UNIT-LINKED LIFE INSURANCE POLICIES: OPTIMAL HEDGING IN PARTIALLY OBSERVABLE MARKET MODELS

CLAUDIA CECI, KATIA COLANERI, AND ALESSANDRA CRETAROLA

Abstract. In this paper we investigate the hedging problem of a unit-linked life insurance contract via the local risk-minimization approach, when the insurer has a restricted information on the market. In particular, we consider an endowment insurance contract, that is a combination of a term insurance policy and a pure endowment, whose final value depends on the trend of a stock market where the premia the policyholder pays are invested. To allow for mutual dependence between the financial and the insurance markets, we use the progressive enlargement of filtration approach. We assume that the stock price process dynamics depends on an exogenous unobservable stochastic factor that also influences the mortality rate of the policyholder. We characterize the optimal hedging strategy in terms of the integrand in the Galtchouk-Kunita-Watanabe decomposition of the insurance claim with respect to the minimal martingale measure and the available information flow. We provide an explicit formula by means of predictable projection of the corresponding hedging strategy under full information with respect to the natural filtration of the risky asset price and the minimal martingale measure. Finally, we discuss applications in a Markovian setting via filtering.

Keywords: Unit-linked life insurance contract; progressive enlargement of filtration; partial information; local risk-minimization.

JEL Classification: C02; G11; G22.

AMS Classification: 91B30; 60G35; 60G40; 60J60.

1. Introduction

Unit-linked life insurance contracts are life insurance policies whose benefits depend on the performance of a certain stock or a portfolio traded in the financial market. For the last years these contracts have experienced a clamorous success, driven by low interest rates, which have considerably reduced the returns of the classic management, and the new Solvency II rules on the insurance regulatory capital, which made the unit-linked much more affordable for the companies, in terms of lower absorption of capital. According to Gantenbein and Mata [29, Chapter 10], the

Claudia Ceci (✉), Department of Economics, University "G. D'Annunzio" of Chieti-Pescara, Viale Pindaro, 42, I-65127 Pescara, Italy.

Katia Colaneri, Department of Economics, University of Perugia, Via Alessandro Pascoli, 20, I-06123 Perugia, Italy.

Alessandra Cretarola, Department of Mathematics and Computer Science, University of Perugia, Via Luigi Vanvitelli, 1, I-06123 Perugia, Italy.

E-mail addresses: c.ceci@unich.it, katia.colaneri@unipg.it, alessandra.cretarola@unipg.it.

unit-linked life insurance policy is "basically a mixed life insurance that combines term coverage with a saving and an investment component". Unlike the traditional mixed life insurance, in these contracts premia are invested by the insurance company in the financial market on behalf of the policyholder who decides how to invest the capital. Among these products, we may distinguish at least three different kinds of policies based on the payoff structure:

- *pure endowment* contract that promises to pay an agreed amount if the policyholder is still alive on a specified future date;
- *term insurance* contract that pays the benefit if the policyholder dies before the policy term;
- *endowment insurance* contract which is a combination of the above contracts and guarantees that benefits will be paid by the insurance company, either at the policy term or after the insured death.

The goal of this paper is to find an optimal hedging strategy for a given endowment insurance contract in a general intensity-based model where the mortality intensity, as well as the drift in the risky asset price dynamics affecting the benefits for the policyholder, is *not observable* by the insurance company, and *mutual dependence* between the stock price trend and the insurance portfolio is allowed. To the best of our knowledge, this is the first time that the problem of hedging a unit-linked life insurance policy is studied under partial information, without assuming independence between the financial and the insurance markets.

Precisely, we propose a suitable combined financial-insurance market model, where the financial market consists of a riskless asset, whose discounted price is equal to 1, and a risky asset, with discounted price process denoted by S. The price process S is represented by a geometric diffusion, whose drift depends on an exogenous unobservable stochastic factor X, correlated with S. The insurance company issues an endowment insurance contract with maturity of T years for an individual whose remaining lifetime is represented by a random time τ.

Modeling the death time of an individual is a fundamental issue to be addressed when dealing with insurance problems. Here, we propose a modeling framework for life insurance liabilities that is also well suited to describe defaultable claims, as the time of death can be handled in a similar manner to the default time of a firm. Then, we take the analogies between mortality and credit risk into account and follow the intensity-based approach of reduced-form methodology, see e.g. Bielecki and Rutkowski [7] and references therein. The death time τ is described by a nonnegative random variable, which is not necessarily a stopping time with respect to the initial filtration $\mathbb{F}$ generated by the underlying Brownian motions driving the dynamics of the pair (S, X). As mentioned above, we do not assume independence between the random time of death and the financial market, and characterize our setting via the progressive enlargement of filtration approach, see the seminal works by Jeulin and Yor [32], Jeulin [31], Jeulin and Yor [33]. This technique is widely applied to reduced-form models for credit risk, as in Bielecki et al. [8, 9, 11], Elliott et al. [25], Kusuoka [35]. Moreover, applications to insurance problems can be found in Biagini et al. [6], Barbarin [2], Choulli et al. [21], Li and Szimayer [37] in a complete information setting.

Here, we consider an enlargement of the filtration $\mathbb{F}$ to make τ a stopping time and we denote it by $\mathbb{G}$. The available information to the insurance company is represented by a subfiltration $\widetilde{\mathbb{G}}$ of $\mathbb{G}$, which contains the natural filtration of S and ensures that τ is still a stopping time. This means that, at any time t, the insurer may observe the risky asset price and knows if the policyholder is still alive or not.

The endowment insurance contract can be treated as a contingent claim in the hybrid market model given by the financial securities and the insurance portfolio, and the objective is to study the hedging problem for the insurance company. Analogously to Bielecki et al. [10], Biagini and Cretarola [4], we assume that hedging stops after the earlier between the policyholder death τ and the maturity T: this allows to work with stopped price processes and guarantees that the stopped Brownian motions, that drive the financial market, are also Brownian motions with respect to the enlarged filtration. As a consequence, we do not need to assume the *martingale invariance property*, also known as *H-hypothesis*, see e.g. Bielecki and Rutkowski [7]. Since the underlying market is incomplete due to the mortality risk and the presence of the unobservable stochastic factor X, it is necessary to select one of the techniques for pricing and hedging in incomplete markets. Then, we choose, among the quadratic hedging methods, the *local risk-minimization* approach (see e.g. Schweizer [48] for further details). The idea of this technique is to find an optimal hedging strategy that perfectly replicates the given contingent claim with minimal cost, within a wide class of admissible strategies that in general might not necessarily be self-financing. Locally risk-minimizing hedging strategies can be characterized via the Föllmer-Schweizer decomposition of the random variable representing the payoffs of the given contingent claim, see e.g. Schweizer [47, 48] for the full information case and Ceci et al. [16, 19] under incomplete information. This quadratic hedging approach has been successfully applied to the hedging problem of insurance products, see e.g. Biagini et al. [6, 5], Choulli et al. [21], Dahl and Møller [22], Møller [39, 40], Vandaele and Vanmaele [51] for the complete information case and Ceci et al. [18] under partial information.

In this paper, we introduce the stopped Föllmer-Schweizer decomposition under partial information and in Proposition 4.10 we characterize the optimal hedging strategy in terms of the integrand in this decomposition. In this sense, we extend Biagini and Cretarola [4, Proposition 3.7] to the partial information framework. We also introduce the corresponding Galtchouk-Kunita-Watanabe decomposition with respect to the minimal martingale measure. In Theorem 4.16, we provide equivalence of these decompositions and, using the result stated in Proposition 4.15, the relation between the optimal hedging strategy under partial information and that under full information via predictable projections. In the case where the mortality intensity has a Markovian dependence on the unobservable stochastic factor X, we can compute the optimal hedging strategy in a more explicit form by means of filtering problems. Pricing and hedging problems for contingent claims under incomplete information using filtering techniques have been studied in credit risk context, in Frey and Runggaldier [27], Frey and Schmidt [28], Tardelli [50] and in the insurance framework in Ceci et al. [18] under the hypothesis of independence between the financial and the insurance markets.

