Research Article

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Hedging the Risk of Wind Power Production Using Dispatchable Energy Source

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Abstract: In this paper we advance a nonlinear optimization problem for hedging wind power variability by using a dispatchable energy source (DES) like gas. The model considers several important aspects such as modeling of wind power production, electricity price, nonlinear penalization scheme for energy underproduction and interrelations among the considered variables. Results are given in terms of optimal cogeneration policy with DES. The optimal policy is interpreted and analyzed in different penalization scenarios and related to a 48 MW hypothetical wind park. The model is suitable for integration of wind energy especially for isolated grids. Some probabilistic results for special moments of a Log-Normal distribution are obtained; they are necessary for the evolution of the optimal policy.

Keywords: Optimization, Copula Function, Weibull Distribution, Electricity Price, Wind Energy

MSC 2010: 90B25

1 Introduction

In the last years the use of renewable energy sources, such as wind, biomass or solar, has increased consistently due to numerous benefits related to their use, see e.g. [9]. The most important benefit is that they may help mitigating climate changes and guarantee energy to low prices by limiting the use of fossil fuels. All these reasons have shifted the attention of the international debate to the urgent need to revise the current structure of the global energy system. In particular, wind energy assumed a dominant role because wind parks occupy limited areas, they do not produce toxic gases and have low installation and maintenance costs. Besides, unlike other renewable sources, the area occupied by a wind farm can be easily restored to renew the pre-existing conditions and wind turbines can face a very long life cycle before being abandoned. Nevertheless, the main problem of wind energy is wind variability that makes the wind power production uncertain and highly variable. To handle this problem different solutions have been implemented. A first chance is represented by the possibility of the wind power producer to subscribe an insurance contract with a dispatchable energy producer in order to immunize its power production against the volatility of wind speed (see [2]). Another financial instrument that can provide a solution to the variability of wind production is a call option that gives the buyer the right, but not the obligation, to buy the electricity at a certain predetermined strike price instead of a spot price. The seller (the wind farm), receives in exchange a premium fee that is the call price (see [5]). A third opportunity is represented by the storage system based on the coordination of wind power generation with reserves in form of dispatchable energy sources which can be part of the energy

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portfolio of the WPP or can be bought on the market (see e.g. [3, 6, 7, 10]). In recent articles ([12, 13]) the authors proposed a methodology based on the optimization of the stochastic LCOE and introducing a scheme for managing production costs of a generation portfolio of dispatchable and non-dispatchable energy sources.

In this paper, we further analyze the benefits of developing a strategy that optimally coordinates the production of wind power with DES. In particular, our model relies on an optimal compensation scheme of wind energy using DES. The model can be applied in cases where market mechanisms of compensation are not present as in isolated grid systems (see [11]). The methodology used is an extension of that advanced in [3]. In details, firstly we assume that the electricity price is a random variable with a Log-Normal probability distribution which implies that negative prices are not admitted. It is reasonable to only think to positive prices because negative prices rarely occur and they are the result of market imbalances. Secondly, the mathematical model presents a penalty in case of underproduction with respect to the quantity to deliver by contract. In the present paper the penalty is an increasing function of the energy not supplied and it is described by a general power function of parameter α. This is a further generalization of the case presented in [3] that considered only a linear penalty function. Thirdly, the quantity of energy produced by wind is modeled by means of a random variable with a mixed discrete-continuous distribution. The choice of a mixed distribution permits us to consider also the effects of wind speeds lower than the cut-in speed and of wind speeds greater than the cut-off speed. Indeed, in cases of excessively low and excessively high speeds the blades do not work due to an insufficient thrust of wind or to preventive switch-off to avoid structural damages, respectively. This aspect was ignored in [3] where only a continuous distribution of wind power production was considered. To model the dependence structure between wind power production and energy prices, we use a Fairlie–Gumbel–Morgenstern (FGM) copula. All previous aspects are used to define the expected profit function which is maximized under a budget constraint and under a fixed quantity of energy to be supplied thus obtaining the optimal quantity to produce with other dispatchable sources. To determine the optimal policy, we need to solve a nonlinear optimization problem that requires the computation of a special moment transform of Log-Normal distribution that at the authors' knowledge represents a new result.

The paper is structured as follows: Section 2 presents the general model, the optimization problem and its solutions. In Section 3 the model is applied to a case of a supposed wind farm located in Sardinia with local electricity prices. The paper ends with the conclusion and some suggestions for further research. The appendix provides mathematical proofs of the results needed in solving the optimization problem.

2 Problem Statement and Its Solution

At the current time $t = 0$, a wind park operator offers a certain amount K of energy to be placed on the market for the following period, say time $t = 1$. Typically, energy markets are characterized by a penalization system that punishes operators who place a quantity of energy different than the promised one. In details, if a WPP does not provide K, he incurs a penalty. The main reason for which such imbalances occur is wind variability and in order to hedge this risk, at the actual time ($t = 0$), the WPP decides to coordinate its production with DES such as gas. The mathematical modelization of the advanced problem starts from the adoption of three specific hypotheses which consider rigorously the aspects described above.

Hypothesis H1. Let π_e be the electricity price at time $t = 1$. At current time $t = 0$, this price is unknown and can be considered a nonnegative random variable. We assume that π_e has a Log-Normal distribution, ($\pi_e \sim$ Log-Normal(μ, σ^2)) with cumulative distribution function

$$F_{\pi_e}(x) := \mathbb{P}[\pi_e \le x] = \Phi\Big(\frac{\ln(x) - \mu}{\sigma\sqrt{2}}\Big),$$

where Φ is the cumulative distribution function of the standard normal distribution. The choice of a Log-Normal distribution is made in order to have the possibility to obtain quasi-explicit solutions of the optimization problem, in contrast with general distributions it can be solved only numerically. Moreover, this improves the results of D'Amico, Petroni and Sobolewski [3] which were based on a Normal distribution because in Section 3 we will show that Log-Normal distribution fits electricity prices better than the Normal distribution. It should be noted that in literature other distributions, such as the Generalized Pareto distribution (see e.g. [15]) or the Box-Cox Power Exponential distribution (see e.g. [4]), are often applied to investigate the electricity price change.

Let π_g be the cost of producing one unit of energy by DES at current time $t = 0$. This price is a known nonnegative quantity. As a consequence the energy produced by the wind farm is more remunerative as compared to DES because it does not need the purchase of any fuel.

Hypothesis H2. Let us denote by We and P_g the quantities of energy produced by the wind and DES, respectively. Define the energy not supplied by

$$\mathrm{ENS} := (K - (\mathrm{We} + P_g))^+ = \max(0, K - (\mathrm{We} + P_g)). \tag{2.1}$$

According to formula (2.1), the WPP agrees to offer K on the market and try to do this using wind power We (which is random) and the quantity P_g to be optimally determined using DES. If the total production $(\mathrm{We} + P_g)$ is unable to reach K, then there will be an energy not supplied ENS. If ENS is greater than zero, the WPP will suffer a loss of $\widetilde{\mathcal{C}} \geq 0$ Euros, where $\widetilde{\mathcal{C}}$ is a function of the energy not supplied, i.e.

$$\widetilde{\mathcal{C}} = C \cdot (\mathrm{ENS})^{\alpha} \quad \text{with } \alpha > 0, C \geq 0.$$

Note that if $\alpha = 1$, we recover the penalization scheme discussed in [3]. Furthermore, we assume that an overproduction of energy exceeding K is not sold at the market and therefore is lost. This is done only for easiness of exposition but is not a loss of generality because by means of a simple translation in the profit function we can recover the considered case.

Obviously, We is a nonnegative random variable because its value at $t = 1$ cannot be known at $t = 0$ due to many features such as wind speed, wind speed share, wind direction, thermal stratification, and so on, which have random outcomes at time $t = 1$. In contrast, P_g is a quantity that should be optimally determined in order to maximize the benefit of the WPP and $P_g \in [0, K]$. In general We has a mixed discrete-continuous distribution

$$F_{\mathrm{we}}(p) = \begin{cases} 0 & \text{if } p < 0, \\ a + (1-a)F(p) & \text{if } p \geq 0 \text{ and } a \in [0,1], \end{cases}$$

where $F(p)$ is an absolutely continuous cumulative distribution function (cdf) of a random variable W and a is a point mass at zero, i.e. $\mathrm{We} \sim a\delta_0 + (1-a)W$. The use of a mixed distribution allows us to consider also the case in which the wind speed is too strong or too weak. Indeed, a wind speed greater than the cut-off speed constitutes a risk of damage to the rotor of the blade and as a consequence the blade is switched in a standstill state. In the opposite case, when the wind speed is lower than the cut-in speed the blade cannot rotate and generate power. Moreover, this choice is supported by our analysis from which it emerges that a is a non-negligible probability as it will be shown in the application. Note that we have a generalization of the case discussed in [3] in which We has a Weibull distribution, which is recovered whenever $a = 0$ and $F(p)$ is a Weibull cdf. It should be remarked that the random variable We admits a probability density function

$$f_{\mathrm{we}}(p) = \frac{\partial F_{\mathrm{we}}(p)}{\partial p} \quad \text{for all } p > 0.$$

Hypothesis H3. Let $F_{(\pi_e,\mathrm{we})}(x, p) = \mathbb{P}[\pi_e \leq x, \mathrm{We} \leq p]$ be the joint cumulative distribution function of the wind power production and energy price at time $t = 1$. In general, the two random variables We and π_e are not independent. A convenient way to represent the joint distribution $F_{(\pi_e,\mathrm{we})}(x, p)$ is by means of a copula function, see e.g. [8]. In this paper, according to [3] we consider a Fairlie–Gumbel–Morgenstern (FGM) copula that is applied to the marginal distributions $F_{\mathrm{we}}(p)$ and $F_{\pi_e}(x)$ such that

$$F_{(\mathrm{we},\pi_e)}(p, x) = \mathcal{C}(F_{\mathrm{we}}(p), F_{\pi_e}(x)).$$

The copula function can be estimated from the available data. In our case we need to consider a copula function that allows for the quasi-explicit calculations of the optimal solutions and is in agreement with the real data. For this reason we confine our attention to the FGM copula. The FGM family of copulas is the only one which is a quadratic polynomial in u and v,

$$\mathcal{C}_\theta(u,v) = uv + \theta uv(1-u)(1-v), \quad \theta \in [-1,1],$$

where θ is the dependence parameter. If $\theta = 0$, then we recover the case of independence between the two random variables.

