

A Review of Non-Markovian Models for the Dynamics of Credit Ratings

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Abstract

This survey reviews the growing literature on Markovian and non-Markovian models for modeling the dynamics of credit ratings. Credit rating is a measure of the creditworthiness of a firm, i.e., it is an evalu-

ation of its likelihood of default. The level of credit ratings varies with respect to time due to random credit risk and thus need to be modeled by an appropriate stochastic process. Models based on Markov chains have been proposed in the literature and are widely used due to mathematical simplicity. However, many empirical evidences suggest the non-suitability of the Markov process to model rating dynamics. To overcome the limitations of Markovian models, several non-Markovian models based on semi-Markov process and Markov regenerative processes have been proposed in the literature. In this article, we give a review of the models proposed in the literature. Further, empirical applications on the real data are presented to compare various modeling approaches.

Keywords: Credit Ratings, Markov process, semi-Markov process, Markov regenerative process, Probability of default

1 Introduction

Banks and other financial institutions face various types of financial risks namely market risk, operational risk and credit risk. The credit risk, also known as risk of default, has become one of the most important issue in modern financial world. It is the risk faced by the lender that arises from a borrower who may not be able to repay the loan at maturity. Credit risk analysis consists of finding the default probability of the borrower and study various problems related to the pricing of derivatives, to the pricing of risky bonds etc. There are two categories of credit risk models namely structural models (also known as firm-value models) and reduced form models (also known as intensity based models). The first category of models is pioneered by Merton [2] who considered the total firm value of the firm and defines default when the firm value falls below a default barrier at maturity. This basic model has further been extended by incorporating many other factors like stochastic default barriers, variability in the interest rates etc. In conclusion, these models provide a mechanism of default in terms of the relation between the firm value (assets) and the liabilities at any time t .

On the other hand, the second class of models, known as intensity based models, does not specify the mechanism of default, i.e., how default occurs, but models it as a first jump time of a counting process. These models were introduced by Jarrow and Turnbull [15] and extended by Lando [18], and Duffie and Singleton [4]. This class of models has become very popular among the practitioners due to mathematical tractability and to the fact that they do not need specific knowledge of the company's value but they rely only on the mod-

eling of the probability of default. A comprehensive treatment of Markovian models of credit rating dynamics is given in Trueck and Rachev [23].

As already pointed out, essentially, credit risk modeling consists of computing the probability of default of a firm going into debt. In order to compute the default probability, various parameters associated with the firm, for instance its credit rating or its asset value, can be used. Credit ratings is one of the most important parameter associated with the firm that quantifies the risk associated with the firm. Credit ratings are issued to a firm by various international organizations like Moody's, Fitch, Standard & Poor's, etc. Higher the credit rating, more credit worthy is the firm. Further, credit ratings serve as an important output to various market models of credit risk and their level changes from time to time since the risk associated to a firm is dynamic. Therefore, there is a need to model accurately the dynamics of credit ratings.

In this direction, Jarrow et al. [15] proposed a Markov chain model, called migration models, in order to study the term structure of credit spreads and model the dynamics of credit ratings. Many other articles implemented the same approach to generate the transition probability matrices of the credit ratings (Hu et al. [14], Nickell et al. [19]). However several articles (Nickell et al. [19], Kavvathas [16], Lando and Skodeberg [18]) observed the empirical behavior of the credit ratings and suggest that the Markov model is inappropriate to model the credit rating dynamics. The main limitations of the Markov models are

1. Downward momentum: It is observed that probability that the next rating change will be a downward change given that the previous change is also downward is high as compared to the other case.
2. Duration: The transition probability depends on the time spent by a firm in a rating since its assignment. The time spent does not follow exponential distribution which is the case in Markov chains.
3. Time non-homogeneity: It is observed that the rating dynamics at two different points of time are different and hence in order to model credit ratings, a time non-homogenous framework is required. In general, transition probabilities varies with the state of the economy (i.e. recession or economic expansion).
4. Ageing effect: It means that the rating migrations depend on the total length of time since the firm received its first credit rating.