The paper is organized as follows. In Section 2 we introduce the combined financial-insurance market model in a partial information scenario via progressive enlargement of filtrations. The semimartingale decompositions of the stopped risky asset price process with respect to the enlarged filtrations $\mathbb{G}$ and $\widetilde{\mathbb{G}}$ respectively, can be found in Section 3. In Section 4 we provide a closed formula for the locally risk-minimizing hedging strategy under incomplete information for the given endowment insurance contract by means of predictable projections. Finally, in Section 5 we discuss the problem in a Markovian framework, where the mortality intensity depends on the unobservable stochastic factor and apply the filtering approach to compute the optimal hedging strategy. In addition, we address the issue of the hazard process and the martingale hazard process of τ under restricted information in Appendix A. Some technical results on the optional and predictable projections under partial information and certain proofs can be found in Appendix B.

2. The setting

We consider the problem of an insurance company that wishes to hedge a unit-linked life insurance contract. The value of the policy depends on the performance of the underlying stock or portfolio traded on the financial market as well as the remaining lifetime of the policyholder. Therefore, the insurer is exposed to both financial and mortality risks. The nature of the problem suggests to construct a combined financial-insurance market model and treat the life insurance policy as a contingent claim. We will define the suitable modeling framework via the progressive enlargement of filtration approach, which allows for possible dependence between the financial market and the insurance portfolio. As first step, we introduce the underlying financial market model.

2.1. **The financial market model.** Let $W = \{W_t,\ t \in [0,T]\}$ and $B = \{B_t,\ t \in [0,T]\}$, with $W_0 = B_0 = 0$, be two independent one dimensional Brownian motions on the complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, with T denoting a fixed and finite time horizon. We define the filtration $\mathbb{F} = \{\mathcal{F}_t,\ t \in [0,T]\}$, by

$$\mathbb{F} = \mathbb{F}^W \vee \mathbb{F}^B,$$

where $\mathbb{F}^W$ and $\mathbb{F}^B$ denote the natural filtrations of the processes W and B, respectively. In addition, we assume that $\mathbb{F}$ satisfies the usual hypotheses of completeness and right continuity.

We consider a simple financial market which consists of one riskless asset whose price process is assumed to be equal to 1 at any time, and one risky asset whose (discounted) price process $S = \{S_t,\ t \in [0,T]\}$ evolves according to the following stochastic differential equation

$$\mathrm{d}S_t = S_t\left(\mu(t, S_t, X_t)\mathrm{d}t + \sigma(t, S_t)\mathrm{d}W_t\right), \quad S_0 = s_0 \in \mathbb{R}^+, \tag{2.1}$$

where $X = \{X_t,\ t \in [0,T]\}$ is an unobservable exogenous stochastic factor satisfying

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + a(t, X_t)\left[\rho \mathrm{d}W_t + \sqrt{1-\rho^2}\mathrm{d}B_t\right], \quad X_0 = x_0 \in \mathbb{R}, \tag{2.2}$$

with $\rho \in [0,1]$. Here, μ, b are $\mathbb{R}$-valued measurable functions and σ, a are $\mathbb{R}^+$-valued measurable functions such that the system of equations (2.1) and (2.2) admits a unique strong solution, see for instance Øksendal [42, Chapter 5]. This also implies that the pair (S, X) is an $(\mathbb{F}, \mathbf{P})$-Markov process.

We assume that the following conditions are in force throughout the paper:

Assumption 2.1.
(i) $\mathbb{E}\left[\int_0^T \big(|\mu(u, S_u, X_u)| + \sigma^2(u, S_u)\big)\,\mathrm{d}u\right] < \infty$;
(ii) $\left|\dfrac{\mu(t, S_t, X_t)}{\sigma(t, S_t)}\right| \leq c$, *$\mathbf{P}$-a.s., for every $t \in [0,T]$, with c being a positive constant.*

Under Assumption 2.1 the set of all equivalent martingale measures for S is non-empty and contains more than a single element, since X does not represent the price of any tradeable asset, and therefore the financial market is incomplete.

Precisely, every equivalent probability measure $\mathbf{Q}$ has the following density $L^{\mathbf{Q}} = \{L_t^{\mathbf{Q}},\ t \in [0,T]\}$, given by

$$L_t^{\mathbf{Q}} := \left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_t} = \mathcal{E}\left(\int_0^{\cdot} -\frac{\mu(u, S_u, X_u)}{\sigma(u, S_u)}\mathrm{d}W_u + \int_0^{\cdot} \psi_u^{\mathbf{Q}}\mathrm{d}B_u\right)_t, \quad t \in [0,T],$$

where $\psi^{\mathbf{Q}} = \{\psi_t^{\mathbf{Q}},\ t \in [0,T]\}$ is an $\mathbb{F}$-predictable process such that $L^{\mathbf{Q}}$ turns out to be an $(\mathbb{F}, \mathbf{P})$-martingale. Here $\mathcal{E}(Y)$ denotes the Doléans-Dade exponential of an $(\mathbb{F}, \mathbf{P})$-semimartingale Y. The choice $\psi_t^{\mathbf{Q}} = 0$, for every $t \in [0,T]$, corresponds to the so-called *minimal martingale measure* for S (see e.g. Föllmer and Schweizer [26]), denoted by $\widehat{\mathbf{P}}$, whose density process $L = \{L_t,\ t \in [0,T]\}$, is defined by

$$L_t := \left.\frac{\mathrm{d}\widehat{\mathbf{P}}}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_t} = \mathcal{E}\left(\int_0^{\cdot} -\frac{\mu(u, S_u, X_u)}{\sigma(u, S_u)}\mathrm{d}W_u\right)_t, \quad t \in [0,T]. \tag{2.3}$$

Condition (ii) of Assumption 2.1 implies that L is a square integrable $(\mathbb{F}, \mathbf{P})$-martingale. As a consequence of the Girsanov Theorem, we get that the process $\widehat{W} = \{\widehat{W}_t,\ t \in [0,T]\}$, given by

$$\widehat{W}_t := W_t + \int_0^t \frac{\mu(u, S_u, X_u)}{\sigma(u, S_u)}\mathrm{d}u, \quad t \in [0,T], \tag{2.4}$$

is an $(\mathbb{F}, \widehat{\mathbf{P}})$-Brownian motion and S is an $(\mathbb{F}, \widehat{\mathbf{P}})$-local martingale, since it is the Doléans-Dade exponential of the $(\mathbb{F}, \widehat{\mathbf{P}})$-local martingale $\left\{\int_0^t \sigma(u, S_u)\mathrm{d}\widehat{W}_u,\ t \in [0,T]\right\}$.

Note that, since X is not directly observable, the available information on the financial market for the insurance company is brought by the natural filtration of the risky asset price process S, that is, $\mathbb{F}^S = \{\mathcal{F}_t^S,\ t \in [0,T]\}$, with $\mathcal{F}_t^S := \sigma\{S_u,\ 0 \leq u \leq t\}$, for each $t \in [0,T]$.

2.2. **The combined financial-insurance market model.** Now, we extend the financial market model by also including an individual to be insured. Let τ be the remaining lifetime of an individual with age a. Here, τ is a nonnegative random variable $\tau : \Omega \to [0, T] \cup \{+\infty\}$ satisfying $\mathbf{P}(\tau = 0) = 0$ and $\mathbf{P}(\tau > t) > 0$, for every $t \in [0, T]$. Since, we only consider a single policyholder we omit the dependence on the age.