From the definition of the FGM copula we immediately obtain

$$F_{(\text{we},\pi_e)}(p,x) = F_{\text{we}}(p)F_{\pi_e}(x)[1+\theta(1-F_{\text{we}}(p))(1-F_{\pi_e}(x))],$$

and by differentiation for all $p > 0$ and all $x \geq 0$,

$$f_{(\text{we},\pi_e)}(p,x) = f_{\text{we}}(p)f_{\pi_e}(x)[1+\theta(1-2F_{\text{we}}(p))(1-2F_{\pi_e}(x))],$$

while for $p = 0$ and all $x \geq 0$ we have that

$$\begin{aligned} f_{(\text{we},\pi_e)}(0,x) &= \Big[\frac{\partial\mathcal{C}}{\partial v}(F_{\text{we}}(0), F_{\pi_e}(x)) - \frac{\partial\mathcal{C}}{\partial v}(F_{\text{we}}(0^-), F_{\pi_e}(x))\Big]f_{\pi e}(x) \\ &= [\mathcal{C}_{we|\pi_e}(F_{\text{we}}(0) \mid F_{\pi_e}(x)) - \mathcal{C}_{we|\pi_e}(F_{\text{we}}(0^-) \mid F_{\pi_e}(x))]f_{\pi e}(x) \\ &= \Delta\mathcal{C}_{we|\pi_e}(F_{\text{we}}(0) \mid F_{\pi_e}(x))f_{\pi e}(x) \\ &= \Delta\mathcal{C}_{we|\pi_e}(a \mid F_{\pi_e}(x))f_{\pi e}(x). \end{aligned}$$

In order to define the optimization problem rigorously we introduce the profit function as the random variable $\mathcal{R}$ defined by

$$\mathcal{R} := \mathcal{R}_> + \mathcal{R}_=,$$

where

$$\begin{aligned} \mathcal{R}_> := {}& \chi\{\text{We} + P_g \geq K\}\chi\{\text{We} > 0\}(\pi_e K - \pi_g P_g) \\ &+ \chi\{\text{We} + P_g < K\}\chi\{\text{We} > 0\}[\pi_e(\text{We} + P_g) - \pi_g P_g - C(K - (\text{We} + P_g))^\alpha] \end{aligned} \tag{2.2}$$

and

$$\mathcal{R}_= := \chi\{\text{We} = 0\}[\pi_e P_g - \pi_g P_g - C(K - P_g)^\alpha], \tag{2.3}$$

where $\chi(A)$ is the indicator function of event A.

Relation (2.2) asserts that if the total energy produced is greater than K, then the WPP has the inflow $\pi_e K$ derived from the selling of the produced energy but, at the same time, suffers the loss $\pi_g P_g$ which is due to the need to buy P_g. Otherwise, when the total energy produced is lower than K, the WPP has an inflow $\pi_e(\text{We} + P_g)$, a cost $\pi_g P_g$ due to the purchase of energy produced by DES and an additional loss due to the penalization for the energy not supplied, that is $C(K - (\text{We} + P_g))^\alpha$.

Relation (2.3) is the equivalent of relation (2.2) when We $= 0$.

Let $\mathcal{M}$ be the expected profit function, i.e. $\mathcal{M} := \mathbb{E}[\mathcal{R}]$. Then we have:

Lemma 1. *Under Assumptions H1–H3, if* $\alpha \geq 1$, *then* $\mathcal{M}(P_g)$ *is a concave function. Whereas if* $0 < \alpha < 1$, *then* $\mathcal{M}(P_g)$ *is a concave function if only if the following condition is verified:*

$$\begin{aligned} \mathbb{E}[\pi_e|\text{We} = K - P_g] > \frac{1}{(1-a)f_{\text{we}}(K-P_g)}\Big\{&-aC\alpha(\alpha-1)(K-P_g)^{\alpha-2} \\ &- \int_0^{K-P_g} f_{\text{we}}(p)(1-a)C\alpha(\alpha-1)(K-(p+P_g))^{\alpha-2}\,dp\Big\}. \end{aligned}$$

Proof. See Section A.1. □

A consequence of Lemma 1 is that a critical point belonging to the open interval $(0, K)$ is a maximum. Moreover, as the expected profit is a continuous function of the control variable over the compact set $[0, K]$, based on the Weierstrass's theorem we know that a global maximum and a global minimum should exist.

Now to consider the optimization problem, we introduce the variable ω denoting the initial wealth of the WPP. Then the quantity P_g that the WPP should decide to produce is constrained by the initial wealth. The optimization problem can be formulated as follows:

$$\begin{aligned} \text{Maximize} \quad & \mathcal{M}(P_g) \\ \text{subject to} \quad & h_1(P_g) = \omega - P_g \pi_g \geq 0, \\ & h_2(P_g) = P_g \geq 0, \\ & h_3(P_g) = K - P_g \geq 0. \end{aligned} \tag{2.4}$$

The constraint h_1 states that the WPP cannot buy DES for a value exceeding the initial wealth. Constraints h_2 and h_3 state that the variable P_g belongs to the closed interval $[0, K]$. Inequalities h_1 and h_3 can be combined into a single constraint once we observe that they express the fact that $P_g \leq \frac{\omega}{\pi_g}$ and $P_g \leq K$. Then, if we define $A := \min\{\frac{\omega}{\pi_g}, K\}$, we can rewrite the optimization problem as follows:

$$\begin{aligned} \text{Maximize} \quad & \mathcal{M}(P_g) \\ \text{subject to} \quad & h_1(P_g) = A - P_g \geq 0, \\ & h_2(P_g) = P_g \geq 0, \end{aligned}$$

where h_1 replaces the constraints h_1 and h_3 in formula (2.4).

To solve the optimization problem, we write the Lagrangian function

$$\mathcal{L} = \mathcal{M}(P_g) + \lambda_1 h_1(P_g) + \lambda_2 h_2(P_g),$$

where λ_1, λ_2 are the Lagrange's multipliers. The application of Kuhn–Tucker's theorem, see e.g. [16], gives us the following system of equations:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial P_g} = \left[C\alpha \displaystyle\int_0^{K-P_g} f_{\text{we}}(p)(K-(p+P_g))^{\alpha-1}\, dp + aC\alpha(K-P_g)^{\alpha-1} \right] \\ \qquad + \left[\displaystyle\int_0^{K-P_g} f_{\text{we}}(p)\mathbb{E}[\pi_e | \text{We} = p]\, dp + a\mathbb{E}[\pi_e | \text{We} = 0] \right] - \pi_g(1+a) \\ \qquad - \lambda_1 + \lambda_2 = 0, \\ \dfrac{\partial \mathcal{L}}{\partial \lambda_1} = h_1(P_g) = A - P_g \geq 0, \\ \dfrac{\partial \mathcal{L}}{\partial \lambda_2} = h_2(P_g) = P_g \geq 0, \\ \lambda_1 \geq 0, \quad \lambda_2 \geq 0, \quad \lambda_1(A - P_g) = 0, \quad \lambda_2 P_g = 0. \end{cases} \tag{2.5}$$

The solution of system (2.5) passes through the consideration of three different cases:

- Case I.1: $P_g = 0$ and $\lambda_1 = 0$,
- Case I.2: $\lambda_2 = 0$ and $P_g = A$,
- Case I.3: $\lambda_1 = \lambda_2 = 0$.

We proceed to analyze each one of the former cases and to determine and discuss corresponding solutions.

2.1 Case I.1: $P_g = 0$ and $\lambda_1 = 0$

If we consider $P_g = 0$, it follows that $\lambda_1 = 0$. If we substitute these two values into the first equation of the system (2.5) we obtain

$$\left[a\mathbb{E}[\pi_e | \text{We} = 0] + \int_0^K f_{\text{we}}(p)\mathbb{E}[\pi_e | \text{We} = p]\, dp \right] - \pi_g(1+a) + \lambda_2 + \left[aC\alpha(K)^{\alpha-1} + C\alpha \int_0^K f_{\text{we}}(K-p)^{\alpha-1} \right] = 0,$$

from which we recover

$$\lambda_2 = \pi_g(1+a) - \left[a\mathbb{E}[\pi_e|\mathrm{We}=0] + \int_0^K f_{\mathrm{we}}(p)\mathbb{E}[\pi_e|\mathrm{We}=p]\,dp\right] - \left[aC\alpha(K)^{\alpha-1} + C\alpha\int_0^K f_{\mathrm{we}}(K-p)^{\alpha-1}\right] \geq 0,$$

where the last inequality holds because $\lambda_2 \geq 0$. These calculations show that the solution $P_g = 0$ can occur only if

$$\pi_g \geq \frac{1}{1+a}\left\{\left[a\mathbb{E}[\pi_e|\mathrm{We}=0] + \int_0^K f_{\mathrm{we}}(p)\mathbb{E}[\pi_e|\mathrm{We}=p]\,dp\right] - \left[aC\alpha K^{\alpha-1} + C\alpha\int_0^K f_{\mathrm{we}}(p)(K-p)^{\alpha-1}\,dp\right]\right\} \tag{2.6}$$
$$=: T(0;a,\alpha).$$

The interpretation is that it is optimal not to produce energy by DES only if the price of this one π_g, exceeds the upper threshold $T(0;\alpha,a)$ or likewise if the unit marginal cost of DES is greater than the unit marginal revenue. Besides, it should be remarked that the upper threshold depends on the wind power production, on the quantity K of energy to be supplied by the contract and on the joint probability density function $f_{\pi_e,we}$.