To overcome these limitations of the Markovian setup, D'Amico et al. [7] proposed a time homogeneous semi-Markov model to model the credit rating dynamics and considered this setup as a reliability problem. They suggested that the semi-Markov framework permits to overcome the constraints of the

Markov models. Later, D'Amico et al. [11] considered non-homogeneous in time semi-Markov models with initial and final backward and forward processes to study the effect of times after last transition and before the next transition on the transition probabilities. Further, in order to address the ageing effect, D'Amico et al. [10] proposed a discrete time non-homogeneous semi Markov model with an age index to model the credit rating dynamics. In order to address the downward rating momentum, D'Amico et al. [8] applied semi-Markov processes (SMP) with an extended state space to account for downward rating momentum. Vasileiou and Vassiliou [25] proposed an inhomogeneous semi-Markov model for the term structure of credit risk spreads and later Vassiliou and Vasileiou [26] studied the asymptotic behaviour of the survival probabilities in an inhomogeneous semi-Markov model for the migration process in credit risk. Fuzzy semi-Markovian based models of credit rating dynamics were provided by Vassiliou and Vasileiou [24]. The book by D'Amico et al. [6] contains a comprehensive treatment of today's state of the art in semi-Markov models of credit rating dynamics.

Recently, Pasricha et al. [20] proposed a more general class of models based on Markov regenerative process (MRGP) to study the downward rating momentum and duration effect. This class of MRGP is a generalization of the semi-Markov processes. They argued that since rating momentum exists only in downward direction, Markov property is satisfied only when there is a migration from a given rating to a better rating (i.e, upward movement). This behavior of the ratings can be modeled by MRGP.

The originality of this study is that it represents the first literature review on non-Markovian credit rating dynamics. This survey presents non-Markov credit rating models in an short but yet accessible form reviewing Markov chains, semi-Markov processes and Markov regenerative processes as they are applied to the credit risk problem. We discuss how the different models may overcome the limitations of the Markov chain based models. An empirical study on the real data obtained from S&P compares the performance of the various models in the literature and also compare them with the real behavior of credit ratings.

The rest of the paper is as follows: Section 2 gives a description of Markov chain model for rating dynamics and their generalization obtained by introducing semi-Markov processes and markov regenerative processes. Section 3 discusses how to apply the above models and some results based on an application to real credit rating data. Section 4 concludes the paper.

2 Materials and Methods

In this section we are going to introduce the meaning of credit ratings and to discuss different stochastic models used for describing the dynamic of credit ratings.

The credit ratings issued to a firm by a rating agency gives the creditworthiness of the firm i.e., its capacity to repay the debt. In practice, there are the following ratings given by Standard and Poor's rating agency:

$$S = \{AAA, AA, A, BBB, BB, B, CCC, D\}.$$

The bonds having rating above BB are investment grade bonds whereas those having BB or below BB are speculative bonds and state D corresponds to default. Consider a firm that starts in some rating $i \in S$. Since, the creditworthiness of the firm changes over time, the level of rating changes too. Therefore, it needs to be modeled by using an appropriate stochastic process. Different choices are possible, they are reviewed in next subsections.

2.1 Markov Chain Models

Let $\{X(t), t \geq 0\}$ be the stochastic process where $X(t)$ represents the rating of the firm at time t . We can observe that this is a discrete state continuous time stochastic process. Jarrow et al. [15] considered the credit rating process $\{X(t), t \geq 0\}$ to be a continuous-time Markov chain. The generator matrix describing the process dynamic can be obtained from the historical data and hence the default probabilities of different credit ratings can be obtained as well as other financial indicators, see e.g. WeiBbach and Mollenhauer [27] and D'Amico [5].

To better understand limitations of the Markov chain model it is worth to introduce it in a more rigorous way. Let $\mathbf{P}(t) = \{p_{ij}(t)\}_{i,j \in S}$ be the transition probability function of the markov chain. The elements of this matrix are defined as follows:

$$p_{ij}(t) := \mathbb{P}\{X(t) = j | X(0) = i\}.$$

The transition probability matrix can be obtained for any time $t \geq 0$ by computing the matrix exponential on the generator matrix \mathbf{A} :

$$\mathbf{P}(t) = e^{t\mathbf{A}} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!}.$$

The elements $\{a_{ij}\}_{i,j \in S}$ of the generator matrix \mathbf{A} express the force of transition departing from any state i and arriving to any state j , i.e.

$$\mathbf{A} = \lim_{r \rightarrow 0} \frac{\mathbf{P}(r) - \mathbf{P}(0)}{r}.$$

In literature, several empirical evidences suggest that Markov process is not appropriate for credit rating modeling. There are four main issues on the unsuitability of the Markov processes for the credit rating evolution namely downward momentum, duration effect, time non-homogeneity, ageing effect. In order to overcome these limitations of Markov models, D'Amico et al. proposed a series of articles based on semi-Markov processes. In the next section, we will briefly discuss these articles.