Then, we define the associated death indicator process as $H = \{H_t,\ t \in [0, T]\}$, where

$$H_t = \mathbf{1}_{\{\tau \le t\}}, \quad t \in [0, T], \tag{2.5}$$

and $\mathbb{F}^H = \{\mathcal{F}_t^H,\ t \in [0, T]\}$ denotes the natural filtration of H. Notice that τ is a stopping time with respect to the filtration $\mathbb{F}^H$, but it is not necessarily a stopping time with respect to the filtration $\mathbb{F}$.

Let $\mathbb{G} = \{\mathcal{G}_t,\ t \in [0, T]\}$ be the enlarged filtration given by

$$\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{F}_t^H, \quad t \in [0, T].$$

This is the smallest filtration which contains $\mathbb{F}$, such that τ is a $\mathbb{G}$-stopping time. In this framework the initial market might be correlated with the time of death τ. The connection between the financial market and τ is expressed in terms of the conditional distribution of τ given $\mathcal{F}_t$, for every $t \in [0, T]$, defined as the process $F = \{F_t,\ t \in [0, T]\}$ given by

$$F_t = \mathbf{P}(\tau \le t | \mathcal{F}_t) = \mathbb{E}\left[H_t | \mathcal{F}_t\right], \quad t \in [0, T]. \tag{2.6}$$

Notice that, $0 \le F_t \le 1$ for every $t \in [0, T]$. In the sequel, we will assume that $F_t < 1$ for every $t \in [0, T]$; this excludes the case where τ is an $\mathbb{F}$-stopping time, see e.g. Bielecki and Rutkowski [7] for further details.

In the following we define the so-called hazard process of the random time τ.

Definition 2.2. *The* $\mathbb{F}$-hazard process *of* τ *under* $\mathbf{P}$ *is the nonnegative process* $\Gamma = \{\Gamma_t,\ t \in [0, T]\}$ *defined by*

$$\Gamma_t = -\ln(1 - F_t), \quad t \in [0, T]. \tag{2.7}$$

In this paper we assume that Γ has a density, i.e. $\Gamma_t = \int_0^t \gamma_u \mathrm{d}u$, for every $t \in [0, T]$, for some nonnegative $\mathbb{F}$- predictable process $\gamma = \{\gamma_t,\ t \in [0, T]\}$ such that $\mathbb{E}\left[\int_0^T \gamma_u \mathrm{d}u\right] < \infty$. The process γ is known as the $\mathbb{F}$-*mortality intensity* or the $\mathbb{F}$-*mortality rate* and the $\mathbb{F}$-*survival process* is given by $\mathbf{P}(\tau > t | \mathcal{F}_t) = e^{-\int_0^t \gamma_u \mathrm{d}u}$, $t \in [0, T]$.

Now, we introduce the $(\mathbb{F}, \mathbb{G})$-martingale hazard process associated with τ.

Definition 2.3. *An $\mathbb{F}$-predictable, right-continuous, increasing process $\Lambda = \{\Lambda_t,\ t \in [0, T]\}$ is called an* $(\mathbb{F}, \mathbb{G})$-martingale hazard process *of the random time τ if and only if the process*

$$M_t = H_t - \Lambda_{t \wedge \tau}, \quad t \in [0, T],$$

is a $(\mathbb{G}, \mathbf{P})$-martingale.

Remark 2.4. *It is well known that in general, the $\mathbb{F}$-hazard process and the $(\mathbb{F}, \mathbb{G})$-martingale hazard process do not coincide. Nevertheless, the existence of the $\mathbb{F}$-mortality intensity ensures that the process F is continuous and increasing. Then, by Bielecki and Rutkowski [7, Proposition 6.2.1] we get that Γ is also an $(\mathbb{F}, \mathbb{G})$-martingale hazard process, and consequently, the process $M = \{M_t,\ t \in [0,T]\}$ defined by*

$$M_t = H_t - \Gamma_{t\wedge\tau} = H_t - \int_0^{t\wedge\tau} \gamma_u \mathrm{d}u = H_t - \int_0^t \lambda_u \mathrm{d}u, \quad t \in [0,T], \tag{2.8}$$

where $\lambda_t = \gamma_t \mathbf{1}_{\{\tau \geq t\}} = \gamma_t(1 - H_{t^-})$, is a $(\mathbb{G}, \mathbf{P})$-martingale. Furthermore, by Dellacherie and Meyer [23, Chapter 6.78], τ is a totally inaccessible $\mathbb{G}$-stopping time.

We assume that the insurance company issues a unit-linked life insurance policy. Precisely, we consider an endowment insurance contract with maturity of T years which can be defined as follows.

Definition 2.5. *An endowment insurance contract is characterized by a a triplet (ξ, Z, τ), where*

- *the random variable $\xi \in L^2(\mathcal{F}_T^S, \mathbf{P})$[1] is the amount paid at maturity T, if the policyholder is still alive at time T;*
- *the process $Z = \{Z_t,\ t \in [0,T]\}$ represents the amount which is immediately paid at death-time τ before maturity T; here, Z is assumed to be square integrable and $\mathbb{F}^S$-predictable;*
- *the random variable τ is the time of death.*

Remark 2.6. *If $Z = 0$ the endowment insurance contract reduces to the so-called* pure endowment contract*, which pays out the amount ξ in case of survival until T, whereas, if $\xi = 0$ we obtain the payoff of a* term insurance contract*, that provides the amount Z_τ at the random time τ in case of death before time T.*

We denote by $N = \{N_t,\ t \in [0,T]\}$ the process that models the payment stream arising from the endowment insurance contract, i.e.

$$N_t = Z_\tau \mathbf{1}_{\{\tau \leq t\}} = \int_0^t Z_s \mathrm{d}H_s, \quad 0 \leq t < T, \quad \text{and} \quad N_T = \xi \mathbf{1}_{\{\tau > T\}}, \quad t = T. \tag{2.9}$$

2.3. **The information levels.** We consider a scenario where the insurance company does not have a complete information on the market. Precisely, we assume that it can observe neither the stochastic factor X affecting the behavior of the risky asset price process S nor the Brownian motions W and B which drive the dynamics of the pair (S, X). In particular, this implies that the insurer does not know completely the $\mathbb{F}$-mortality rate γ of τ. For instance, γ may be dependent on the unobservable stochastic factor X, that is $\gamma_t = \gamma(t, X_t)$, for each $t \in [0,T]$, with γ being a nonnegative measurable function. This special case will be discussed in Section 5. At any time t,

[1]For any σ-algebra $\mathcal{H}$, the set $L^2(\mathcal{H}, \mathbf{P})$ denotes the space of all $\mathcal{H}$-measurable random variables ζ such that $\mathbb{E}\left[|\zeta|^2\right] < \infty$.

the insurer may observe the risky asset price and knows if the policyholder died or not. Hence, the available information is described by the filtration $\widetilde{\mathbb{G}} = \{\widetilde{\mathcal{G}}_t,\ t \in [0,T]\}$, given by

$$\widetilde{\mathcal{G}}_t := \mathcal{F}_t^S \vee \mathcal{F}_t^H, \quad t \in [0,T].$$

Since $\mathbb{F}^S \subseteq \mathbb{F}$, we have

$$\widetilde{\mathbb{G}} \subseteq \mathbb{G}.$$

We assume throughout the paper that all filtrations satisfy the usual hypotheses of completeness and right-continuity. Some results about the hazard process and the martingale hazard process of τ under partial information can be found in Appendix A.

In the sequel we will address the hedging problem of the endowment insurance contract (ξ, Z, τ) in a partial information setting characterized by the information flow $\widetilde{\mathbb{G}}$. Since hedging stops either at time T or τ, whichever comes first, it makes sense to consider the stopped discounted price process. This also implies that we can work without assuming the so-called martingale invariance property between filtrations $\mathbb{F}$ and $\mathbb{G}$, which establishes that every $\mathbb{F}$-martingale is also a $\mathbb{G}$-martingale. The martingale invariance property is frequently assumed when considering enlargement of filtrations. To the best of our knowledge there are only a few papers in the literature where this hypothesis is not imposed, see for instance Barbarin [3], Choulli et al. [21] in the insurance framework and Biagini and Cretarola [4] in the credit risk setting.