We can represent the upper threshold $T(0;a,\alpha)$ more explicitly if we use the adopted parameterization. Indeed, (2.6) is equivalent to

$$\begin{aligned}\pi_g \geq \frac{1}{1+a}\Bigg\{&C\alpha\left[aK^{\alpha-1} + (1-a)\int_0^K f(p)(K-p)^{\alpha-1}\,dp\right] + \int_0^{+\infty} x\Delta C_{we|\pi_e}(a \mid F_{\pi_e}(x))f_{\pi e}(x)\,dx \\ &+ \int_0^K\int_0^{+\infty} xf_{\pi e}(x)f(p)(1-a)\Big[1+\theta(1-2F(p))(1-2F_{\pi_e}(x)) - 2a\theta(1-F(p))(1-2F_{\pi_e}(x))\Big]\,dx\,dp\Bigg\}.\end{aligned} \tag{2.7}$$

Now observe that the third addendum on the right-hand side of (2.7) can be expressed as follows:

$$\begin{aligned}&\int_0^K\int_0^{+\infty} xf_{\pi e}(x)f(p)(1-a)[1+\theta(1-2F(p))(1-2F_{\pi_e}(x)) - 2a\theta(1-F(p))(1-2F_{\pi_e}(x))]\,dx\,dp \\ &= (1-a)\int_0^K\int_0^{+\infty} xf_{\pi e}(x)f(p)\,dx\,dp + \int_0^K\int_0^{+\infty} x(1-a)f_{\pi e}(x)f(p)\theta[(1-2F(p))(1-2F_{\pi_e}(x))]\,dx\,dp \\ &\quad - \int_0^K\int_0^{+\infty} x(1-a)f_{\pi e}(x)f(p)2a\theta[(1-F(p))(1-2F_{\pi_e}(x))]\,dx\,dp \\ &= (1-a)F(K)\mathbb{E}[\pi_e] + (1-a)\theta[(F(K)-F^2(K))(\mathbb{E}[\pi_e] - 2\mathbb{E}[\pi_e F_{\pi_e}(\pi_e)])] \\ &\quad - 2a(1-a)\theta\left(\int_0^K f(p)(1-F(p))\,dp\right)\left(\int_0^{+\infty} xf_{\pi_e}(x)(1-2F_{\pi_e}(x))\,dx\right) \\ &= (1-a)F(K)\mathbb{E}[\pi_e] + (1-a)\theta[(F(K)-F^2(K))(\mathbb{E}[\pi_e] - 2\mathbb{E}[\pi_e F_{\pi_e}(\pi_e)])] \\ &\quad - 2a\theta(1-a)[F(K) - 0.5F^2(K)][\mathbb{E}[\pi_e] - 2\mathbb{E}[\pi_e F_{\pi_e}(\pi_e)]].\end{aligned} \tag{2.8}$$

Finally, by substitution of (2.8) into (2.7) we obtain that the solution of Case I.1 suggests $P_g = 0$ whenever

$$\begin{aligned}\pi_g \geq \frac{1}{1+a}\Bigg\{&C\alpha[aK^{\alpha-1} + (1-a)F(K)\mathbb{E}[(K-W)^{\alpha-1}|W \leq K]] \\ &+ \int_0^{+\infty} x\Delta C_{we|\pi_e}(a \mid F_{\pi_e}(x))f_{\pi e}(x)\,dx + (1-a)F(K)\mathbb{E}[\pi_e] \\ &+ (1-a)\theta[F(K)-F^2(K)][\mathbb{E}[\pi_e] - 2\mathbb{E}[\pi_e F_{\pi_e}(\pi_e)]] \\ &- 2a(1-a)\theta[F(K) - 0.5F^2(K)][\mathbb{E}[\pi_e] - 2\mathbb{E}[\pi_e F_{\pi_e}(\pi_e)]]\Bigg\}.\end{aligned} \tag{2.9}$$

The right-hand side of inequality (2.9) can be evaluated whenever $\mathbb{E}[\pi_e F_{\pi_e}(\pi_e)]$ is known. Next proposition shows how to get this expectation.

Proposition 2. *If* $\pi_e \sim \text{Log-Normal}(\mu, \sigma^2)$, *then*

$$\mathbb{E}[\pi_e F_{\pi_e}(\pi_e)] = e^{\mu}\left\{\sum_{k\in E}\frac{\sigma^k}{k!}I_k^P + \sum_{k\in O}\frac{\sigma^k}{k!}I_k^D\right\},$$

where E is the set of even numbers and O is the set of odd numbers. Moreover,

$$I_k^P := \frac{1}{2}\prod_{i\in O_k} i, \quad \text{where } I_0^P = 0.5,$$

with $O_k = \{i \in \mathbb{N} : i \leq k \text{ with } i \text{ odd}\}$. *Whereas*

$$I_k^D := \frac{1}{\sqrt{4\pi}}T_k$$

with $T_k = \sum_{j=1}^{\frac{k+1}{2}} t_{k,j}$, *where*

$$\begin{aligned} t_{1,1} &= 1, \\ t_{i,1} &= \prod_{h\in O_{i-1}}\frac{(i-1)-h}{2} && \text{for } i \text{ odd}, \\ t_{i,j} &= t_{i-2,j-1}(i-1) && \text{for all } j = 2,\ldots,\frac{i+1}{2}. \end{aligned}$$

Proof. See Section A.2. □

2.2 Case I.2: $\lambda_2 = 0$ and $P_g = A$

This case splits into two subcases either according to the two possible values of A, i.e. $A = K$ or $A = \frac{\omega}{\pi_g}$. Thus, let us consider $\lambda_2 = 0$ and $A = P_g$, a substitution of these values into the first equation of system (2.5) gives

$$\begin{aligned} &\left[a\mathbb{E}[\pi_e|\text{We} = 0] + \int_0^{K-A} f_{\text{we}}(p)\mathbb{E}[\pi_e|\text{We} = p]\,dp\right] - \pi_g(1+a) - \lambda_1 \\ &+ \left[aC\alpha(K-A)^{\alpha-1} + C\alpha\int_0^{K-A} f_{\text{we}}(K-(p+A))^{\alpha-1}\,dp\right] = 0, \end{aligned}$$

from which we recover

$$\begin{aligned} \lambda_1 = &\left[a\mathbb{E}[\pi_e|\text{We} = 0] + \int_0^{K-A} f_{\text{we}}(p)\mathbb{E}[\pi_e|\text{We} = p]\,dp\right] \\ &- \pi_g(1+a) + \left[aC\alpha(K-A)^{\alpha-1} + C\alpha\int_0^{K-A} f_{\text{we}}(K-(p+A))^{(\alpha-1)}\,dp\right] \geq 0. \end{aligned}$$

It follows that

$$\begin{aligned} \pi_g \leq \frac{1}{1+a}&\left\{a\mathbb{E}[\pi_e|\text{We} = 0] + \int_0^{K-A} f_{\text{we}}(p)\mathbb{E}[\pi_e|\text{We} = p]\,dp\right. \\ &\left. + aC\alpha(K-A)^{\alpha-1} + C\alpha\int_0^{K-A} f_{\text{we}}(K-(p+A))^{(\alpha-1)}\,dp\right\} =: T(A;\alpha,a). \end{aligned} \tag{2.10}$$

The interpretation is that, if the price of energy produced by DES, π_g, is below a lower threshold $T(A;\alpha,a)$, then it is preferable to produce a quantity $P_g = A$ of energy using this source. As already remarked, A is the minimum between $\frac{\omega}{\pi_g}$ and K thus it can assume two values: either $A = K$ or $A = \frac{\omega}{\pi_g}$. In the case $A = K$ by substitution into (2.10) we get

$$\pi_g \leq \left[\frac{a\mathbb{E}[\pi_e|\mathrm{We}=0]}{1+a}\right].$$

We can ignore this solution since the price of energy to be produced by DES will never reach zero or become negative. In the case $A = \frac{\omega}{\pi_g}$ the lower threshold is not nullified and the condition $\pi_g \leq T(\frac{\omega}{\pi_g};\alpha,a)$ suggests purchasing DES using all the initial wealth ω, i.e. $P_g = \frac{\omega}{\pi_g}$. Now considering the adopted parameterization, the solution can be written in the following way: If $A = K$, we have

$$\begin{aligned}\pi_g &\leq \frac{a\mathbb{E}[\pi_e|\mathrm{We}=0]}{1+a}\\ &= \frac{1}{1+a}\left\{\int_0^{+\infty} x\Delta\mathfrak{C}_{we|\pi_e}(a \mid F_{\pi_e}(x))f_{\pi e}(x)\,dx\right\},\end{aligned}$$

if $A = \frac{\omega}{\pi_g}$, we have

$$\begin{aligned}\pi_g &\leq T\left(\frac{\omega}{\pi_g};\alpha,a\right)\\ &= \frac{1}{1+a}\left\{aC\alpha\left(K-\frac{\omega}{\pi_g}\right)^{\alpha-1} + C\alpha(1-a)F\left(K-\frac{\omega}{\pi_g}\right)\mathbb{E}\left[\left(K-\frac{\omega}{\pi_g}-W\right)^{\alpha-1}\middle|\mathrm{We}\leq K-\frac{\omega}{\pi_g}\right]\right.\\ &\quad + \int_0^{+\infty} x\Delta\mathfrak{C}_{we|\pi_e}(a \mid F_{\pi_e}(x))f_{\pi e}(x)\,dx + (1-a)F\left(K-\frac{\omega}{\pi_g}\right)\mathbb{E}[\pi_e]\\ &\quad + (1-a)\theta\left[F\left(K-\frac{\omega}{\pi_g}\right) - F^2\left(K-\frac{\omega}{\pi_g}\right)\right]\left[\mathbb{E}[\pi_e] - 2\mathbb{E}[\pi_e F_{\pi_e}(\pi_e)]\right]\\ &\quad \left. - 2a(1-a)\theta\left[F\left(K-\frac{\omega}{\pi_g}\right) - 0.5F^2\left(K-\frac{\omega}{\pi_g}\right)\right]\left[\mathbb{E}[\pi_e] - 2\mathbb{E}[\pi_e F_{\pi_e}(\pi_e)]\right]\right\},\end{aligned}$$

which again can be completely evaluated using Proposition 2.