2.2 Semi-Markov Models

2.2.1 Time Homogeneous semi-Markov Model

D'Amico et al. [7] proposed a semi-Markov model to overcome these issues. We give a brief overview of semi-Markov credit rating model proposed by D'Amico et al. [7]. First, we give the definition of Markov renewal sequence. A sequence of the random variables $\{(X_n, T_n), n = 0, 1, \dots\}$ is called a Markov renewal sequence if

1. $T_0 = 0, T_{n+1} \geq T_n; X_n \in S = \{0, 1, 2, \dots\}$
2. $\forall n \geq 0,$
 $\mathbb{P}\{X_{n+1} = j, T_{n+1} - T_n \leq t \mid X_n = i, T_n, X_{n-1}, T_{n-1}, \dots, X_0, T_0\}$
 $= \mathbb{P}\{X_{n+1} = j, T_{n+1} - T_n \leq t \mid X_n = i\}$ (Markov property)
 $= \mathbb{P}\{X_1 = j, T_1 - T_0 \leq t \mid X_0 = i\}$ (time homogeneity)

The kernel $\mathbf{Q}(t) = [Q_{i,j}(t)]$ associated with the process is defined by

$$Q_{ij}(t) = \mathbb{P}\{X_{n+1} = j, T_{n+1} - T_n \leq t \mid X_n = i\}, \quad i, j \in S, t \geq 0.$$

and it follows that

$$p_{ij} = \lim_{t \rightarrow \infty} Q_{ij}(t), \quad i, j \in S.$$

where $\mathbf{P} = [p_{ij}]_{i,j \in S}$ is the one-step transition probability matrix of the embedded Markov chain with state space S .

We define the probability that the process will leave state $i, i \in S$ in time t ,

$$H_i(t) = \mathbb{P}\{T_{n+1} - T_n \leq t \mid X_n = i\}.$$

It can be observed that

$$H_i(t) = \sum_{j \in S} Q_{ij}(t).$$

Next, we define the distribution function of the waiting time in each state i , given that the next state is known:

$$G_{ij}(t) = \mathbb{P}\{T_{n+1} - T_n \leq t \mid X_n = i, X_{n+1} = j\}, \quad i, j \in S, t \geq 0.$$

These probabilities can be obtained as follows

$$G_{ij}(t) = \begin{cases} \frac{Q_{ij}(t)}{p_{ij}} & \text{if } p_{ij} \neq 0 \\ 1 & \text{if } p_{ij} = 0 \end{cases}.$$

The main difference between a continuous-time Markov chain and a SMP is in the distribution functions $G_{ij}(t)$. In a Markov environment this distribution function has to be a cumulative distribution function of negative exponential. On the other hand, in the semi-Markov case the distribution functions $G_{ij}(t)$ can be of cumulative distribution function of any general distribution. Thus, SMP accounts for the effect of duration inside a rating class.

Now, we can define the homogeneous semi-Markov process $\{Z(t), t \geq 0\}$, which represents, for each waiting time, the state occupied by the process, i.e.,

$$Z(t) = X_{N(t)} \quad \text{where } N(t) = \max\{n : T_n \leq t\}.$$

The transition probabilities for $\{Z(t), t \geq 0\}$ are defined by

$$\phi_{ij}(t) = \mathbb{P}\{Z(t) = j \mid Z(0) = i\}, i, j \in S, t \geq 0.$$

We collect them in a matrix of function: $\Phi(t) = (\phi_{i,j}(t))_{i,j \in S}$.

They can be obtained by solving the Markov renewal equation

$$\Phi(t) = \mathbf{E}(t) + (\mathbf{Q} * \Phi)(t),$$

or

$$\phi_{ij}(t) = \delta_{ij}(1 - H_i(t)) + \sum_{\gamma \in S} \int_0^t \phi_{\gamma j}(t - y) dQ_{i\gamma}(y), \quad i, j \in S. \quad (1)$$

where δ_{ij} represents Kronecker delta, $\mathbf{E}(t) = [E_{ij}(t)]_{i,j \in S}$ is a diagonal matrix defined as

$$E_{ij}(t) = \begin{cases} 1 - H_i(t) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

The credit rating of a firm gives its reliability degree or credit worthiness. For example, in the case of Standard & Poor's, there are the eight different classes of rating and so the set of states S can be denoted by

$$S = \{AAA, AA, A, BBB, BB, B, CCC, D\}.$$

The credit risk problem can be positioned in the reliability environment with states $\{1, 2, \dots, 8\}$ with 1 representing *AAA* and 8 representing the default state, i.e., state *D*. Considering this idea, D'Amico et al. [7] proposed semi-Markov based modeling approach for credit rating dynamics since it will address the limitation of Markov model which says that the time spent inside a

rating is an exponential distribution. In the credit risk environment, the first part of equation (3) can be interpreted as the probability that the rating organization does not give a new rating evaluation till time t . In the second part of above equation, $Q_{i\gamma}(y)$ represents the probability that firm will get a rating γ in time y and then firm will migrate to rating j in time $(t - y)$ following one of the possible paths.