3. The semimartingale decompositions of the stopped risky asset price process

In this section we provide the semimartingale decomposition of the stopped price process $S^\tau = \{S_{t\wedge\tau},\ t \in [0,T]\}$ with respect to the information flows $\mathbb{G}$ and $\widetilde{\mathbb{G}}$ respectively, and we show that, under suitable conditions, S^τ satisfies the so-called *structure condition* with respect to both $\mathbb{G}$ and $\widetilde{\mathbb{G}}$ on the stochastic interval $[\![0, \tau \wedge T]\!]$, see e.g. Schweizer [46, Section 1, page 1540] for further details.

The structure condition of the stopped price process is a relevant tool for the computation of the minimal martingale measure and the orthogonal decompositions that allow to characterize the optimal hedging strategy under full and partial information. Moreover, the semimartingale decomposition of S^τ with respect to the information flow $\widetilde{\mathbb{G}}$ allows to reduce the hedging problem under partial information to a full information problem where all involved processes are $\widetilde{\mathbb{G}}$-adapted.

Remark 3.1. *Recall that if the process F given in* (2.6) *is increasing, for any given $\mathbb{F}$-predictable $(\mathbb{F}, \mathbf{P})$-martingale, $m = \{m_t,\ t \in [0,T]\}$, the stopped process $m^\tau = \{m_{t\wedge\tau},\ t \in [0,T]\}$ is a $(\mathbb{G}, \mathbf{P})$-martingale, see Bielecki and Rutkowski [7, Lemma 5.1.6].*

Since F is increasing in our setting, both processes $W^\tau = \{W_{t\wedge\tau},\ t \in [0,T]\}$ and $B^\tau = \{B_{t\wedge\tau},\ t \in [0,T]\}$ are $(\mathbb{G}, \mathbf{P})$-martingales.

Moreover, by Lévy's Theorem we also obtain that W^τ and B^τ are $(\mathbb{G}, \mathbf{P})$-Brownian motions on $[\![0, \tau \wedge T]\!]$ and, as a consequence, the integral processes $\left\{\int_0^t \varphi_s \mathrm{d}W_s^\tau,\ t \in [0,T]\right\}$ and

$\left\{\int_0^t \varphi_s \mathrm{d}B^\tau_s,\ t \in [0,T]\right\}$ *are* $(\mathbb{G}, \mathbf{P})$*-(local) martingales for any* $\mathbb{G}$*-predictable process* $\varphi = \{\varphi_t,\ t \in [0,T]\}$.

By Remark 3.1, we get that the stopped price process S^τ is a $(\mathbb{G}, \mathbf{P})$-semimartingale, decomposable as the sum of a locally square integrable $(\mathbb{G}, \mathbf{P})$-local martingale and a $(\mathbb{G}, \mathbf{P})$-predictable process of finite variation, both null at zero, i.e.

$$S^\tau_t = s_0 + \int_0^{t\wedge\tau} S^\tau_u \mu(u, S^\tau_u, X^\tau_u)\mathrm{d}u + \int_0^{t\wedge\tau} S^\tau_u \sigma(u, S^\tau_u)\mathrm{d}W^\tau_u, \quad t \in [0,T],$$

where

$$X^\tau_t = x_0 + \int_0^{t\wedge\tau} b(u, X^\tau_u)\mathrm{d}u + \int_0^{t\wedge\tau} a(u, X^\tau_u)\left[\rho \mathrm{d}W^\tau_u + \sqrt{1-\rho^2}\mathrm{d}B^\tau_u\right], \quad t \in [0,T].$$

Since S^τ is $\widetilde{\mathbb{G}}$-adapted, then it also admits a semimartingale decomposition with respect to the information flow $\widetilde{\mathbb{G}}$, which will be computed below by means of the (stopped) innovation process I^τ, defined in equation (3.1).

Given any subfiltration $\mathbb{H} = \{\mathcal{H}_t,\ t \in [0,T]\}$ of $\mathbb{G}$, we will use the notation ${}^{o,\mathbb{H}}Y$ (respectively ${}^{p,\mathbb{H}}Y$) to indicate the optional (respectively predictable) projection of a given $\mathbf{P}$-integrable, $\mathbb{G}$-adapted process Y with respect to $\mathbb{H}$ and $\mathbf{P}$, defined as the unique $\mathbb{H}$-optional (respectively $\mathbb{H}$-predictable) process such that ${}^{o,\mathbb{H}}Y_{\widehat{\tau}} = \mathbb{E}\left[Y_{\widehat{\tau}}|\mathcal{H}_{\widehat{\tau}}\right]$ $\mathbf{P}$-a.s. (respectively ${}^{p,\mathbb{H}}Y_{\widehat{\tau}} = \mathbb{E}\left[Y_{\widehat{\tau}}|\mathcal{H}_{\widehat{\tau}-}\right]$ $\mathbf{P}$-a.s.) on $\{\widehat{\tau} < \infty\}$ for every $\mathbb{H}$-optional (respectively $\mathbb{H}$-predictable) stopping time $\widehat{\tau}$.

Moreover, in the sequel we denote by ${}^{o,\widetilde{\mathbb{G}}}\mu$, ${}^{p,\widetilde{\mathbb{G}}}\mu$, the optional projection and the predictable projection respectively of the process $\{\mu(t, S^\tau_t, X^\tau_t),\ t \in [0,T]\}$ with respect to the information flow $\widetilde{\mathbb{G}}$.

Lemma 3.2. *Under Assumption 2.1, the process* $I^\tau = \{I^\tau_t,\ t \in [0,T]\}$ *defined by*

$$I^\tau_t := W^\tau_t + \int_0^{t\wedge\tau} \frac{\mu(u, S^\tau_u, X^\tau_u) - {}^{p,\widetilde{\mathbb{G}}}\mu_u}{\sigma\left(u, S^\tau_u\right)}\mathrm{d}u, \quad t \in [0,T], \tag{3.1}$$

is a $(\widetilde{\mathbb{G}}, \mathbf{P})$*-Brownian motion on* $[\![0, \tau \wedge T]\!]$.

The proof is postponed to Appendix B.2.

Lemma 3.2 allows to get the following $\widetilde{\mathbb{G}}$-semimartingale decomposition of S^τ,

$$S^\tau_t = s_0 + \int_0^{t\wedge\tau} S^\tau_u\, {}^{p,\widetilde{\mathbb{G}}}\mu_u \mathrm{d}u + \int_0^{t\wedge\tau} S^\tau_u\, \sigma(u, S^\tau_u)\mathrm{d}I^\tau_u, \quad t \in [0,T],$$

i.e. the sum of a locally square integrable $(\widetilde{\mathbb{G}}, \mathbf{P})$-local martingale and a $(\widetilde{\mathbb{G}}, \mathbf{P})$-predictable process of finite variation both null at zero.