2.3 Case I.3: $\lambda_1 = \lambda_2 = 0$

In the case in which $\lambda_1 = \lambda_2 = 0$, a substitution of these values into the first equation of system (2.5) gives

$$\begin{aligned}&\left[a\mathbb{E}[\pi_e|\mathrm{We}=0] + \int_0^{K-P_g} f_{\mathrm{we}}(p)\mathbb{E}[\pi_e|\mathrm{We}=p]\,dp\right] - \pi_g(1+a)\\ &\quad + \left[aC\alpha(K-P_g)^{\alpha-1} + C\alpha\int_0^{K-P_g} f_{\mathrm{we}}(K-p-P_g)^{\alpha-1}\,dp\right] = 0,\end{aligned}$$

from which we recover

$$\begin{aligned}T(P_g;\alpha,a) &:= \frac{1}{1+a}\left[a\mathbb{E}[\pi_e|\mathrm{We}=0] + \int_0^{K-P_g} f_{\mathrm{we}}(p)\mathbb{E}[\pi_e|\mathrm{We}=p]\,dp\right]\\ &\quad + \left[aC\alpha(K-P_g)^{\alpha-1} + C\alpha\int_0^{K-P_g} f_{\mathrm{we}}(K-p-P_g)^{\alpha-1}\,dp\right] = \pi_g,\end{aligned}$$

or more simply

$$T(P_g;\alpha,a) = \pi_g.$$

In particular, with the adopted parameterization we have

$$\begin{aligned}\pi_g &= aC\alpha(K-P_g)^{\alpha-1} + C\alpha(1-a)F(K-P_g)\mathbb{E}[(K-P_g-W)^{\alpha-1}|W \le K-P_g] \\ &\quad + \int_0^{+\infty} x\Delta\mathcal{C}_{we|\pi_e}(a \mid F_{\pi_e}(x))f_{\pi e}(x)\,dx + (1-a)F(K-P_g)\mathbb{E}[\pi_e] \\ &\quad + (1-a)\theta[(F(K-P_g)-F^2(K-P_g))(\mathbb{E}[\pi_e]-2\mathbb{E}[\pi_e F_{\pi_e}(\pi_e)])] \\ &\quad - 2a(1-a)\theta[(F(K-P_g)-0.5F^2(K-P_g))(\mathbb{E}[\pi_e]-2\mathbb{E}[\pi_e F_{\pi_e}(\pi_e)])].\end{aligned} \tag{2.11}$$

Observe that if the function $T(P_g;\alpha,a)$ is decreasing in its argument P_g, the optimal solution is given by

$$P_g = T^{-1}(\pi_g; a, \alpha). \tag{2.12}$$

Thus it is important to discuss the existence of the inverse function in order to solve equation (2.11) and to evaluate its solution (2.12). To this end, we proceed to compute the first-order derivative of the function $T(P_g, \alpha, a)$:

$$\frac{\partial T}{\partial P_g} = \frac{1}{1+a}\left[-aC\alpha(\alpha-1)(K-P_g)^{\alpha-2} - f_{\mathrm{we}}(K-P_g)\mathbb{E}[\pi_e|\mathrm{We}=K-P_g] - C\alpha(\alpha-1)\int_0^{K-P_g} f_{\mathrm{we}}(K-p-P_g)^{\alpha-2}\,dp\right].$$

Note that:
- If $\alpha \ge 1$, we have $\frac{\partial T}{\partial P_g} < 0$ so the inverse function exists and the optimal solution is given by formula (2.12).
- If $0 < \alpha < 1$, we have

  $$\frac{-aC\alpha(\alpha-1)(K-P_g)^{\alpha-2} - C\alpha(\alpha-1)\int_0^{K-P_g} f_{\mathrm{we}}(p)(K-p-P_g)^{\alpha-2}\,dp}{1+a} > 0$$

  and

  $$\frac{-f_{\mathrm{we}}(K-P_g)\mathbb{E}[\pi_e|\mathrm{We}=K-P_g]}{1+a} < 0.$$

  In order to obtain invertibility, it is necessary to impose a condition on C. In particular, $\frac{\partial T}{\partial P_g} \lesseqgtr$ according to whether

  $$C \gtreqless \frac{-f_{\mathrm{we}}(K-P_g)\mathbb{E}[\pi_e|\mathrm{We}=K-P_g]}{(\alpha-1)\alpha[a(K-P_g)^{\alpha-2} + \int_0^{K-P_g} f_{\mathrm{we}}(p)(K-(p+P_g))^{\alpha-2}\,dp]}.$$

  Also in this case (2.12) provides the optimal solution.

3 Application

In this section it is shown an application of the model just described to a non-isolated system in which there is a penalization scheme. However, it is necessary to remark that the results are particulary useful for isolated grids. The model is applied to an hypothetical wind farm of 48 MW rated power and electricity prices traded at IPEX (Italian Power Exchange).

In details wind data have been downloaded from NASA's MERRA-2 database[1] and then converted into wind energy using a power curve (see Figure 1) assuming for each turbine:
- Geographical coordinates: 39.5 N (latitude) and 8.75 E (longitude),
- Hub height: 95 m,
- Rated power: 2 MW,
- Cut-in wind speed: 13 m/s,
- Cut-out wind speed: 25 m/s,
- Rated wind speed: 13 m/s.

1 See https://gmao.gsfc.nasa.gov/reanalysis/MERRA-2.

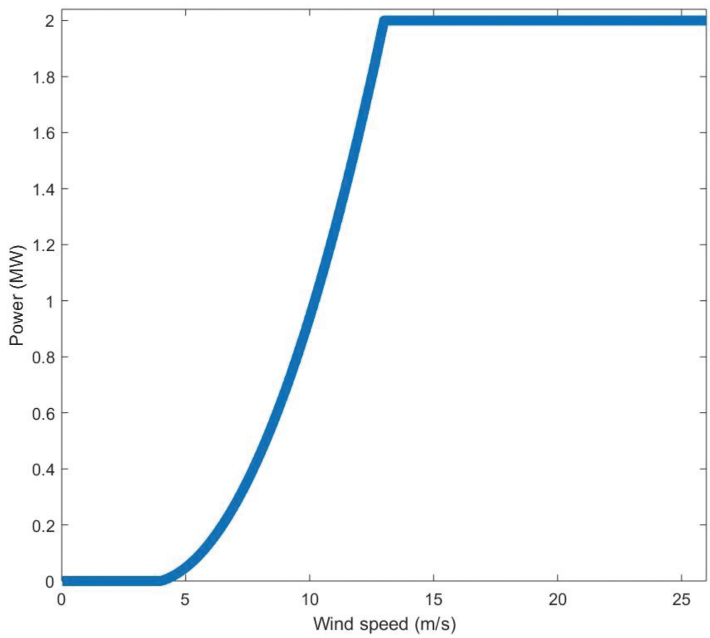

Figure 1: Power curve of a 2MW wind turbine.

In our wind farm we assume that there are 24 independent wind turbines; accordingly, the total wind power production is given by multiplying the numbers of turbines with the unitary wind power production. This is done to simplify the analysis because in a real situation shear effects and geomorphological structures of the land are amid the aspects that may induce correlation among the turbines.

Data on electricity prices have been downloaded from the official manager of the market GME.[2] All the data refer to a period of ten years (from 2008 to 2018 for a total of 87647 observations) and the resolution is 1 h. The unit of energy produced is MWh. The unit of electricity prices is €/MWh.

Firstly, we checked the daily correlation between the electricity price and the wind power production time series, see Table 1. We report the hourly correlation values, the corresponding p-values (which assesses the significance for the correlations) and the 95 %-Confidence Intervals. All of the correlations are negative and in line with our choice of the FGM copula that does not allow the reproduction of strong correlations. Since our application is based on a daily scale we estimated the daily correlation that is equal to -0.11936.

In this application we assume that the wind power production, has a mixed discrete-continuous distribution with the following cdf:

$$F_{\mathrm{we}}(p) = \begin{cases} 0 & \text{if } p < 0, \\ a + (1-a)\left(1 - e^{-(\frac{p}{\lambda})^{\gamma}}\right) & \text{if } p \geq 0,\ a \in [0,1], \end{cases}$$

where $F_{\mathrm{we}}(p)$ has a Weibull distribution for his continuous part, i.e. in the interval $(0, +\infty)$, and a is a point mass at zero. The estimation of the parameter a is equal to 0.2623 with the corresponding 95 %-Confidence Interval $[0.2418, 0.2829]$ and it is computed as the frequency of the null wind power production. In general, it is possible to use other distributions to shape wind energy production. A fairly common choice is to model the body of the distribution with a Weibull and its tails with a Generalized Pareto distribution to adequately consider extreme wind speed events (see e.g. [1]). In Figure 2 we can see the probability plot when the wind power production is positive. For the electricity price we have decided to use a Log-Normal distribution because this distribution has a better fit than the Normal one as we can see from Figure 3 and it allows a quasi-explicit computation of the optimal policy.

