate of Theorem 3.4 is tight.

LEMMA 3.5 (see [20, pp. 208 and 210]). *If b and σ are time independent, D is convex, and $\Delta_{p+1}\eta$ is a standard Gaussian random variable, then for any $q\in\mathbb{N}$*

$$\sup_{h<1}\ \sup_{t\leq T,x\in\bar{D}}\ \mathbb{E}_{t,x}\left[\left(\xi_T^h\right)^q\right]=C_q(T)<+\infty\quad\forall q>0,$$

where $C_q(T)$ is a constant depending only on T, D, q, and the coefficients of (1.1).

THEOREM 3.6. *If b and σ are time independent, D is convex, and $\Delta_{p+1}\eta$ is a standard Gaussian random variable, then*

$$\sup_{t\leq T,x\in\bar{D}}\left|\mathbb{E}_{t,x}\left[F_{t,T}(X,\xi)\right]-\mathbb{E}_{t,x}\left[F_{t,T}^h(X^h,\xi^h)\right]\right|\longrightarrow 0\quad\textit{for}\quad h\to 0.$$

If, furthermore, u_s is a Hölder continuous function of s of exponent $1/2$ and $u_{x_ix_j}$ is a Lipschitz continuous function of x for $i,j=1,\dots,d$, then for every $\epsilon>0$ there is a constant $C^\epsilon(T)$, depending only on T,D, the coefficients of (1.1), *and the data of* (1.2), *such that, for $h<1$,*

$$\sup_{t\leq T,x\in\bar{D}}\left|\mathbb{E}_{t,x}\left[F_{t,T}(X,\xi)\right]-\mathbb{E}_{t,x}\left[F_{t,T}^h(X^h,\xi^h)\right]\right|\leq C^\epsilon(T)h^{1/2-\epsilon}.$$

Proof. For any $q\in\mathbb{N}$

$$\mathbb{E}\left[\left(\sqrt{h}H_N\right)^{2q}\right]\leq\mathbb{E}\left[\omega_B(h,T)^{2q}\right],\tag{3.17}$$

where B is a standard Brownian motion and

$$\omega_B(h,T)=\sup_{\substack{0\,\leq\,s_1\,\leq\,s_2\,\leq\,T\\ s_2\,-\,s_1\,\leq\,h}}|B_{s_1}-B_{s_2}|.$$

By Lemma 3 in the Appendix of [20], for every $\epsilon>0$ there is a constant C^ϵ such that the right-hand side of (3.17) is bounded by $C^\epsilon h^{q-\epsilon}$. Then the assertion follows from Lemmas 3.1, 3.2, and 3.5. ∎

Remark 3.2. In Theorems 3.4 and 3.6 we have proved (1.9) by means of Lemma 3.1, which requires that there exists a classical solution u of (1.3). However, at least in the pure Neumann case ($\partial_1 D=\emptyset$), giving up uniformity in t and x, (1.9) holds much more in general. In fact the stochastic process (X^h,ξ^h) defined by (1.5) always converges in law to the solution (X,ξ) of the SDE of coefficients b and σ with normal reflection in $\bar{D}$. (This is proved in [20] if b and σ are time independent, D is convex, and $\Delta_{p+1}\eta$ is a standard Gaussian random variable, while if $\Delta_{p+1}\eta$ is bounded it can be shown by standard weak convergence techniques and results on the Skorohod problem, such as, for instance, the ones in [2].) Then, for any f and g_2, (1.9) (without uniformity in t and x) follows from Lemmas 3.3 and 3.5.

4. Numerical tests. For our numerical tests we considered the two-dimensional diffusion process with the same coefficients as the first example in [23], namely,

$$b(s,x) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad \sigma(s,x) = \begin{pmatrix} 0 & \frac{\sin(x_1+x_2)}{\sqrt{s+1}} \\ \frac{\cos(x_1+x_2)}{\sqrt{s+1}} & 0 \end{pmatrix}, \tag{4.1}$$

and normal reflection in $\bar{D} = [-L, L] \times [-L, L]$. With the choice

$$\partial_1 D = \emptyset, \quad f(x) = x_1^2 + x_2^2, \quad c(s,x) = 0, \quad \lambda(s,x) = 0, \quad g_2(s,x) = -2L, \tag{4.2}$$