2.2.2 Duration Dependent semi-Markov Models

D'Amico et al. [11, 10, 9] proposed a series of papers in order to completely study the effect of duration inside a rating on the transition probabilities within a non-homogeneous semi-Markov model. In order to introduce a non-homogeneous semi-Markov model we need to consider a non-homogeneous semi-Markov kernel, $\mathbf{Q}(s, t)$. The elements of the kernel gives the joint probability of visiting with next transition within time t rating class j given that the rating occupied at current time s is equal to i , in formula

$$Q_{ij}(s, t) = \mathbb{P}\{X_{n+1} = j, T_{n+1} \leq t \mid X_n = i, T_n = s\}. \quad (2)$$

Relation (2) shows that the probability to migrate from state i at time s to state j at time t depends also on the time s and not only on the state i as it was for the homogeneous model. In turn, all probabilities defined in the homogeneous case can be analogously defined considering an explicitly dependence on the current time s . In particular the transition probabilities for the non-homogeneous semi-markov process $Z(t) := X_{N(t)}$ are defined by

$$\phi_{ij}(s, t) = \mathbb{P}\{Z(t) = j \mid Z(s) = i\}, i, j \in S, 0 \leq s \leq t.$$

We collect them in a matrix of function: $\Phi(s, t) = (\phi_{i,j}(s, t))_{i,j \in S}$. Thus, the transition probability function will satisfy a non-homogeneous Markov Renewal equation:

$$\Phi(s, t) = \mathbf{E}(s, t) + (\mathbf{Q} * \Phi)(s, t),$$

or

$$\phi_{ij}(s, t) = \delta_{ij}(1 - \sum_{k \in S} Q_{ik}(s, t)) + \sum_{\gamma \in S} \int_0^t \phi_{\gamma j}(y, t) dQ_{i\gamma}(s, y), i, j \in S. \quad (3)$$

where δ_{ij} represents Kronecker delta, $\mathbf{E}(s, t) = [E_{ij}(s, t)]_{i,j \in S}$ is a diagonal matrix defined as

$$E_{ij}(s, t) = \begin{cases} 1 - \sum_{k \in S} Q_{ik}(s, t) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

The effect of the duration inside a state on the transition probabilities can explicitly be considered by means of the probabilities of being in rating j at time t given that firm had rating i at time s , but it entered in this state at time l and remained in state i until time s , without any other transition, i.e., have an age of $s - l$. The recurrence time processes gives complete information of duration inside a state and hence, gives more accurate estimates of transition probabilities.

D'Amico et al. [11, 10, 9] proposed a semi-Markov process (SMP) model by taking into account the recurrence times, thus addressing the ageing effect in credit rating dynamics. The non-homogeneity of the model addresses the issue of time dependence of rating evaluation. In the sequel, we give a brief overview of duration dependent non-homogeneous semi-Markov models.

We first define recurrence time processes. Let $\{N(t), t \geq 0\}$ be a renewal process with renewal epochs as $\{T_1, T_2, \dots\}$. Define

$$B(t) = t - T_{N(t)}, \quad (4)$$

$$F(t) = T_{N(t)+1} - t. \quad (5)$$

Here, $B(t)$ represents the time elapsed since the most recent renewal at or before time t . It is called age process or backward recurrence time process. Similarly, $F(t)$ represents time from t until first renewal after t . It is called residual process or forward recurrence time process. We can define the age process and residual process at initial and final time for time non-homogeneous SMP since it effects the transition probabilities of $\{Z(t), t \geq 0\}$. Define the transition probabilities considering recurrence times as follows

$$\phi_{ij}^{BF}(s; \tilde{a}, t, \tilde{b}) = P(Z(t) = j, B(t) \leq t - \tilde{a}, F(t) \leq \tilde{b} - t \mid Z(s) = i, T_{N(s)} = s). \quad (6)$$