In Table 2 and in Table 3 we show the estimates of all the parameters that we will use to test the model.

2 See https://www.mercatoelettrico.com.

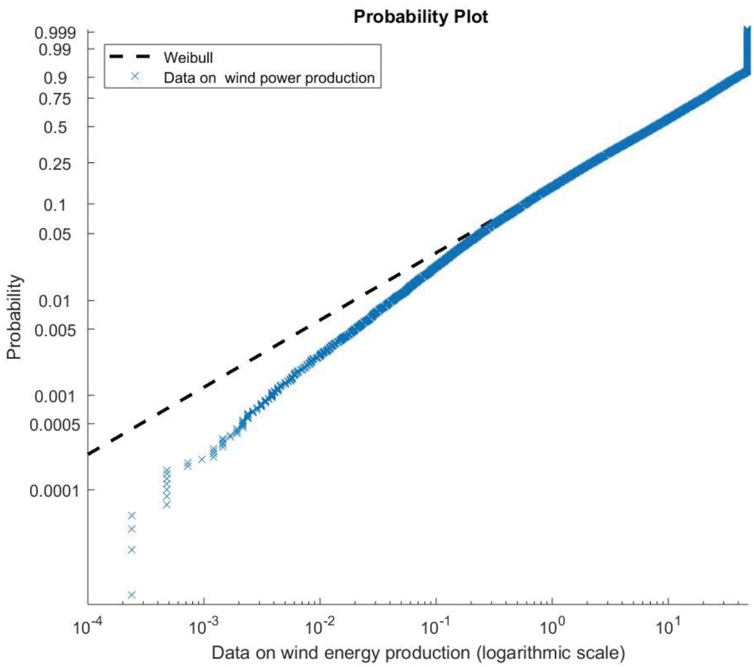

Figure 2: The probability plot compares the distribution of the data on wind power production to the Weibull theoretical distribution.

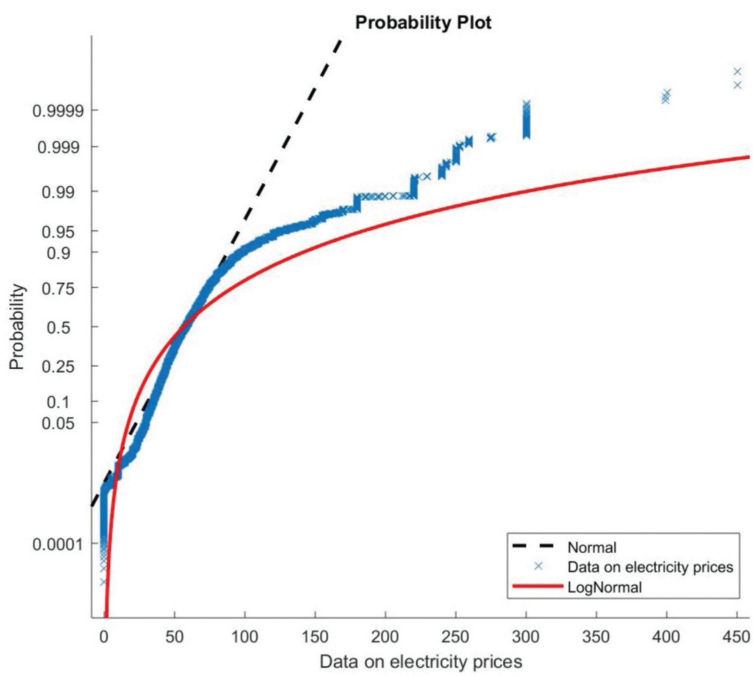

Figure 3: The probability plot compares the distribution of the data on electricity prices to the Normal and Log-Normal theoretical distributions.

In the following we show the optimal policies obtained by solving equations (2.9), (2.10) and (2.12). All the results are achieved by setting the parameter C as the average electricity price and estimating the parameter θ using MLE in Matlab. Moreover, we fix α greater than 1 in order to adequately penalize the greater deviations from the target, more than the smaller ones. We use equation (2.9) to find the values of π_g for which it is convenient to produce energy only using wind as a function of K. In Figure 4 we have four graphs,

Daily Hour	Correlation	p-Value	95 %-Confidence Interval
1	−0.1588	4.76e−22	[−0.1902, −0.1270]
2	−0.1639	2.12e−23	[−0.1953, −0.1321]
3	−0.1682	1.41e−24	[−0.1995, −0.1365]
4	−0.1704	3.32e−25	[−0.2018, −0.1388]
5	−0.1602	1.98e−22	[−0.1917, −0.1285]
6	−0.1023	5.84e−10	[−0.1343, −0.0701]
7	−0.0834	4.43e−07	[−0.1156, −0.0511]
8	−0.0919	2.62e−08	[−0.1240, −0.0597]
9	−0.0827	5.56e−07	[−0.1148, −0.0511]
10	−0.0951	8.37e−09	[−0.1272, −0.0629]
11	−0.1072	8.24e−11	[−0.1392, −0.0750]
12	−0.1123	1.02e−11	[−0.1442, −0.0801]
13	−0.1167	1.48e−12	[−0.1486, −0.0846]
14	−0.1139	5.08e−12	[−0.1458, −0.0818]
15	−0.1107	1.97e−11	[−0.1426, −0.0786]
16	−0.1131	7.09e−12	[−0.1450, −0.0810]
17	−0.0862	1.80e−07	[−0.1183, −0.0539]
18	−0.0845	3.15e−07	[−0.1166, −0.0522]
19	−0.0974	3.67e−09	[−0.1294, −0.0652]
20	−0.1251	3.31e−14	[−0.1569, −0.0930]
21	−0.1322	−1.05e−15	[−0.1639, −0.1002]
22	−0.1268	1.46e−14	[−0.1586, −0.0947]
23	−0.1235	6.86e−14	[−0.1533, −0.0914]
24	−0.1381	5.21e−17	[−0.1698, −0.1061]

Table 1: Daily correlation values between electricity prices and wind power production, p-values, 95 %-Confidence Intervals.

Parameter	Estimate	95 %-Confidence Interval	Standard Error
λ [MW]	11.91	[11.79, 12.03]	0.0613
γ	0.800	[0.795, 0.805]	0.0026
a	0.26	[0.24, 0.28]	0.0105

Table 2: Parameters of the mixed discrete-continuous distribution for the wind power production.

Parameter	Estimate	95 %-Confidence Interval	Standard Error
μ [€/MWh]	4.034	[4.029, 4.038]	0.0024
σ [€/MWh]	0.719	[0.715, 0.723]	0.0017

Table 3: Parameters of the Log-Normal distribution for the electricity prices.

each obtained with a different level of α ($\alpha = 1$, $\alpha = 1.1$, $\alpha = 1.3$, $\alpha = 1.5$). The optimal region of each graph, suggests that, by setting a certain value of K, if α increases the optimal region becomes smaller. This means that as α rises, the WPP will be willing to pay more for DES to avoid the loss. In general, even if K increases, in addition to α, the WPP will be available to pay DES more in order to have no penalty.

In Figure 5 we show the results of using equation (2.10) for two different values of K and for two levels of α, as a function of π_g and ω. In this figure we have four graphs, each shows the optimal region where it is beneficial to use all of WPP's wealth to buy energy produced by DES. We can note two kind of behaviors: firstly, the optimal region increases when K increases and secondly, for given levels of K, the optimal area increases when α increases. The first observation means that if the energy to be served augments, it is difficult to cover the production of energy using only wind but it is necessary using also DES. The second one says that if the penalty increases, the WPP will buy more DES to avoid it even if the price of gas rises. As a matter of example, for $K = 33$ MWh, given $\pi_g = 50$ MWh $\cdot$ € the WPP should use his initial wealth on up 3200 MWh $\cdot$ € when $\alpha = 1$ and on up 5500 MWh $\cdot$ € when $\alpha = 1.3$.

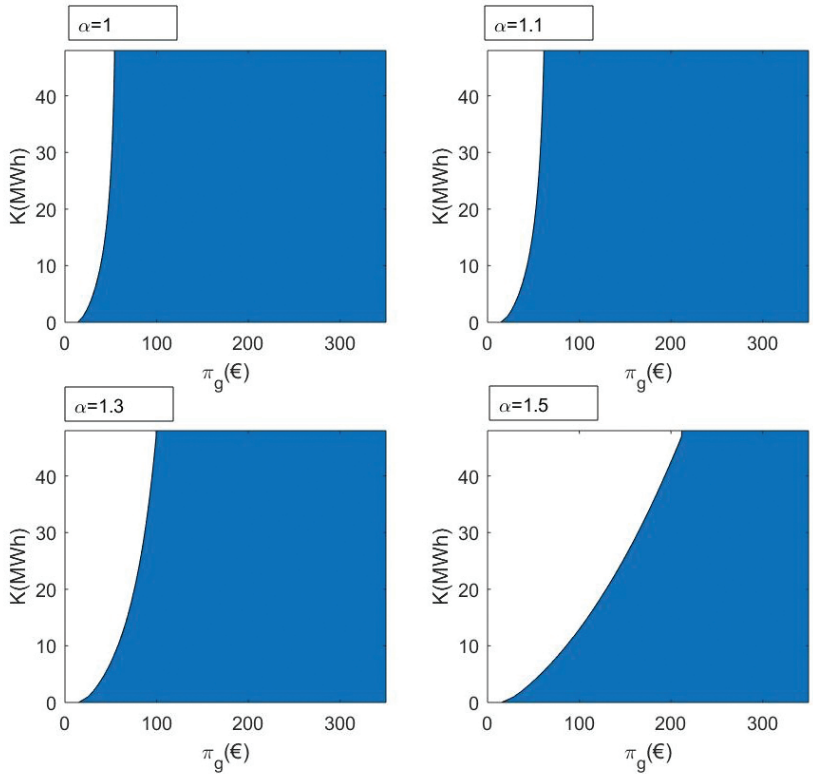

Figure 4: Combinations of (π_g,K), with different levels of α, for which it is optimal to produce energy using wind farm only, i.e. $P_g = 0$.