Moreover, S^τ satisfies the structure condition with respect to both the filtrations $\mathbb{G}$ and $\widetilde{\mathbb{G}}$. Precisely,

$$S_t^\tau = s_0 + M_t^{\mathcal{G}} + \int_0^{t\wedge\tau} \alpha_u^{\mathcal{G}} \mathrm{d}\langle M^{\mathcal{G}}\rangle_u, \quad t \in [0,T],$$

$$S_t^\tau = s_0 + M_t^{\widetilde{\mathcal{G}}} + \int_0^{t\wedge\tau} \alpha_u^{\widetilde{\mathcal{G}}} \mathrm{d}\langle M^{\widetilde{\mathcal{G}}}\rangle_u, \quad t \in [0,T],$$

where $M^{\mathcal{G}} = \{M_t^{\mathcal{G}},\ t \in [0,T]\}$ and $M^{\widetilde{\mathcal{G}}} = \{M_t^{\widetilde{\mathcal{G}}},\ t \in [0,T]\}$ are the locally square integrable $(\mathbb{G}, \mathbf{P})$-local martingale and $(\widetilde{\mathbb{G}}, \mathbf{P})$-local martingale respectively, given by

$$M_t^{\mathcal{G}} := \int_0^{t\wedge\tau} S_u^\tau \sigma(u, S_u^\tau)\mathrm{d}W_u^\tau, \quad M_t^{\widetilde{\mathcal{G}}} := \int_0^{t\wedge\tau} S_u^\tau \sigma(u, S_u^\tau)\mathrm{d}I_u^\tau, \quad t \in [0,T], \tag{3.2}$$

and $\alpha^{\mathcal{G}} = \{\alpha_t^{\mathcal{G}},\ t \in [0,T]\}$ and $\alpha^{\widetilde{\mathcal{G}}} = \{\alpha_t^{\widetilde{\mathcal{G}}},\ t \in [0,T]\}$ are the $\mathbb{G}$-predictable and $\widetilde{\mathbb{G}}$-predictable processes, respectively given by

$$\alpha_t^{\mathcal{G}} := \frac{\mu(t, S_t^\tau, X_t^\tau)}{S_t^\tau \sigma^2(t, S_t^\tau)}, \quad \alpha_t^{\widetilde{\mathcal{G}}} := \frac{{}^{p,\widetilde{\mathbb{G}}}\mu_t}{S_t^\tau \sigma^2(t, S_t^\tau)}, \quad t \in [0,T].$$

4. Local risk-minimization for payment streams under partial information

The combined financial-insurance market model outlined in Section 2 is not complete. Indeed, the number of random sources is larger than the number of tradeable risky assets due to the presence of a totally inaccessible death time. Moreover, additional randomness is brought here by the unobservable stochastic factor X. Then, a replicating strategy, which is at the same time self-financing, may not exist in this framework. In this section we look for a locally risk-minimizing hedging strategy under restricted information for the payment stream associated with the endowment insurance contract (ξ, Z, τ), and discuss the relation with the corresponding optimal hedging strategy under full information.

Remark 4.1. *Since $F_t < 1$ for all $t \in [0,T]$, for every $\mathbb{G}$-predictable (respectively $\widetilde{\mathbb{G}}$-predictable) process Y there is an $\mathbb{F}$-predictable (respectively $\mathbb{F}^S$-predictable) process $\widetilde{Y}$ such that $\widetilde{Y}_t \mathbf{1}_{\{\tau\geq t\}} = Y_t \mathbf{1}_{\{\tau\geq t\}}$ $\mathbf{P}$-a.s. for every $t \in [0,T]$ (see e.g. Dellacherie et al. [24, Paragraph 75, part a), page 186] or Blanchet-Scalliet and Jeanblanc [12, page 147]).*

Remark 4.1 allows to consider the following classes of admissible hedging strategies under full and partial information.

Definition 4.2. *The space $\Theta^{\mathbb{F},\tau}$ consists of all $\mathbb{R}$-valued $\mathbb{F}$-predictable processes $\theta = \{\theta_t,\ t \in [\![0, T\wedge\tau]\!]\}$ satisfying*

$$\mathbb{E}\left[\int_0^{T\wedge\tau} (\theta_u \sigma(u, S_u^\tau) S_u^\tau)^2 \,\mathrm{d}u + \left(\int_0^{T\wedge\tau} |\theta_u\, \mu(u, S_u^\tau, X_u^\tau) S_u^\tau| \mathrm{d}u\right)^2\right] < \infty.$$

Definition 4.3. *The space* $\Theta^{\mathbb{F}^S,\tau}$ *consists of all* $\mathbb{R}$*-valued* $\mathbb{F}^S$*-predictable processes* $\theta = \{\theta_t,\ t \in [\![0, T\wedge\tau]\!]\}$ *satisfying*

$$\mathbb{E}\left[\int_0^{T\wedge\tau} (\theta_u \sigma(u, S_u^\tau) S_u^\tau)^2 \,\mathrm{d}u + \left(\int_0^{T\wedge\tau} |\theta_u \ {}^{p,\widetilde{\mathbb{G}}}\mu_u \ S_u^\tau| \mathrm{d}u\right)^2\right] < \infty.$$

Remark 4.4. *Notice that for* $\theta \in \Theta^{\mathbb{F},\tau}$ *(respectively* $\theta \in \Theta^{\mathbb{F}^S,\tau}$*), we get*

(i) $\displaystyle\int_0^{t\wedge\tau} \theta_u \mathrm{d}S_u = \int_0^t \theta_u \mathrm{d}S_u^\tau$, *for every* $t \in [0, T]$*, see Dellacherie and Meyer [23, Chapter VIII, equation 3.3];*

(ii) *the integral process* $\left\{\displaystyle\int_0^t \theta_u \mathrm{d}S_u^\tau,\ t \in [0, T]\right\}$ *is a* $(\mathbb{G}, \mathbf{P})$*-semimartingale (respectively* $(\widetilde{\mathbb{G}}, \mathbf{P})$*-semimartingale), see Prokhorov and Shiryaev [44, Chapter 3.II].*

Definition 4.5. *An* $(\mathbb{F}, \mathbb{G})$*-strategy (respectively* $(\mathbb{F}^S, \widetilde{\mathbb{G}})$*-strategy) is a bidimensional process* $\varphi = (\theta, \eta)$ *where* $\theta \in \Theta^{\mathbb{F},\tau}$ *(respectively* $\theta \in \Theta^{\mathbb{F}^S,\tau}$*) and* η *is a real-valued* $\mathbb{G}$*-adapted (respectively* $\widetilde{\mathbb{G}}$*-adapted) process such that the associated value process* $V(\varphi) := \theta S^\tau + \eta$ *is right-continuous and square integrable over* $[\![0, T\wedge\tau]\!]$.

Note that the first component θ of the $(\mathbb{F}, \mathbb{G})$-strategy (respectively $(\mathbb{F}^S, \widetilde{\mathbb{G}})$-strategy), which represents the number of risky assets in the portfolio, is $\mathbb{F}$-predictable (respectively $\mathbb{F}^S$-predictable), while the amount η invested in the risk-free asset is $\mathbb{G}$-adapted (respectively $\widetilde{\mathbb{G}}$-adapted). This reflects the natural situation where a trader invests in the risky asset according to her/his knowledge on the asset prices before the death of the policyholder and rebalances the portfolio also upon the death information.

Following Schweizer [49], we assign to each admissible strategy a cost process.

Definition 4.6. *The* cost process $C(\varphi)$ *of an* $(\mathbb{F}, \mathbb{G})$*-strategy (respectively* $(\mathbb{F}^S, \widetilde{\mathbb{G}})$*-strategy)* $\varphi = (\theta, \eta)$ *is given by*

$$C_t(\varphi) := N_t + V_t(\varphi) - \int_0^t \theta_u \mathrm{d}S_u^\tau, \quad t \in [\![0, T\wedge\tau]\!], \tag{4.1}$$

where N *is defined in* (2.9).

An $(\mathbb{F}, \mathbb{G})$*-strategy (respectively* $(\mathbb{F}^S, \widetilde{\mathbb{G}})$*-strategy)* φ *is called* mean-self-financing *if its cost process* $C(\varphi)$ *is a* $(\mathbb{G}, \mathbf{P})$*-martingale (respectively* $(\widetilde{\mathbb{G}}, \mathbf{P})$*-martingale).*

It is well known in the literature (see e.g. Møller [40], Schweizer [49], Biagini and Cretarola [4]) that a natural extension of the local risk-minimization approach to payment streams requires to look for admissible strategies φ satisfying the 0-achieving property, that is,

$$V_{\tau\wedge T}(\varphi) = 0, \quad \mathbf{P}-\text{a.s.}.$$

Then, by Schweizer [49, Theorem 1.6], we provide the following equivalent definition of locally risk-minimizing strategy.