These probabilities can be obtained as follows:

Theorem 1: For $i, j \in S$ and for $s < \tilde{a} < t < \tilde{b}$, we have

$$\begin{aligned} \phi_{ij}^{BF}(s; \tilde{a}, t, \tilde{b}) &= \delta_{ij} \mathbf{1}_{\{\tilde{a}=s\}} (H_i(s, \tilde{b}) - H_i(s, t)) \\ &\quad + \sum_{m \in S} \int_s^t \phi_{mj}^{BF}(\theta; \tilde{a}, t, \tilde{b}) dQ_{im}(s, \theta). \end{aligned} \quad (7)$$

Next, we define a SMP with recurrence times (age and residual life) both at initial and final times.

Definition 2: For $i, j \in S$ and for $a < s < b < \tilde{a} < t < \tilde{b}$ such that $1 - H_i(a, b) > 0$, define the following transition probabilities with age and residual life at initial and final time

$${}^{bf} \phi_{ij}^{BF}(a, s, b; \tilde{a}, t, \tilde{b}) = P(Z(t) = j, B(t) \leq t - \tilde{a}, F(t) \leq \tilde{b} - t \mid Z(s) = i, B(s) = s - a, F(s) > b - s).$$

These probabilities can be obtained as follows:

Theorem 2: For $i, j \in S$ and for $a < s < b < \tilde{a} < t < \tilde{b}$ such that $1 - H_i(a, b) > 0$, we have

$$\begin{aligned} {}^{bf}\phi_{ij}^{BF}(a, s, b; \tilde{a}, t, \tilde{b}) &= \delta_{ij} 1_{\{\tilde{a}=a\}} \frac{(H_i(a, \tilde{b}) - H_i(a, t))}{1 - H_i(a, b)} \\ &+ \sum_{m \in S} \int_b^t \phi_{mj}^{BF}(\theta; \tilde{a}, t, \tilde{b}) \frac{dQ_{im}(a, \theta)}{1 - H_i(a, b)}, \end{aligned} \quad (8)$$

where $\phi_{mj}^{BF}(\theta; \tilde{a}, t, \tilde{b})$ can be obtained from Equation (7). Relation (8) expresses the probability of being in rating class j at time t with elapsed time in this rank less or equal than $t - \tilde{a}$ and residual life in this lower than $\tilde{b} - t$ given that the entrance in the rating $Z(s) = i$ was at time a and the next transition after time s was at a time greater than b . These duration dependent transition probabilities allow us to understand how the transition probabilities among rating classes are perturbed by imposing some constraints on the duration of occupancy of the ratings at starting time s and arriving time t of evaluation.

2.2.3 Semi-Markov Model with Extended State Space

In order to take into account the downward problem, D'Amico et al. [8] introduced a modified semi-Markov model by extending the state space. This extended state space permits a method for obtaining a model that describes simultaneously the duration problem, the dependence of the rating evaluation on the chronological time and the downward effect.

They introduced another six states so that the state space becomes

$$S = \{AAA, AA, AA-, A, A-, BBB, BBB-, BB, BB-, B, B-, CCC, CCC-, D\}.$$

For example, the state BBB is divided into BBB and $BBB-$. The firm will receive a rating assignment equal to BBB if it made a transition from a lower rating rank while, on the other hand, it will be in the class $BBB-$ if it arrived in this rating from a better rating (a downward transition). Any one of the first 13 states can be considered as a state where the firm is still able to repay its debt and the last one is the only bad state denoting the default of the firm. Then, the Equations (7), (8) can be solved considering the extended state space. The different probability values of $\phi_{ij}^{BF}(s; \tilde{a}, t, \tilde{b})$ and ${}^{bf}\phi_{ij}^{BF}(a, s, b; \tilde{a}, t, \tilde{b})$ solves the downward problem considering at the same time the other non-Markovian effects.

2.3 Markov Regenerative Model

A major limitation of the above advanced technique consisting in the extension of the state space is as follows: the number of parameters (i.e., the transition probabilities) to be estimated increases, however, the credit rating data in some cases could not be enough to estimate a larger number of parameters. A possible solution is to combine various rating categories to lower the number of parameters so that the available data can be used for the estimation. Therefore, due to limited data availability, extending the state space is not always an appropriate choice to address the downward rating momentum. To overcome this limitation, Pasricha et al. [20] proposed a more general class of models based on Markov regenerative process (MRGP) to study the downward rating momentum and duration effect.