the functional (1.2) becomes

$$F_{t,T}(X,\xi) = |X_T|^2 + 2L\xi_T. \tag{4.3}$$

Since the domain $[-L, L] \times [-L, L]$ satisfies the assumptions made in section 2, by Theorem 2.5 the expectation of $F_{t,T}(X,\xi)$ is given by the solution of problem (1.3), which is, irrespectively of $L > 0$,

$$u(t,x) = x_1^2 + x_2^2 + \log(T+1) - \log(t+1). \tag{4.4}$$

The discretization scheme (1.5) has been implemented with pseudorandom variables $\{\Delta_{p+1}\eta\}_{0 \le p \le N-1}$ uniformly distributed over $\{(-1,-1), (-1,1), (1,-1), (1,1)\}$. The expectation of the functional $F^h_{t,T}(X^h, \xi^h)$ has been approximated by the arithmetic mean over $m = 10,000$ independent paths. The starting time $t = 0$ and the starting point $x^0 = (1,1)$ have been set, and the behavior of the error

$$e(h,T) = \mathbb{E}_{0,x^0}\left[F_{0,T}(X,\xi)\right] - \frac{1}{m}\sum_{i=1}^{m} F^h_{0,T}(X^{h,i}, \xi^{h,i}) \tag{4.5}$$

has been analyzed both with respect to T, for fixed h, and with respect to h, for fixed T. The values $L = 10, 1.2, 1$ have been considered. A large number of experiments were made, by performing double precision Fortran computations on the IBM RISC/6000-390 of the Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate of the University of Padua. The workstation was equipped with an IBM AIX XL Fortran v. 02 compiler. The discrete uniform pseudorandom variables $\{\Delta_{p+1}\eta\}_{0 \le p \le N-1}$ were generated by using the routine DURAND from library ESSL (see reference [6]). The most significant results, which are discussed below, were obtained for $0 \le T \le 10$ and for the values $h = 0.000625$, 0.00125, 0.0025, 0.005, 0.01, 0.02 (some results for $h = 0.04$ are also reported).

Figures 4.1 and 4.2 show the behavior of $e(h,T)$ versus T, $0 \le T \le 10$, when $L = 10$ is set. In this case the starting point $x^0 = (1,1)$ is so far from the boundary that none of the simulated paths hits ∂D before $T = 10$. For $h = 0.01$, the behavior of the error compares very well with that given for the same value of h in the second figure of the first example in [23]. The error is negative and is linearly decreasing with respect to T, a phenomenon which is explained in [23]. No other values of h are considered for the Euler scheme in [23]. The curves in Figure 4.1 exhibit large

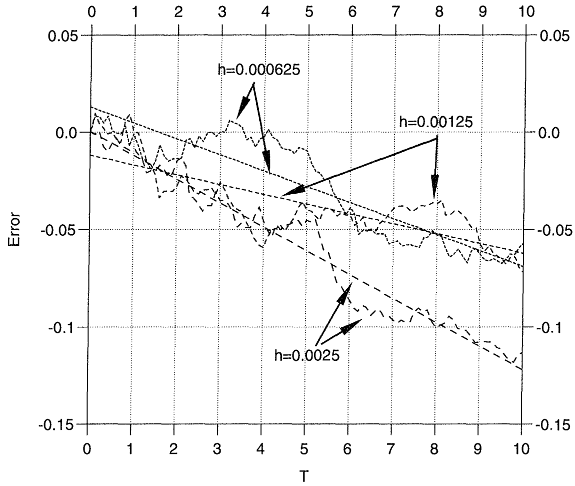

FIG. 4.1. *The error* $e(h, T)$ *plotted vs* T. $L = 10$ *is set. The least squares fitting straight lines are also shown.*

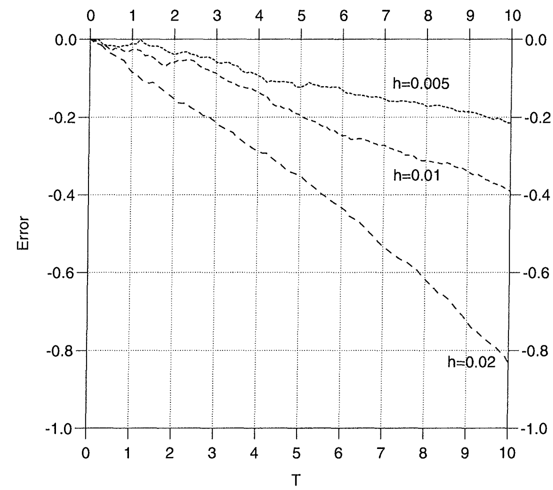

FIG. 4.2. *The error* $e(h, T)$ *plotted vs* T. $L = 10$ *is set. Note that the vertical scale is different from the one in Figure* 1.