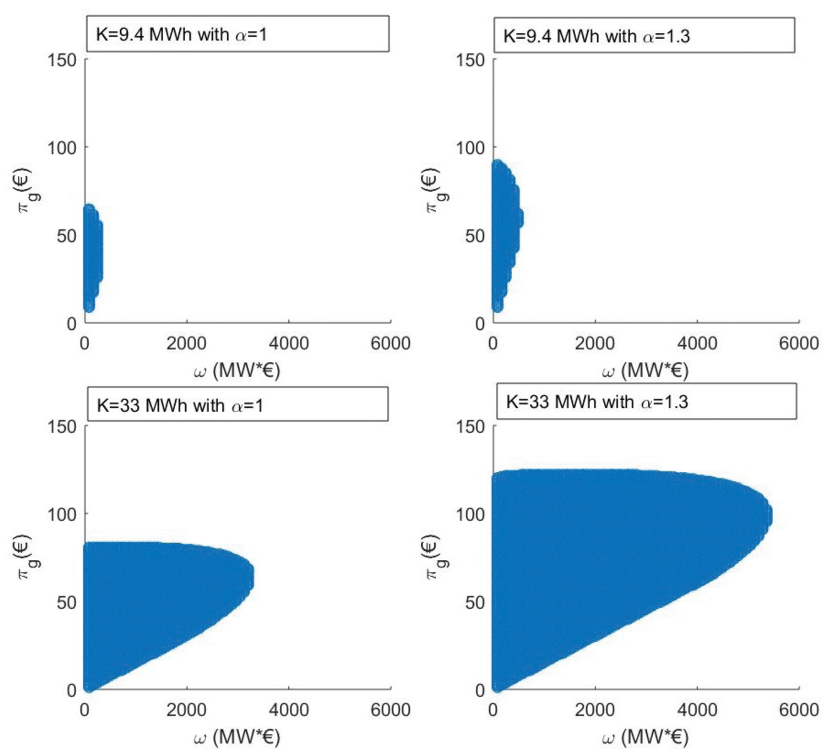

Figure 5: Combinations of (ω,π_g) for which it is optimal to produce power using the total wealth of the WPP, i.e. $P_g = \frac{\omega}{\pi_g}$. Cases: $K = 9.4$ MWh with $\alpha = 1$ (top-left-panel), $K = 9.4$ MWh with $\alpha = 1.3$ (top-right-panel), $K = 33$ MWh with $\alpha = 1$ (bottom-left-panel), $K = 33$ MWh with $\alpha = 1.3$ (bottom-right-panel)

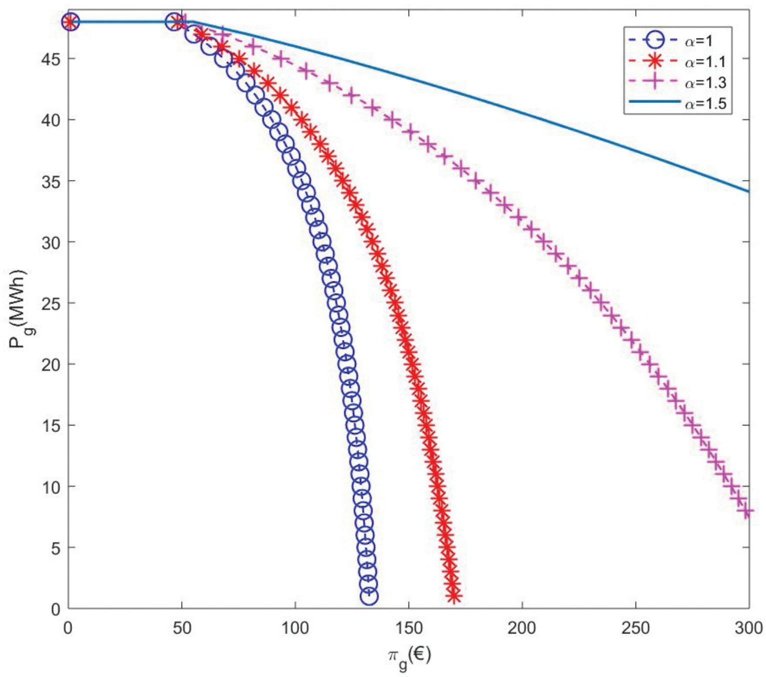

Figure 6: Optimal quantity of P_g depending on the level of π_g for $K = 48$ MWh. Cases: $\alpha = 1$ (blue curve), $\alpha = 1.1$ (red curve), $\alpha = 1.3$ (purple curve), $\alpha = 1.5$ (light blue curve).

In Figure 6 we show the results of using equation (2.12) when $K = 48$ MWh and for different levels of α ($\alpha = 1$, $\alpha = 1.1$, $\alpha = 1.3$, $\alpha = 1.5$), as a function of P_g and π_g. It can be noticed that as α increases the curves move up. If we fix a level of π_g, when α increases the optimal policy for the WPP is to expand the quantity of DES bought to avoid the increasing of the potential penalty.

4 Conclusion

In this paper, we presented an optimization problem for the optimal coordination of wind energy and energy production through a dispatchable energy source (DES).

The nonlinear optimization problem considers the amount of energy P_g to be produced by DES as a control variable. Depending on the random wind power production, the electricity prices, the penalization scheme and their inter-dependencies, we determine quasi-explicit optimal policies given an initial wealth and the total quantity of energy K to be offered on the market.

We demonstrated the validity of our approach with an illustrative application to a hypothetical wind farm located in Sardinia. Electricity prices data have been downloaded from the GME official site while wind energy data have been obtained according to wind speeds downloaded from NASA's MERRA-2 database.

Moreover, the proposed methodology could be extended under three different scenarios. Firstly, other distributions to model both electricity prices and wind power production could be used. In particular, it might be interesting to apply distributions belonging to the Extreme Value Theory (EVT) class such as the Generalized Pareto distribution since it allows to consider extreme variations in both electricity prices and wind speed. Secondly, it could be adopted in a multi-periodical setting in which the WPP decides repeatedly the quantity of energy to produce using DES including the possibility of borrowing money such that his total profit is maximized over the considered time interval. Thirdly, these techniques could be practiced considering also K as a control variable. Within this last framework, the expected profit function is a function of both the P_g and K variables and the target of the WPP is to determine the optimal combination of P_g and K in order to maximize the expected profit function.

A Proofs

A.1 Proof of Lemma 1

Let $\mathbb{M}_{>} := \mathbb{E}[R_{>}]$ and $\mathbb{M}_{=} := \mathbb{E}[R_{=}]$. Then

$$\begin{aligned}\mathcal{M} &= \mathbb{M}_{>} + \mathbb{M}_{=}\\ &= \mathbb{E}[\chi\{\mathrm{We} + P_g \geq K\}\chi\{\mathrm{We} > 0\}(\pi_e K - \pi_g P_g)] + \mathbb{E}[\chi\{\mathrm{We} + P_g < K\}\chi\{\mathrm{We} > 0\}[\pi_e(\mathrm{We} + P_g) - \pi_g P_g\\ &\qquad - C(K - (\mathrm{We} + P_g))^{\alpha}] + \mathbb{E}[\chi\{\mathrm{We} = 0\}[\pi_e P_g - \pi_g P_g - C(K - P_g)^{\alpha}],\end{aligned}$$

where

$$\begin{aligned}\mathbb{E}[\chi\{\mathrm{We} \geq K - P_g\}\chi\{\mathrm{We} > 0\}\pi_e K] &= K \int_{K-P_g}^{+\infty} \left(\int_0^{+\infty} x f_{(\mathrm{we},\pi_e)}(x, p)\, dx \right) dp,\\ -\mathbb{E}[\chi\{\mathrm{We} \geq K - P_g\}\chi\{\mathrm{We} > 0\}\pi_g P_g] &= -\int_{K-P_g}^{+\infty} \pi_g P_g f_{\mathrm{we}}(x)\, dx = -\pi_g P_g \overline{F}_{\mathrm{we}}(K - P_g),\\ \mathbb{E}[\chi\{\mathrm{We} < K - P_g\}\chi\{\mathrm{We} > 0\}\pi_e(\mathrm{We} + P_g)] &= \int_0^{K-P_g} \int_0^{+\infty} x(p + P_g) f_{(\mathrm{we},\pi_e)}(x, p)\, dx\, dp,\\ -\mathbb{E}[\chi\{\mathrm{We} < K - P_g\}\chi\{\mathrm{We} > 0\}[\pi_g P_g + C(K - (\mathrm{We} + P_g))^{\alpha}] &= -\int_0^{K-P_g} f_{\mathrm{we}}(p)[\pi_g P_g + C(K - (p + P_g))^{\alpha}]\, dp,\\ \mathbb{E}[\chi\{\mathrm{We} = 0\}[\pi_e P_g - \pi_g P_g - C(K - P_g)^{\alpha}] &= a P_g \mathbb{E}[\pi_e | \mathrm{We} = 0] - a[\pi_g P_g + C(K - P_g)^{\alpha}].\end{aligned} \tag{A.1}$$