In this section, a Markov regenerative process (MRGP) is described briefly followed by the credit rating model based on MRGP. Let $\{X_n, n = 0, 1, \dots\}$ be a sequence of random variables with state space S . Let $\{N(t), t \geq 0\}$ be a counting process generated by the sequence $\{T_n, n = 0, 1, \dots\}$. The stochastic process $\{Z(t), t \geq 0\}$ where

$$Z(t) := X_{N(t)}, t \geq 0,$$

is called an MRGP if

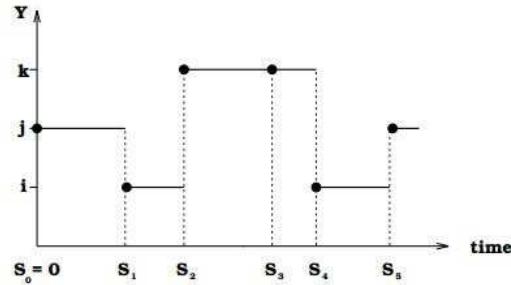
1. There exists $S' \subset S$ such that $\{X_n, n = 0, 1, \dots\}$ is a time homogeneous Markov chain with the state space S' .
2. $\forall n \geq 0, i, j \in S'$, we have

$$\begin{aligned} P\{X_{n+1} = j, T_{n+1} - T_n \leq t \mid X_n = i, T_n, X_{n-1}, T_{n-1} \dots, X_0, T_0\} = \\ P\{X_{n+1} = j, T_{n+1} - T_n \leq t \mid X_n = i\} & \quad \text{(Markov property)} \\ P\{X_1 = j, T_1 \leq t \mid X_0 = i\} & \quad \text{time homogeneity.} \end{aligned}$$

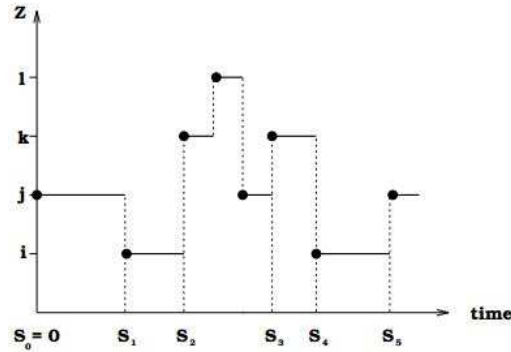
3. $P\{Z(T_n + t) = j \mid X_n = i\} = P\{Z(t) = j \mid X_0 = i\}$,

From the above definition, one can easily observe that every SMP is an MRGP and both the stochastic processes allow an arbitrary distribution for the sojourn times unlike the exponential sojourn time in Markov environment. Further, Figure 1 presents the sample paths of an MRGP and an SMP. The main difference between these two processes can be seen by observing their sample paths and comparing the sequence T_n of regeneration time points to the sequence obtained from the state transition instants of the process. Since not every transition in an MRGP is a regeneration point, one can address the downward rating momentum by considering the rating upgrade as a regeneration time epoch and hence in between two regenerations (i.e., rating upgrades),

the Markov property is not satisfied. In other words, we can say that the underlying process $\{Z(t), t \geq 0\}$ does not satisfy the Markovian property at the time instants when the firm faces a rating downgrade. Hence, the regeneration instances T_n exactly corresponds to the times of entering state i from a state j such that $j < i$ and the state of the process can change between the regeneration time points T_n and T_{n+1} due to rating downgrades. Therefore, an MRGP gives an appropriate choice to model the dynamics of credit ratings.



(a) semi-Markov process



(b) Markov regenerative process

Figure 1: Sample path of a semi-Markov process (SMP) and a Markov regenerative process (MRGP)

In order to obtain the transition probabilities of the process $\{Z(t), t \geq 0\}$, we define the global kernel, $\mathbf{K}(t)$ and the local kernel, $\mathbf{E}(t)$. Here, the global kernel describes the dynamics of the process immediately after the next regenerative time epoch (i.e., a rating upgrade in credit rating application). On the other hand, the local kernel explains the dynamics of the process between two regeneration epochs (i.e., can address the downward rating momentum).