oscillations. This is due to the fact that for small values of h the error deriving from approximating $\mathbb{E}_{0,x^0}\left[F^h_{0,T}(X^h, \xi^h)\right]$ by an arithmetic mean becomes relevant relative to the error deriving from approximating $\mathbb{E}_{0,x^0}\left[F_{0,T}(X, \xi)\right]$ by $\mathbb{E}_{0,x^0}\left[F^h_{0,T}(X^h, \xi^h)\right]$.

Figures 4.3 and 4.4 show the behavior of the error $e(h, T)$ versus T, $0 \leq T \leq 10$, when $L = 1$ is set. The initial point x^0 is now on the boundary, and thus the simulated trajectories undergo a large number of reflections at the beginning. Inspecting Figures 4.3 and 4.4, we see that $e(h, T)$ is positive and grows with T, for small values of T; as T becomes larger, either $e(h, T)$ remains quite unchanged (when $h = 0.00125$, 0.0025, 0.005), or it decreases, in the end oscillating around zero (for $h = 0.000625$), or approaching larger and larger negative values ($h = 0.01$, 0.02).

Figures 4.5 and 4.6 show the behavior of the error $e(h, T)$ when $L = 1.2$ is set, i.e., the starting point $x^0 = (1, 1)$ is "near to," but not on the boundary, thus producing

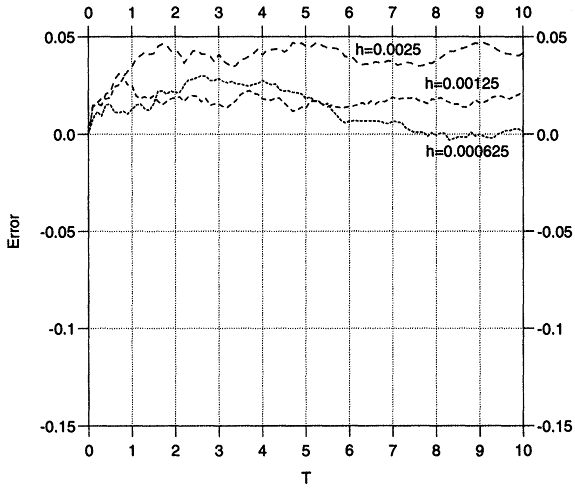

FIG. 4.3. *The same as Figure* 4.2, *setting* $L = 1$. *Note that the vertical scale is the same as in Figure* 4.1.

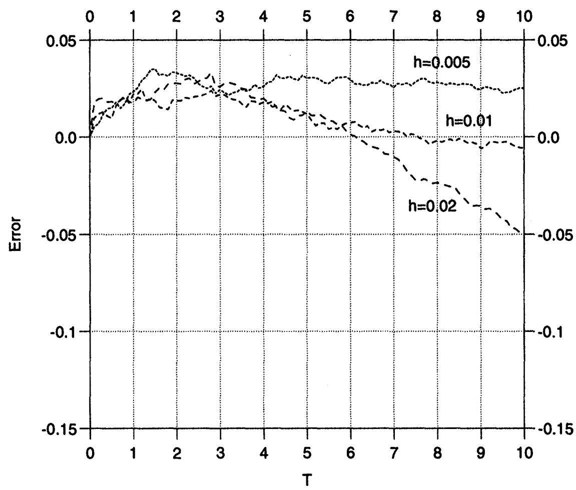

FIG. 4.4. *The same as Figure* 4.3. $L = 1$ *is set.*

fewer reflections than in the previous case. A mixture of the behaviors detected for $L = 1$ and $L = 10$ can be recognized. In particular, when $h = 0.01$ (Figure 4.6), $e(h, T)$ initially oscillates with T, but then starts to decrease and finally linearly runs toward larger negative values.

Now let us study how $e(h, T)$ changes with h.