Definition 4.7. *Let N be the payment stream given in* (2.9) *associated with the endowment insurance contract (ξ, Z, τ). We say that an $(\mathbb{F}, \mathbb{G})$-strategy (respectively $(\mathbb{F}^S, \widetilde{\mathbb{G}})$-strategy) φ is $(\mathbb{F}, \mathbb{G})$-*locally risk-minimizing *(respectively* $(\mathbb{F}^S, \widetilde{\mathbb{G}})$-locally risk-minimizing*) for N if*

(i) *φ is 0-achieving and mean-self-financing;*
(ii) *the cost process $C(\varphi)$, defined in* (4.1)*, is strongly orthogonal to the $\mathbb{G}$-martingale part $M^{\mathcal{G}}$ (respectively $\widetilde{\mathbb{G}}$-martingale part $M^{\widetilde{\mathcal{G}}}$) of S^τ, both given in* (3.2).

Locally risk-minimizing hedging strategies can be characterized via the Föllmer-Schweizer decomposition of payment streams associated with life insurance contracts under partial information.

Taking Remark 4.1 into account, we give the following definitions of stopped Föllmer-Schweizer decompositions of a square integrable random variable with respect to $\mathbb{G}$ and $\widetilde{\mathbb{G}}$.

Definition 4.8 (Stopped Föllmer-Schweizer decomposition with respect to $\mathbb{G}$)**.** *Given a random variable $\zeta \in L^2(\mathcal{G}_T, \mathbf{P})$, we say that ζ admits a* stopped Föllmer-Schweizer decomposition with respect to $\mathbb{G}$, *if there exist a process $\theta^{\mathcal{F}} \in \Theta^{\mathbb{F},\tau}$, a square integrable $(\mathbb{G}, \mathbf{P})$-martingale $A^{\mathcal{G}} = \{A^{\mathcal{G}}_t,\ t \in [\![0, T \wedge \tau]\!]\}$ null at zero, strongly orthogonal to the martingale part of S^τ, $M^{\mathcal{G}}$, given in* (3.2)*, and $\zeta_0 \in \mathbb{R}$ such that*

$$\zeta = \zeta_0 + \int_0^T \theta^{\mathcal{F}}_u \mathrm{d}S^\tau_u + A^{\mathcal{G}}_{T\wedge\tau}, \quad \mathbf{P} - a.s.. \tag{4.2}$$

Definition 4.9 (Stopped Föllmer-Schweizer decomposition with respect to $\widetilde{\mathbb{G}}$)**.** *Given a random variable $\zeta \in L^2(\widetilde{\mathcal{G}}_T, \mathbf{P})$, we say that ζ admits a* stopped Föllmer-Schweizer decomposition with respect to $\widetilde{\mathbb{G}}$, *if there exist a process $\theta^{\mathcal{F}^S} \in \Theta^{\mathbb{F}^S,\tau}$, a square integrable $(\widetilde{\mathbb{G}}, \mathbf{P})$-martingale $A^{\widetilde{\mathcal{G}}} = \{A^{\widetilde{\mathcal{G}}}_t,\ t \in [\![0, T \wedge \tau]\!]\}$ null at zero, strongly orthogonal to the martingale part of S^τ,$M^{\widetilde{\mathcal{G}}}$, given in* (3.2)*, and $\zeta_0 \in \mathbb{R}$ such that*

$$\zeta = \zeta_0 + \int_0^T \theta^{\mathcal{F}^S}_u \mathrm{d}S^\tau_u + A^{\widetilde{\mathcal{G}}}_{T\wedge\tau}, \quad \mathbf{P} - a.s.. \tag{4.3}$$

Under Assumption 2.1, the *mean variance tradeoff processes* $K = \{K_t,\ t \in [0, T]\}$ and $\widetilde{K} = \{\widetilde{K}_t,\ t \in [0, T]\}$ under $\mathbb{G}$ and $\widetilde{\mathbb{G}}$, respectively defined by

$$K_t := \int_0^t (\alpha^{\mathcal{G}}_u)^2 \mathrm{d}\langle M^{\mathcal{G}}\rangle_u, \qquad \widetilde{K}_t := \int_0^t (\alpha^{\widetilde{\mathcal{G}}}_u)^2 \mathrm{d}\langle M^{\mathcal{G}}\rangle_u, \quad t \in [0, T],$$

are bounded uniformly in t and ω. Boundedness of the mean variance tradeoff processes and Remark 4.1 guarantee the existence of decompositions (4.2) and (4.3) for every $\zeta \in L^2(\widetilde{\mathcal{G}}_T, \mathbf{P}) \subseteq L^2(\mathcal{G}_T, \mathbf{P})$, see e.g. Schweizer [46, Section 5] and references therein. Other classes of sufficient conditions for the existence of the Föllmer-Schweizer decompositions can be found e.g. in Schweizer [47], Monat and Stricker [41], Choulli et al. [20] and Ceci et al. [16].

Let N be the payment stream associated with the endowment insurance contract (ξ, Z, τ) given in (2.9) and consider the random variable

$$N_{T\wedge\tau} = \xi \mathbf{1}_{\{\tau > T\}} + Z_\tau \mathbf{1}_{\{\tau \le T\}}. \tag{4.4}$$

Note that $N_{T\wedge\tau} \in L^2(\widetilde{\mathcal{G}}_T, \mathbf{P}) \subseteq L^2(\mathcal{G}_T, \mathbf{P})$ since $\xi \in L^2(\mathcal{F}^S_T, \mathbf{P})$ and Z is a $\mathbf{P}$-square integrable and $\mathbb{F}^S$-predictable process. Then, $N_{T\wedge\tau}$ admits a stopped Föllmer-Schweizer decomposition with respect to both $\mathbb{G}$ and $\widetilde{\mathbb{G}}$, i.e.

$$N_{T\wedge\tau} = \zeta_0 + \int_0^T \theta^{\mathcal{F}}_u \mathrm{d}S^\tau_u + A^{\mathcal{G}}_{T\wedge\tau}, \quad \mathbf{P}-\text{a.s.}, \tag{4.5}$$

$$N_{T\wedge\tau} = \zeta_0 + \int_0^T \theta^{\mathcal{F}^S}_u \mathrm{d}S^\tau_u + A^{\widetilde{\mathcal{G}}}_{T\wedge\tau}, \quad \mathbf{P}-\text{a.s.}. \tag{4.6}$$

The following proposition gives a characterization of the optimal hedging strategy.