1. **Global Kernel:** The global kernel $\mathbf{K}(t) = [K_{ij}(t)]_{i,j \in S'}$ associated with the process is defined as

$$K_{ij}(t) = P\{Z(T_1) = j, T_1 \leq t | Z(0) = i\}, \quad i, j \in S', \quad t \geq 0.$$

One can observe that the one-step transition probabilities $P = [p_{ij}]_{i,j \in S'}$ of the embedded Markov chain can be obtained as follows

$$p_{ij} = \lim_{t \rightarrow \infty} K_{ij}(t), \quad i, j \in S'.$$

2. **Local Kernel:** The local kernel $\mathbf{E}(t) = [E_{ij}(t)]_{i \in S', j \in S}$ is defined as follows

$$E_{ij}(t) = P\{Z(t) = j, T_1 > t | Z_0 = i\}, \quad i \in S', j \in S, t \geq 0$$

Further, the transition probabilities for the process $\{Z(t), t \geq 0\}$ are defined by

$$V_{ij}(t) = P\{Z(t) = j | Z(0) = i\}, \quad i \in S', j \in S, t \geq 0.$$

These can be obtained by solving the generalized Markov renewal equation [17]

$$V_{ij}(t) = E_{ij}(t) + \sum_{\gamma \in S'} \int_0^t V_{\gamma j}(t-y) dK_{i\gamma}(y), \quad i \in S', j \in S. \quad (9)$$

In credit rating dynamics framework, we can understand first term in Equation (9) as the probability of a firm facing a rating downgrade before any rating upgrade given it started in a rating category i at time 0. On the other hand, the second term in Equation (9) as the probability that the firm will upgrade to rating γ in time y and then it will migrate to rating j in the remaining time $t - y$ following some trajectory. In order to apply MRGP to model the dynamics of the credit ratings, we need to obtain the global and local kernel in credit ratings framework. From Pasricha et al. [20], we give the global and local kernel in the framework of credit rating modeling.

The global kernel $\mathbf{K}(t) = [K_{ij}(t)]_{i,j \in S'}$ can be obtained as follows

$$K_{ij}(t) = P(Z(T_1) = j, T_1 \leq t | Z(0) = i) = \begin{cases} G_{ij}(t) \cdot p_{ij} & \text{if } p_{ij} \neq 0 \\ 0 & \text{if } p_{ij} = 0 \end{cases}.$$

For each $i \in S'$, $E_{ij}(t), j \in S$ of the MRGP describes the behavior of rating evolution between two regeneration epochs as to how the rating moves to a lower rating before going to upper rating and is given by

$$E_{ij}(t) = \begin{cases} 0 & \text{if } i > j \\ \phi_{ij}^{(i)}(t) \times (1 - \sum_{k \in S'} K_{ik}(t)) & \text{if } i \leq j \end{cases}.$$

where $\phi_{ij}^{(i)}(t)$ are the transition probabilities of the subordinated SMP which accounts for the downward momentum. For initial state i at time 0, the transition probabilities $\phi_{ij}^{(i)}(t)$ can be obtained solving renewal equation in semi-Markov framework by considering a process with state space $\{i, i+1, \dots, 8\}$ with only possible transitions to be downward transitions.

3 Results and Discussion

In this section, we present the computational analysis of the proposed model on real data of Standard and Poor's rating agency. In order to show the applicability of non-Markov credit rating models, we first describe the methodology to estimate the model parameters. Finally, we present an application of the real data and compare the default distribution obtained using Markov and non-Markov models.

3.1 Parameter Estimation

In case of continuous time-homogeneous Markov models, the only parameter need to be estimated is the generator matrix of the Markov process, see e.g. Albert (1962) [1] and more recently Sadek and Linnios (2005) [21]. On the other hand, for time homogeneous non-Markov models, we need to estimate $G_{ij}(t)$, $i, j \in S$ along with the transition probability matrix \mathbf{P} of the embedded Markov chain $\{X_n\}$. Similarly, for time non-homogeneous models, the parameters that need to be estimated are $\mathbf{P}(s)$ and $G_{ij}(s, t)$, $j \in S$ for different initial times s and final times t . The methodology proposed in the articles considered in this review paper is as follows:

- (a) Estimation of $\mathbf{P}(s)$: For a fixed s , using the credit rating history at time s , the number of transitions from i to j with next jump are collected and assigned to a frequency matrix. Then, by normalizing the obtained frequency matrix, the probabilities $p_{ij}(s)$, $i, j \in S$ are calculated. The procedure is repeated for each $s \in [0, T]$.
- (b) Estimation of $G_{ij}(s, t)$, $j \in S$: For fixed s and t , in order to estimate the distribution functions of time spent inside a rating i given the next rating is j before time t , the following steps are followed:
 - (i) Firstly, identify those transitions which have initial rating i at time s and final rating j before time t , i.e., in a time duration of $t - s$.
 - (ii) For all the identified transitions, the number of time points (i.e. quarters) are calculated and a histogram is plotted to identify the most closely fit distribution.
 - (iii) Then, parameters of best fit distributions are estimated using maximum likelihood estimation.
 - (iv) Further, apply the Kolmogorov-Sminrov test (KS test) to statistically test the best fit distribution.