Figures 4.7, 4.8, 4.9, and 4.10 show the values of $e(h, T)$ versus $h = 0.000625$, 0.00125, 0.0025, 0.005, 0.01, 0.02, 0.04 for $L = 1, 1.2, 10$ and $T = 2, 10$. From (3.12) and from the results given in [23] we guessed that

$$e(h, T) \simeq -c_1(T)h + c_2(T)\sqrt{h} \tag{4.6}$$

for suitable $c_1 > 0$, $c_2 > 0$.

The dashed curves in Figures 4.7, 4.8, 4.9, and 4.10 are obtained by computing the coefficients c_1 and c_2 in (4.6) by the least squares method (see Table 4.1). For $T = 2$,

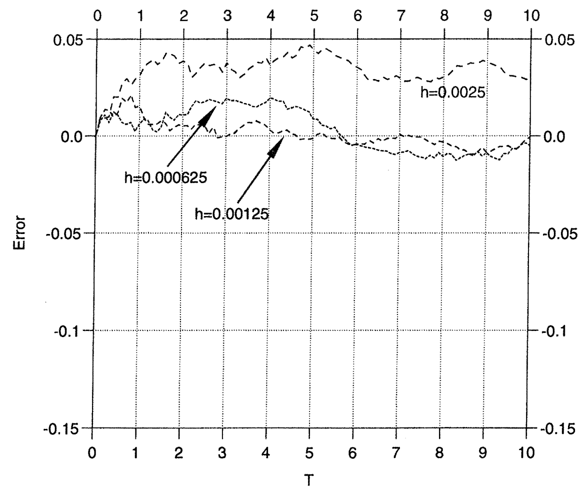

FIG. 4.5. *The same as Figure* 4.3, *setting* $L = 1.2$.

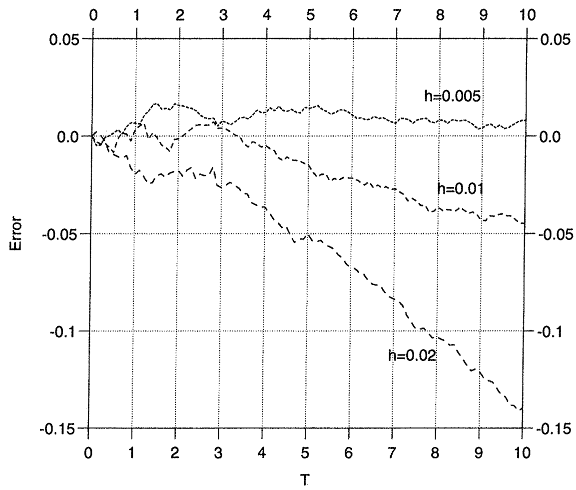

FIG. 4.6. *The same as Figure* 4.3, *setting* $L = 1.2$.

$L = 10$, a negative c_2 value was obtained, probably due to approximation errors. Computed values of $e(h,T)$ and least squares smoothing compare well, suggesting that equation (4.6) provides a good qualitative analysis of the variation of the error with respect to h. We are currently working on a rigorous proof of (4.6) and studying how c_1 and c_2 depend on T. c_1 and c_2 depend also on the domain D and the starting point x^0, in particular, in this example, on the parameter L. The larger L is, the smaller $c_2(T,L)$ should be with respect to $c_1(T,L)$. This could also explain why the error for $L = 1$ and $L = 1.2$ is much smaller, in absolute value, than for $L = 10$, namely because, for $L = 1$ and $L = 1.2$, a compensation of errors occurs. Another explanation might be that every time the simulated path hits the boundary, the normal derivative of u is evaluated exactly.

The results shown up to now were obtained by generating pseudorandom variables $\{\Delta_{p+1}\eta^h\}_{0\leq p\leq N-1}$ from a discrete uniform distribution. When Gaussian pseudoran-