Now we calculate the first-order derivative of $\mathcal{M}$ with respect to the variable P_g. In order to do this, we need to evaluate the derivatives of all the five addenda in formula (A.1) using Leibnitz's formula for differentiation under an integral sign. Let us start computing the derivative of the first addendum of (A.1). We have

$$\frac{\partial K \int_{K-P_g}^{+\infty} (\int_0^{+\infty} x f_{(\mathrm{we},\pi_e)}(x, p)\, dx)\, dp}{\partial P_g} = K\mathbb{E}[\pi_e | \mathrm{We} = K - P_g] f_{\mathrm{we}}(K - P_g). \tag{A.2}$$

The derivatives of the second addendum is

$$-\frac{\partial}{\partial P_g} \pi_g P_g \overline{F}_{\mathrm{we}}(K - P_g) = -f_{\mathrm{we}}(K - P_g)\pi_g P_g - \overline{F}_{\mathrm{we}}(K - P_g)\pi_g. \tag{A.3}$$

The derivatives of the third addendum is

$$\begin{aligned}&\frac{\partial}{\partial P_g} \int_0^{K-P_g} \int_0^{+\infty} x(p + P_g) f_{(\mathrm{we},\pi_e)}(x, p)\, dx\, dp\\ &\quad = -K f_{\mathrm{we}}(K - P_g)\mathbb{E}[\pi_e \mid \mathrm{We} = K - P_g] + \int_0^{K-P_g} f_{\mathrm{we}}(p)\mathbb{E}[\pi_e \mid \mathrm{We} = p]\, dp.\end{aligned} \tag{A.4}$$

The derivatives of the fourth addendum is

$$\begin{aligned}&-\frac{\partial}{\partial P_g} \int_0^{K-P_g} f_{\mathrm{we}}(p)[\pi_g P_g + C(K - (p + P_g))^{\alpha}]\, dp\\ &= f_{\mathrm{we}}(K - P_g)[\pi_g P_g + C(K - (K - P_g + P_g))^{\alpha}] - \int_0^{K-P_g} f_{\mathrm{we}}(p)[\pi_g - C\alpha(K - (p + P_g))^{\alpha-1}]\, dp\\ &= f_{\mathrm{we}}(K - P_g)\pi_g P_g - \int_0^{K-P_g} f_{\mathrm{we}}(p)[\pi_g - C\alpha(K - (p + P_g))^{\alpha-1}\, dp].\end{aligned} \tag{A.5}$$

Finally, we report the derivative of the fifth addendum:

$$\frac{\partial}{\partial P_g} aP_g\mathbb{E}[\pi_e|\text{We}=0] - a[\pi_g P_g + C(K-P_g)^\alpha] = a\mathbb{E}[\pi_e|\text{We}=0] - a\pi_g + aC\alpha(K-P_g)^{\alpha-1}. \tag{A.6}$$

The summation of formulas (A.2), (A.3), (A.4), (A.5), (A.6) gives

$$\begin{aligned}\frac{\partial\mathcal{M}(P_g)}{\partial P_g} &= K\mathbb{E}[\pi_e|\text{We}=K-P_g]f_{\text{we}}(K-P_g) - f_{\text{we}}(K-P_g)\pi_g P_g - \overline{F}_{\text{we}}(K-P_g)\pi_g \\ &\quad - Kf_{\text{we}}(K-P_g)\mathbb{E}[\pi_e \mid \text{We}=K-P_g] + \int_0^{K-P_g} f_{\text{we}}(p)\mathbb{E}[\pi_e \mid \text{We}=p]\,dp + f_{\text{we}}(K-P_g)\pi_g P_g \\ &\quad - \int_0^{K-P_g} f_{\text{we}}(p)[\pi_g - C\alpha(K-(p+P_g))^{\alpha-1}\,dp] + a\mathbb{E}[\pi_e|\text{We}=0] - a\pi_g + aC\alpha(K-P_g)^{\alpha-1},\end{aligned}$$

and through some algebraic manipulations we have

$$\begin{aligned}\frac{\partial\mathcal{M}(P_g)}{\partial P_g} &= -\pi_g(1+a) + \int_0^{K-P_g} f_{\text{we}}(p)C\alpha[K-(p+P_g)]^{\alpha-1}\,dp + aC\alpha(K-P_g)^{\alpha-1} \\ &\quad + \int_0^{K-P_g} f_{\text{we}}(p)\mathbb{E}[\pi_e|\text{We}=p]\,dp + a\mathbb{E}[\pi_e|\text{We}=0].\end{aligned}$$

To prove the concavity of $\mathcal{M}(P_g)$, we proceed to compute the second-order derivative:

$$\begin{aligned}\frac{\partial^2\mathcal{M}(P_g)}{\partial P_g^2} &= -aC\alpha(\alpha-1)(K-P_g)^{\alpha-2} - \int_0^{K-P_g} f_{\text{we}}(p)(1-a)C\alpha(\alpha-1)(K-(p+P_g))^{\alpha-2}\,dp \\ &\quad - (1-a)f_{\text{we}}(K-P_g)\mathbb{E}[\pi_e|\text{We}=K-P_g].\end{aligned} \tag{A.7}$$

Note that if $\alpha \geq 1$, the right-hand side of (A.7) is negative and the function $\mathcal{M}(P_g)$ is concave. If $0 < \alpha < 1$, we have concavity only if $\frac{\partial^2\mathcal{M}(P_g)}{\partial P_g^2} < 0$, i.e. when

$$\begin{aligned}\mathbb{E}[\pi_e|\text{We}=K-P_g] > \frac{1}{(1-a)f_{\text{we}}(K-P_g)}\Bigg\{&-aC\alpha(\alpha-1)(K-P_g)^{\alpha-2} \\ &- \int_0^{K-P_g} f_{\text{we}}(p)(1-a)C\alpha(\alpha-1)(K-(p+P_g))^{\alpha-2}\,dp\Bigg\}.\end{aligned}$$

The lemma is proved.

A.2 Proof of Proposition 2

This proof is based on the following two lemmas (Lemma 3 and Lemma 4).

Lemma 3. *Let $\phi(a)$ be the probability density function of a standard Normal distribution. Then:*

- *If k is even, we have*

$$\int_{-\infty}^{+\infty} a^k\phi^2(a)\,da = \frac{1}{2\sqrt{\pi}}\prod_{i\in O_k}\frac{k-i}{2}. \tag{A.8}$$

- *If k is odd, we have*

$$\int_{-\infty}^{+\infty} a^k\phi^2(a)\,da = 0. \tag{A.9}$$

Proof. We start to prove formula (A.8). If k is even, we have

$$\int_{-\infty}^{+\infty} a^k \phi^2(a)\, da = \int_{-\infty}^{+\infty} a^k \frac{1}{2\pi} e^{-a^2}\, da = \frac{1}{2\pi} \int_{-\infty}^{+\infty} a^{k-1}\Big(-\frac{1}{2}\Big)(-2ae^{-a^2})\, da.$$

Using integration by parts, we obtain

$$\begin{aligned}\int_{-\infty}^{+\infty} a^k \phi^2(a)\, da &= \Big(-\frac{1}{4\pi}\Big)\Big\{\lim_{a\to+\infty} \frac{a^{k-1}}{e^{a^2}} - \lim_{a\to-\infty} \frac{a^{k-1}}{e^{a^2}} - (k-1)\int_{-\infty}^{+\infty} a^{k-2} 2\pi \frac{1}{2\pi} e^{-a^2}\, da\Big\} \\ &= \Big(-\frac{1}{4\pi}\Big)\Big\{0 - \lim_{b\to+\infty} \frac{(-b)^{k-1}}{e^{b^2}} - 2\pi(k-1)\int_{-\infty}^{+\infty} a^{k-2}\phi^2(a)\, da\Big\}.\end{aligned}$$

Since $\lim_{b\to+\infty} \frac{-(b^{k-1})}{e^{b^2}} = 0$, we have

$$\int_{-\infty}^{+\infty} a^k \phi^2(a)\, da = \Big(-\frac{1}{4\pi}\Big)\Big\{0 - 2\pi(k-1)\int_{-\infty}^{+\infty} a^{k-2}\phi^2(a)\, da\Big\} = \frac{k-1}{2}\int_{-\infty}^{+\infty} a^{k-2}\phi^2(a)\, da. \tag{A.10}$$

This relation is of recursive type and from [14] we have that for $k = 2$,

$$\int_{-\infty}^{+\infty} a^k \phi^2(a)\, da = \int_{-\infty}^{+\infty} a^2 \phi^2(a)\, da = \frac{1}{4\sqrt{\pi}}.$$

Then according to relation (A.10) for $k = 4$ we get

$$\int_{-\infty}^{+\infty} a^4 \phi^2(a)\, da = \frac{3}{2}\cdot\frac{1}{4\sqrt{\pi}},$$

and in general

$$\int_{-\infty}^{+\infty} a^k \phi^2(a)\, da = \frac{1}{2\sqrt{\pi}} \prod_{i\in O_k} \frac{k-i}{2},$$

where $O_k = \{i \in \mathbb{N} : i \le k \text{ with } i \text{ odd}\}$.