For the duration dependent models, a similar methodology is used by taking the recurrence times into account.

3.2 Applicability on the Real Data

We consider the quarterly credit rating history since 1985 to 2015 issued by Standard & Poor's to the long term issuers. Following the procedure mentioned above, we find that the best fit distribution for the time spent inside a rating given the next rating is Weibull distribution. For the calculations, R software has been used. The estimated one-step transition probability matrix is given in Matrix 1.

Matrix 1. 1 year transition probability matrix \mathbf{P}

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	0.9726	0.0259	0.0010	0.0002	0.0002	0.0000	0.0002	0.0000
AA	0.0013	0.9760	0.0217	0.0008	0.0001	0.0000	0.0000	0.0001
A	0.0002	0.0052	0.9814	0.0124	0.0005	0.0001	0.0001	0.0000
BBB	0.0000	0.0004	0.0088	0.9787	0.0106	0.0010	0.0002	0.0003
BB	0.0001	0.0003	0.0005	0.0131	0.9612	0.0223	0.0017	0.0008
B	0.0000	0.0001	0.0004	0.0004	0.0146	0.9619	0.0192	0.0034
CCC	0.0001	0.0000	0.0006	0.0009	0.0017	0.0481	0.8780	0.0706
D	0	0	0	0	0	0	0	1

Further, in order to compare the different non-Markov models with the real data, we compare the default probabilities given by S&P for (1981-2014) with those obtained by the non-Markov models. We consider the average cumulative default rates by ratings given in “Annual Global Corporate Default Study And Rating Transitions (2014)” by Standard and Poor's. Considering the average transition matrix given in the report for the period of (1981-2014), we obtain the default distribution of a firm using both the models namely MRGP and SMP. For the comparison purposes, we fixed the required parameters, i.e., $G_{ij}(t)$ and p_{ij} for all the models. The results are compared with the real data in the report and the results obtained by implement MRGP and SMP model for each rating category at time 0. Figure 1 presents the results, i.e., the default distribution on the log scale. From the figure, we observe that the results obtained from MRGP gives a better fit as compared to the SMP model.

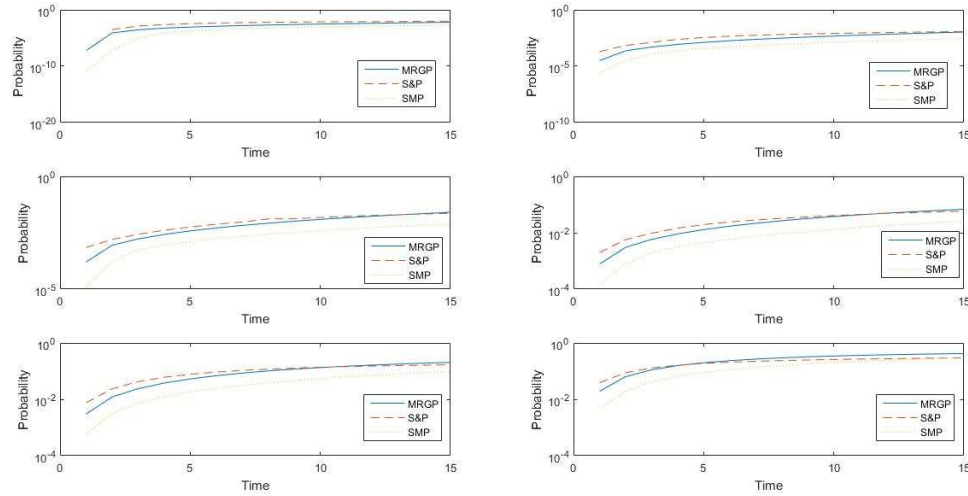


Figure 2: Distribution function of first time of default given initial rating (1981-2014)

4 Conclusion

This survey reviews, for the first time ever, the growing literature on non-Markov models based on semi-Markov process and Markov regenerative process to model the dynamics of credit ratings. Various models of credit rating dynamics proposed in the literature are discussed. Further, the estimation procedure for various parameters of these models is presented with an application on the real data.

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