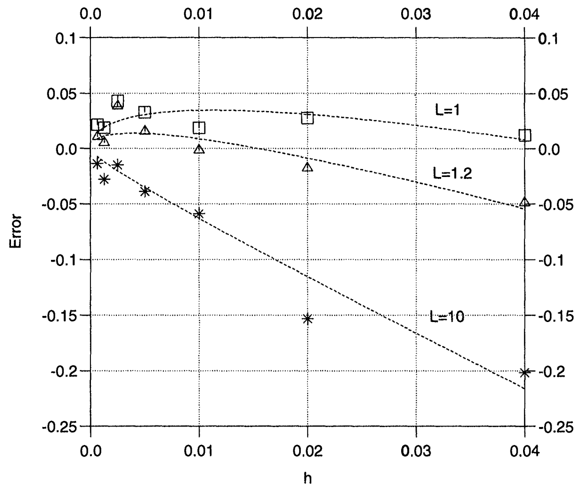

FIG. 4.7. *Values of* $e(h,T)$ *for* $h = 0.000625, 0.00125, 0.0025, 0.005, 0.01, 0.02, 0.04$, *when* $T = 2$ *is set. The cases* $L = 1$ *(□ symbols),* $L = 1.2$ *(△ symbols),* $L = 10$ *(* symbols) are considered. The dashed curves are obtained by a least squares fit of the form* (4.6).

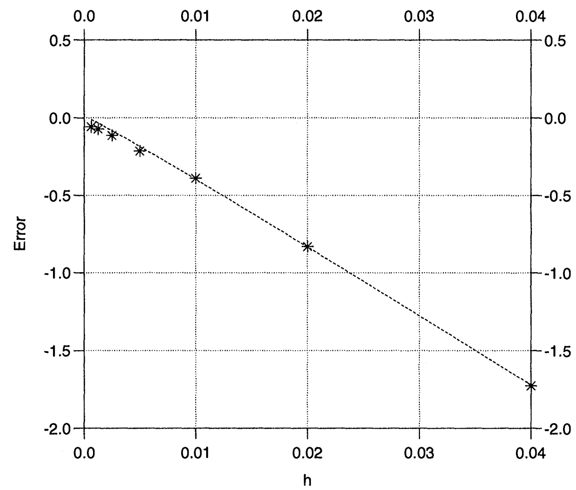

FIG. 4.8. *The same as Figure* 4.7 *for* $T = 10$, $L = 10$. *Note that the vertical scale is different from the one in Figure* 4.7.

dom variables are generated, more CPU time is consumed, but no gain in precision seems to be achieved. Table 4.2 shows the CPU time spent to compute the same quantities by using either uniform (columns labeled "s_U") or normal pseudorandom variables (columns labeled "s_N"). The latter were obtained by calling the routine DNRAND of library ESSL (see reference [6]). The column labeled "s_N/s_U" in Table 4.2 shows the ratios of the "s_N" values divided by the "s_U" values. We see that the overall CPU time consumed using normal pseudorandom variables is 1.2 times larger.

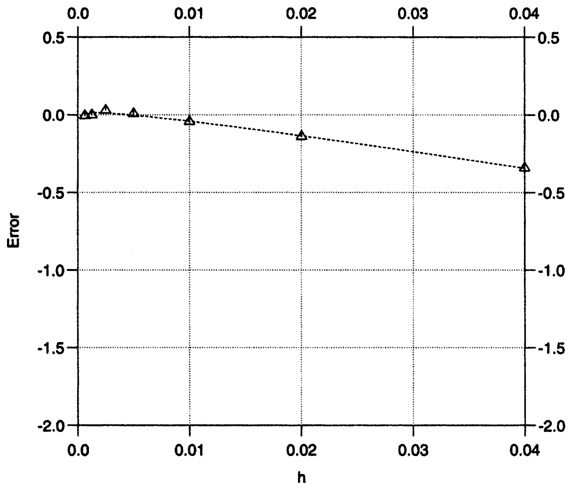

FIG. 4.9. *The same as Figure* 4.8, *for* $T = 10$, $L = 1.2$.

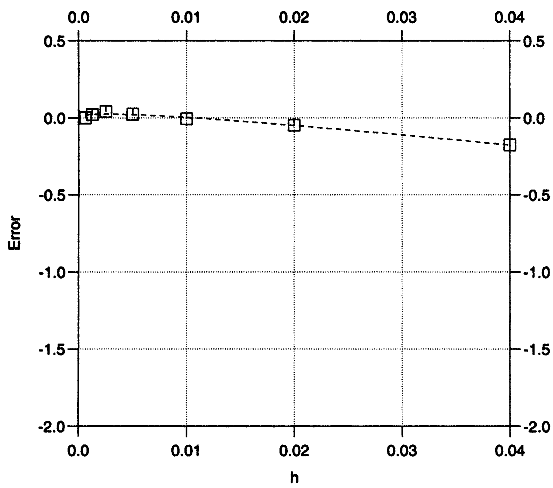

FIG. 4.10. *The same as Figure* 4.8, *for* $T = 10$, $L = 1$.