Let us now prove formula (A.9). If k is odd, we have

$$\int_{-\infty}^{+\infty} a^k \phi^2(a)\, da = \int_{-\infty}^{+\infty} a^k \frac{1}{2\pi} e^{-a^2}\, da, \tag{A.11}$$

and $F(a) = a^k \frac{1}{2\pi} e^{-a^2}$ is an odd function. Thus

$$F(-a) = -a^k \frac{1}{2\pi} e^{-a^2} = -F(a)$$

and for this reason $\int_{-\infty}^{+\infty} F(a)\, da = 0$. □

Lemma 4. *Let $\phi(a)$ and $\Phi(a)$ be the probability density function and the distribution function of a standard Normal distribution, respectively. Then:*

- *If k is even, we have*

$$I_k^P := \int_{-\infty}^{+\infty} a^k \phi(a)\Phi(a)\, da = I_{k-2}^P (k-1) \quad \text{and} \quad I_0^P = 0.5. \tag{A.12}$$

- *If k is odd, we have*

$$I_k^D := \int_{-\infty}^{+\infty} a^k \phi(a)\Phi(a)\, da = (k-1) I_{k-2}^D + \frac{1}{2\sqrt{\pi}} \prod_{i\in O_k} \frac{k-i}{2} = \frac{1}{2\sqrt{\pi}} \sum_{j=1}^{\frac{k+1}{2}} t_{k,j}, \tag{A.13}$$

  and the numbers $t_{k,j}$ are those defined in Proposition 2.

Proof. Consider formula (A.12):

$$\int_{-\infty}^{+\infty} a^k \phi(a)\Phi(a)\,da = \int_{-\infty}^{+\infty} a\phi(a)\Phi(a)a^{k-1}\,da.$$

Set $u'(a) = a\phi(a)$ and $v(a) = \Phi(a)a^{k-1}$. Then $u(a) = -\phi(a)$; indeed,

$$D(-\phi(a)) = D\left(\frac{-1}{\sqrt{2\pi}}e^{-0.5a^2}\right) = a\phi(a).$$

Moreover, $v'(a) = (k-1)a^{k-2}\Phi(a) + a^{k-1}\phi(a)$ and using integrations by parts, we obtain

$$\int_{-\infty}^{+\infty} a^k\phi(a)\Phi(a)\,da = [-\phi(a)a^{k-1}\Phi(a)]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} -\phi(a)[(k-1)a^{k-2}\Phi(a) + a^{k-1}\phi(a)]\,da. \tag{A.14}$$

Note that

$$\lim_{a\to+\infty} -\phi(a)a^{k-1}\Phi(a) = 0$$

because

$$\lim_{a\to+\infty}\frac{a^{k-1}\Phi(a)}{e^{0.5a^2}} \le \lim_{a\to+\infty}\frac{a^{k-1}}{e^{0.5a^2}} = 0.$$

Moreover, for $a > 0$,

$$\frac{a^{k-1}\Phi(a)}{e^{0.5a^2}} \ge 0$$

and

$$\lim_{a\to+\infty}\frac{a^{k-1}\Phi(a)}{e^{0.5a^2}} = 0.$$

Note that changing the variable $x = -a$ we get

$$\begin{aligned}\lim_{a\to-\infty}\frac{a\Phi(a)}{e^{0.5a^2}} &= \lim_{x\to+\infty}\frac{-x\Phi(-x)}{e^{0.5x^2}}\\ &= -\lim_{x\to+\infty}\frac{x\Phi(-x)}{e^{0.5x^2}}\\ &\le \lim_{x\to+\infty}\frac{x}{e^{0.5x^2}} = 0\end{aligned}$$

and the observing that $\frac{x\Phi(-x)}{e^{0.5x^2}} \ge 0$ implies

$$\lim_{x\to+\infty}\frac{x\Phi(-x)}{e^{0.5x^2}} = 0.$$

Consequently, formula (A.14) becomes

$$(k-1)\int_{-\infty}^{+\infty}\phi(a)a^{k-2}\Phi(a)\,da + \int_{-\infty}^{+\infty}\phi(a)^2a^{k-1}\,da.$$

Being k even, $(k-2)$ is even and $(k-1)$ is odd so using the result of Lemma 3, we have

$$\int_{-\infty}^{+\infty}\phi(a)a^k\Phi(a)\,da = (k-1)\int_{-\infty}^{+\infty}a^{k-2}\phi(a)\Phi(a)\,da.$$

If $k = 2$, then

$$\int_{-\infty}^{+\infty}\phi(a)(k-1)a^2\Phi(a)\,da = (2-1)\int_{-\infty}^{+\infty}\phi(a)\Phi(a)\,da = \frac{1}{2}.$$

If $k = 4$, we have

$$\int_{-\infty}^{+\infty}\phi(a)a^4\Phi(a)\,da = (3)\int_{-\infty}^{+\infty}\phi(a)\Phi(a)a^2\,da = \frac{3}{2}.$$

In general, if k is even, then

$$I_k^P = \int_{-\infty}^{+\infty} \phi(a)a^k\Phi(a)\,da = \frac{1}{2}\prod_{i\in O_k} i,$$

or recursively

$$I_k^P = I_{k-2}^P(k-1).$$

Now let us consider formula (A.13):

$$\int_{-\infty}^{+\infty} a^k\phi(a)\Phi(a)\,da = \int_{-\infty}^{+\infty} a\phi(a)\Phi(a)a^{k-1}\,da. \tag{A.15}$$

By using integrations by parts, integral (A.15) becomes

$$\int_{-\infty}^{+\infty} (k-1)\phi(a)\Phi(a)a^{k-2}\,da + \int_{-\infty}^{+\infty} \phi^2(a)a^{k-1}\,da.$$

Being k odd, $(k-1)$ is even and $(k-2)$ is odd so using formula (A.8), we have

$$(k-1)\int_{-\infty}^{+\infty} \phi(a)\Phi(a)a^{k-2}\,da + \frac{1}{2\sqrt{\pi}}\prod_{i\in O_{k-1}}\frac{(k-1-i)}{2}.$$

In general, we have the following recursive relation:

$$\begin{aligned} I_k^D &= (k-1)I_{k-2}^D + \int_{-\infty}^{+\infty} a^{k-1}\phi^2(a)\,da \\ &= (k-1)I_{k-2}^D + \frac{1}{2\sqrt{\pi}}\prod_{i\in O_k}\frac{(k-i)}{2}. \end{aligned}$$

The lemma is proved. □

From a practical point of view it can be interesting to note that coefficients $t_{i,j}$ can be obtained using the triangular array described in Table 4. In the first column of the table called “Row index” there are the rows from 1 to m and in the second column of the table there are coefficients $t_{i,j}$ where i refers to the row and j refers to the column. For example the element $t_{1,1}$ is the first element of the fist row and $t_{3,2}$ is the second element of the third row and so on. In particular, the first elements of each row are obtained with the following formula: $\frac{(m-2)\cdots 1}{2^{(m-1)/2}}$ so the element $t_{3,1}$ comes from $(3-2)\frac{1}{2}$, the element $t_{5,1}$ comes from $(5-2)\frac{1}{2^2}$, etc., until the m-row. Indeed, the first column’s coefficients $t_{i,1}$ are the result of the multiplication of $(m-2)$ and the element of the previous row with the same position, i.e. $t_{(i-1),1}$. As regards all of the other coefficients, they are obtained multiplying $(m-1)$ by the element in the north-west corner. For example, applying this rule we have that the coefficient $t_{11,3}$ comes from $\frac{(11-1)(6)(5)(3)(1)}{(2)(2)(2)}$, where $\frac{(6)(5)(3)(1)}{(2)(2)(2)}$ is $t_{9,2}$; $t_{11,4}$ is the result of the product of $(m-1)$ by $t_{10,3}$, and so on. Finally, to obtain T_k is necessary adding the rows of this triangular array.

Row Index	$t_{i,j}$						
1	1						
3	$\frac{1}{2}$	2					
5	$\frac{3}{4}$	2	8				
7	$\frac{15}{8}$	$\frac{9}{2}$	12	48			
9	$\frac{105}{16}$	15	36	96	384		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
m	$\frac{(m-2)\cdots 1}{2^{(m-1)/2}}$	…	…	…	…	…	…

Table 4: The triangular array shows coefficients $t_{i,j}$.

Proof of Proposition 2. We have

$$\mathbb{E}[\pi_e F_{\pi_e}(\pi_e)] = \int_0^{+\infty} x \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-\frac{(\log(x)-\mu)^2}{2\sigma^2}} \Phi \frac{(\log(x)-\mu)}{\sigma}\, dx.$$

By setting $t = \frac{(\log(x)-\mu)}{\sigma}$, we obtain

$$\mathbb{E}[\pi_e F_{\pi_e}(\pi_e)] = \int_{-\infty}^{+\infty} e^{t\sigma} e^{\mu} \frac{1}{\sqrt{2\pi}} \exp\Big(\frac{-t^2}{2}\Big)\Phi(t)\, dt = e^{\mu} \int_{-\infty}^{+\infty} e^{t\sigma}\phi(t)\Phi(t)\, dt,$$

but $e^{t\sigma} = \sum_{k=0}^{+\infty} \frac{(t\sigma)^k}{k!}$ so we have

$$\mathbb{E}[\pi_e F_{\pi_e}(\pi_e)] = e^{\mu} \sum_{k=0}^{+\infty} \frac{(\sigma)^k}{k!} \int_{-\infty}^{+\infty} t^k \phi(t)\Phi(t)\, dt.$$

Using Lemma 3 and Lemma 4, we have

$$\mathbb{E}[\pi_e F_{\pi_e}(\pi_e)] = e^{\mu}\Big\{\sum_{k\in E} \frac{\sigma^k}{k!} I_k^P + \sum_{k\in O} \frac{\sigma^k}{k!} I_k^D\Big\}. \qquad \square$$

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