Proposition 4.10. *Let N be the payment stream associated with the endowment insurance contract (ξ, Z, τ) and suppose that Assumption 2.1 is in force. Then, N admits an $(\mathbb{F}^S, \widetilde{\mathbb{G}})$-locally risk-minimizing strategy $\varphi^* = (\theta^*, \eta^*)$, explicitly given by*

$$\theta^* = \theta^{\mathcal{F}^S}, \quad \eta^* = V(\varphi^*) - \theta^{\mathcal{F}^S} S^\tau, \tag{4.7}$$

with value process

$$V_t(\varphi^*) = \zeta_0 + \int_0^t \theta^{\mathcal{F}^S}_u \mathrm{d}S^\tau_u + A^{\widetilde{\mathcal{G}}}_t - N_t, \quad t \in [\![0, T\wedge\tau]\!], \tag{4.8}$$

and minimal cost

$$C_t(\varphi^*) = \zeta_0 + A^{\widetilde{\mathcal{G}}}_t, \quad t \in [\![0, T\wedge\tau]\!], \tag{4.9}$$

where ζ_0, $\theta^{\mathcal{F}^S}$ and $A^{\widetilde{\mathcal{G}}}$ are given in decomposition (4.6)*.*

Proof. Under Assumption 2.1 we have that $N_{T\wedge\tau}$ admits a stopped Föllmer-Schweizer decomposition with respect to $\widetilde{\mathbb{G}}$, given by (4.6). Then, the proof follows by that of Biagini and Cretarola [4, Proposition 3.7], by replacing the filtrations $\mathbb{G}$ and $\mathbb{F}$ with $\widetilde{\mathbb{G}}$ and $\mathbb{F}^S$, respectively. Precisely, by (4.6) we get that (4.7) and (4.8) define an $(\mathbb{F}^S, \widetilde{\mathbb{G}})$-strategy with cost (4.9). It is easy to see that $C(\varphi^*)$ is a $(\widetilde{\mathbb{G}}, \mathbf{P})$-martingale and that φ^* is 0-achieving, and therefore φ^* is an $(\mathbb{F}^S, \widetilde{\mathbb{G}})$-locally risk-minimizing strategy. □

4.1. **The optimal strategy via the Galtchouk-Kunita-Watanabe decomposition.** According to the local risk-minimization approach, see, e.g. Schweizer [48], when the underlying risky asset price process is continuous and satisfies the structure condition, the Föllmer-Schweizer decomposition of a given square integrable random variable can be computed by switching to the minimal martingale measure and considering the corresponding Galtchouk-Kunita-Watanabe decomposition. In the following, we revisit this methodology in the combined financial-insurance market model outlined in Section 2.

Definition 4.11. *A martingale measure $\widehat{\mathbf{P}}$ equivalent to $\mathbf{P}$ with square integrable density is called* minimal *for S^τ if any square integrable $(\mathbb{G}, \mathbf{P})$-martingale, which is strongly orthogonal to the martingale part of S^τ, $M^{\mathcal{G}}$ given in* (3.2)*, under $\mathbf{P}$ is also a $(\mathbb{G}, \widehat{\mathbf{P}})$-martingale.*

Define the process $L^\tau = \{L^\tau_t,\ t \in [0,T]\}$ by setting

$$L^\tau_t = \frac{\mathrm{d}\widehat{\mathbf{P}}}{\mathrm{d}\mathbf{P}}\bigg|_{\mathcal{G}_{\tau\wedge t}} := \mathcal{E}\left(-\int_0^\cdot \frac{\mu(u,S^\tau_u,X^\tau_u)}{\sigma(u,S^\tau_u)}\mathrm{d}W^\tau_u\right)_{t\wedge\tau}, \quad t \in [0,T]. \tag{4.10}$$

By condition (ii) of Assumption 2.1, we get that $L^\tau_t \in L^2(\mathcal{G}_t,\mathbf{P})$, for every $t \in [0,T]$.

Applying the results in Ansel and Stricker [1], we get that $\widehat{\mathbf{P}}$, given in (4.10), corresponds to the minimal martingale measure. By the Girsanov theorem the process $\widehat{W}^\tau = \{\widehat{W}^\tau_t,\ t \in [\![0,T\wedge\tau]\!]\}$, defined by

$$\widehat{W}^\tau_t := W^\tau_t + \int_0^{t\wedge\tau} \frac{\mu(u,S^\tau_u,X^\tau_u)}{\sigma(u,S^\tau_u)}\mathrm{d}u, \quad t \in [\![0,T\wedge\tau]\!],$$

is a $(\mathbb{G},\widehat{\mathbf{P}})$-Brownian motion.

Note that L^τ and $\widehat{W}^\tau$ coincide with the processes L and $\widehat{W}$, given in (2.3) and (2.4) respectively, on the stochastic interval $[\![0,T\wedge\tau]\!]$.

Remark 4.12. *We may also define the minimal martingale measure $\widehat{\mathbf{Q}}$ for S^τ with respect to the information flow $\widetilde{\mathbb{G}}$, by setting*

$$\frac{\mathrm{d}\widehat{\mathbf{Q}}}{\mathrm{d}\mathbf{P}}\bigg|_{\widetilde{\mathcal{G}}_{\tau\wedge T}} := \mathcal{E}\left(-\int_0^\cdot \alpha^{\widetilde{\mathcal{G}}}_u \mathrm{d}M^{\widetilde{\mathcal{G}}}_u\right)_{T\wedge\tau} = \mathcal{E}\left(-\int_0^\cdot \frac{{}^{p,\widetilde{\mathbb{G}}}\mu_u}{\sigma(u,S^\tau_u)}\mathrm{d}I^\tau_u\right)_{T\wedge\tau}.$$

Since S^τ has continuous trajectories, $\widehat{\mathbf{Q}}$ coincides with the restriction of $\widehat{\mathbf{P}}$ over $\widetilde{\mathcal{G}}_{\tau\wedge T}$, see, e.g. Ceci et al. [19, Lemma 4.3]. Indeed, by (3.1)

$$I^\tau_t + \int_0^{t\wedge\tau} \frac{{}^{p,\widetilde{\mathbb{G}}}\mu_u}{\sigma(u,S^\tau_u)}\mathrm{d}u = W^\tau_t + \int_0^{t\wedge\tau} \frac{\mu(u,S^\tau_u,X^\tau_u)}{\sigma(u,S^\tau_u)}\mathrm{d}u = \widehat{W}^\tau_t, \quad t \in [\![0,T\wedge\tau]\!],$$

which, therefore, implies that the process $\left\{I^\tau_t + \int_0^{t\wedge\tau} \frac{{}^{p,\widetilde{\mathbb{G}}}\mu_u}{\sigma(u,S^\tau_u)}\mathrm{d}u,\ t \in [\![0,T\wedge\tau]\!]\right\}$ is a $(\widetilde{\mathbb{G}},\widehat{\mathbf{P}})$-Brownian motion since it is $\widetilde{\mathbb{G}}$-adapted.

In the following we show that the Föllmer-Schweizer decomposition of the payment stream N associated with the endowment insurance contract (ξ,Z,τ) indeed coincides with its Galtchouk-Kunita-Watanabe decomposition under the minimal martingale measure, which is easier to characterize.

For reader's convenience, we recall the definition of the Galtchouk-Kunita-Watanabe decomposition of a square integrable random variable, adapted to this setting.

Definition 4.13 (Galtchouk-Kunita-Watanabe decomposition)**.** *Any random variable $\zeta \in L^2(\mathcal{G}_T,\widehat{\mathbf{P}})$ (respectively $\zeta \in L^2(\widetilde{\mathcal{G}}_T,\widehat{\mathbf{P}})$) admits a* Galtchouk-Kunita-Watanabe decomposition *with respect to S^τ under $\widehat{\mathbf{P}}$, that is, it can be uniquely written as*

$$\zeta = \zeta_0 + \int_0^T \bar{\theta}_u \mathrm{d}S^\tau_u + \bar{A}_{T\wedge\tau}, \quad \widehat{\mathbf{P}} - a.s.,$$

where $\zeta_0 \in \mathbb{R}$, $\bar{\theta} = \{\bar{\theta}_t,\ t \in [\![0, T \wedge \tau]\!]\}$ *is a* $\mathbb{G}$*-predictable (respectively* $\widetilde{\mathbb{G}}$*-predictable) process such that* $\widehat{\mathbb{E}}\left[\int_0^{T\wedge\tau} \left(\bar{\theta}_u \sigma(u, S_u^\tau) S_u^\tau\right)^2 \mathrm{d}u\right] < \infty$ *and* $\bar{A} = \{\bar{A}_t,\ t \in [\![0, T \wedge \tau]\!]\}$ *is a* $(\mathbb{G}, \widehat{\mathbf{P}})$*-martingale (respectively* $(\widehat{\mathbb{G}}, \widehat{\mathbf{P}})$*-martingale) null at zero, strongly orthogonal to* S^τ*.*

From now on we work under the following standing assumption.