TABLE 4.1

Coefficients c_1 *and* c_2 *of* (4.6) *computed by the least squares method with the data shown in Figures* 4.7, 4.8, 4.9, 4.10.

$T = 2$		
L	c_1	c_2
1.0	3.054368	0.652123
1.2	3.614892	0.450973
10.0	4.501581	−0.179268
$T = 10$		
L	c_1	c_2
1.0	9.1553	0.950793
1.2	12.9672	0.877067
10.0	46.3092	0.6567402

TABLE 4.2

CPU time (in seconds) spent to perform numerical simulations up to $T = 10$ by generating uniform pseudorandom variables (s_U) or normal pseudorandom variables (s_N). The ratios s_N/s_U are also shown. The cases $L = 1$ and $L = 10$ are considered.

$L = 1$			
h	s_U	s_N	s_N/s_U
0.000625	1508	1800	1.2
0.00125	761	905	1.2
0.0025	382	453	1.2
0.005	191	228	1.2
0.01	96	114	1.2
0.02	48	58	1.2
$L = 10$			
h	s_U	s_N	s_N/s_U
0.000625	1555	1841	1.2
0.00125	777	925	1.2
0.0025	388	461	1.2
0.005	194	230	1.2
0.01	97	115	1.2
0.02	48	57	1.2

TABLE 4.3

Ratios A_N/A_U between the average absolute errors raised using pseudonormal (A_N) or pseudouniform variables (A_U). $T = 10$; the cases $L = 1, 1.2, 10$ are considered. The mean ratios are also shown.

h	$L = 1$	$L = 1.2$	$L = 10$
0.000625	0.1620E+1	0.2259E+1	0.7647E+0
0.00125	0.2021E+1	0.4635E+1	0.1512E+1
0.0025	0.1133E+1	0.8006E+0	0.7197E+0
0.005	0.1754E+1	0.1516E+1	0.9997E+0
0.01	0.3302E+1	0.1152E+1	0.1023E+1
0.02	0.1106E+1	0.8953E+0	0.9463E+0
mean	0.1823E+1	0.1876E+1	0.9942E+0

For given h and L values, let us define the average absolute errors

$$A_N = \frac{1}{T/h} \sum_{p=1}^{T/h} |e_N(h, ph)|, \quad A_U = \frac{1}{T/h} \sum_{p=1}^{T/h} |e_U(h, ph)|,$$

where e_N and e_U denote the errors raised using normal and uniform pseudorandom variables, respectively. Table 4.3 shows the ratios A_N/A_U for $T = 10$, $L = 1, 1.2, 10$. In the case $L = 1$, the average absolute error obtained using pseudonormal variables can be up to three times that for pseudouniform variables, and the mean ratio (with respect to h) is 1.8. In the case $L = 10$ the ratios are practically equal to 1, when $h \geq 0.005$. The cases $L = 10$, $h < 0.005$ are peculiar: as was previously pointed out, in these cases the incidence of the error due to the approximation of an expectation by an arithmetic mean is relevant. The mean ratio is close to 1. When $L = 1.2$, the mean ratio is very close to that for $L = 1$, suggesting again that a poorer precision is likely to be obtained using pseudonormal variables. Summarizing, no gain in precision, or even a loss, is to be expected using pseudonormal variables instead of pseudouniform ones.

Inspecting Table 4.2, we see that for each $h < 0.02$, the time reported for $L = 1$ is smaller than in the case $L = 10$. This is surprising enough, because some additional CPU time is needed to compute the last two summands of (1.7) when a simulated path hits the boundary, and many simulated paths hit the boundary when $L = 1$, while none hit when $L = 10$. Investigating further, we concluded that this result is due to the optimization strategies used on request by the IBM AIX XL Fortran compiler. The CPU times reported in Table 4.2 were obtained running the optimized code. Running the unoptimized compiled code needs up to three times the CPU seconds recorded in Table 4.2, and the computing time turns out to be larger when $L = 1$ than when $L = 10$.

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