Assumption 4.14. *Given in* (2.9) *the payment stream* N *associated with the endowment insurance contract* (ξ, Z, τ)*, we assume that* $N_{T\wedge\tau}$ *given in* (4.4) *is* $\widehat{\mathbf{P}}$*-square integrable and* S^τ *is* $\widehat{\mathbf{P}}$*-locally square integrable.*

Under Assumption 4.14, $N_{T\wedge\tau}$ admits the Galtchouk-Kunita-Watanabe decomposition with respect to S^τ under $(\widetilde{\mathbb{G}}, \widehat{\mathbf{P}})$ given by

$$N_{T\wedge\tau} = \widehat{\mathbb{E}}\left[N_{T\wedge\tau}\right] + \int_0^T \widehat{\theta}_u^{\widetilde{\mathcal{G}}} \mathrm{d}S_u^\tau + \widehat{A}_{T\wedge\tau}^{\widetilde{\mathcal{G}}}, \quad \widehat{\mathbf{P}} - \text{a.s.}, \tag{4.11}$$

where $\widehat{\theta}^{\widetilde{\mathcal{G}}} = \{\widehat{\theta}_t^{\widetilde{\mathcal{G}}},\ t \in [\![0, T \wedge \tau]\!]\}$ is $\widetilde{\mathbb{G}}$-predictable and satisfies $\mathbb{E}\left[\int_0^T \left(\widehat{\theta}_t^{\widetilde{\mathcal{G}}} \sigma(t, S_t^\tau) S_t^\tau\right)^2 \mathrm{d}t\right] < \infty$, $\widehat{A}^{\widetilde{\mathcal{G}}} = \{\widehat{A}_t^{\widetilde{\mathcal{G}}},\ t \in [\![0, T \wedge \tau]\!]\}$ is a $(\widetilde{\mathbb{G}}, \widehat{\mathbf{P}})$-martingale null at time zero, strongly orthogonal to S^τ. By Remark 4.1, it is always possible to replace $\widehat{\theta}^{\widetilde{\mathcal{G}}}$ by an $\mathbb{F}^S$-predictable process $\widehat{\theta}^{\mathcal{F}^S}$ such that $\mathbf{1}_{\{\tau \geq t\}} \widehat{\theta}_t^{\widetilde{\mathcal{G}}} = \mathbf{1}_{\{\tau \geq t\}} \widehat{\theta}_t^{\mathcal{F}^S}$, for each $t \in [0, T]$. Then, equation (4.11) can be written as

$$N_{T\wedge\tau} = \widehat{\mathbb{E}}\left[N_{T\wedge\tau}\right] + \int_0^T \widehat{\theta}_u^{\mathcal{F}^S} \mathrm{d}S_u^\tau + \widehat{A}_{T\wedge\tau}^{\widetilde{\mathcal{G}}}, \quad \widehat{\mathbf{P}} - \text{a.s.}. \tag{4.12}$$

Notice that, since $N_{T\wedge\tau}$ is also $\mathcal{G}_T$-measurable, we can consider the Galtchouk-Kunita-Watanabe decomposition with respect to $(\mathbb{G}, \widehat{\mathbf{P}})$, i.e.

$$N_{T\wedge\tau} = \widehat{\mathbb{E}}\left[N_{T\wedge\tau}\right] + \int_0^T \widehat{\theta}_u^{\mathcal{F}} \mathrm{d}S_u^\tau + \widehat{A}_{T\wedge\tau}^{\mathcal{G}}, \quad \widehat{\mathbf{P}} - \text{a.s.}, \tag{4.13}$$

where $\widehat{\theta}^{\mathcal{F}}$ is an $\mathbb{F}$-predictable process such that $\widehat{\mathbb{E}}\left[\int_0^{T\wedge\tau} \left(\widehat{\theta}_u^{\mathcal{F}} \sigma(u, S_u^\tau) S_u^\tau\right)^2 \mathrm{d}u\right] < \infty$ and $\widehat{A}^{\mathcal{G}} = \{\widehat{A}_t^{\mathcal{G}},\ t \in [\![0, T \wedge \tau]\!]\}$ is a $(\mathbb{G}, \widehat{\mathbf{P}})$-martingale null at zero strongly orthogonal to S^τ.

The following proposition provides a representation of the integrand $\widehat{\theta}^{\mathcal{F}^S}$ in the Galtchouk-Kunita-Watanabe decomposition of $N_{T\wedge\tau}$ under partial information in terms of the corresponding integrand $\widehat{\theta}^{\mathcal{F}}$ in the Galtchouk-Kunita-Watanabe decomposition under full information, and finally Theorem 4.16 gives the characterization of the locally risk-minimizing strategy for the insurance claim (ξ, Z, τ) under partial information.

In the sequel, given any subfiltration $\mathbb{H}$ of $\mathbb{G}$, the notation ${}^{\widehat{p},\mathbb{H}}Y$ refers to the $(\mathbb{H}, \widehat{\mathbf{P}})$-predictable projection of a given $\widehat{\mathbf{P}}$-integrable $\mathbb{G}$-adapted process Y.

Proposition 4.15. *Under Assumptions 2.1 and 4.14, the integrand* $\widehat{\theta}^{\mathcal{F}^S}$ *in decomposition* (4.12) *is given by*

$$\widehat{\theta}^{\mathcal{F}^S}_t = \frac{{}^{\widehat{p},\mathbb{F}^S}\big(\widehat{\theta}^{\mathcal{F}}_t e^{-\int_0^t \gamma_u \mathrm{d}u}\big)}{{}^{\widehat{p},\mathbb{F}^S}\big(e^{-\int_0^t \gamma_u \mathrm{d}u}\big)}, \quad t \in [\![0, T \wedge \tau]\!], \tag{4.14}$$

where the process $\widehat{\theta}^{\mathcal{F}}$ *is the integrand in decomposition* (4.13), *and the* $(\widetilde{\mathbb{G}}, \widehat{\mathbf{P}})$*-martingale* $\widehat{A}^{\widetilde{\mathcal{G}}} = \{\widehat{A}^{\widetilde{\mathcal{G}}}_t,\ t \in [\![0, T \wedge \tau]\!]\}$ *is given by*

$$\widehat{A}^{\widetilde{\mathcal{G}}}_t = \widehat{\mathbb{E}}\left[\widehat{A}^{\mathcal{G}}_t \Big| \widetilde{\mathcal{G}}_t\right] + \widehat{\mathbb{E}}\left[\int_0^t (\widehat{\theta}^{\mathcal{F}}_u - \widehat{\theta}^{\mathcal{F}^S}_u)\mathrm{d}S^\tau_u \Big| \widetilde{\mathcal{G}}_t\right], \quad t \in [\![0, T \wedge \tau]\!]. \tag{4.15}$$

Proof. In virtue of Corollary B.4, if $\widehat{\theta}^{\mathcal{F}^S}$ satisfies (4.14), then

$$\widehat{\theta}^{\mathcal{F}^S}_t = {}^{\widehat{p},\widetilde{\mathbb{G}}}\widehat{\theta}^{\mathcal{F}}_t, \quad t \in [\![0, T \wedge \tau]\!].$$

By decomposition (4.13) we can write

$$N_{T\wedge\tau} = \widehat{\mathbb{E}}\left[N_{T\wedge\tau}\right] + \int_0^T {}^{\widehat{p},\widetilde{\mathbb{G}}}\widehat{\theta}^{\mathcal{F}}_u \mathrm{d}S^\tau_u + \widetilde{A}_{T\wedge\tau} + \widehat{A}^{\mathcal{G}}_{T\wedge\tau}, \quad \widehat{\mathbf{P}} - \text{a.s.}, \tag{4.16}$$

where $\widetilde{A} = \{\widetilde{A}_t,\ t \in [\![0, T \wedge \tau]\!]\}$, given by

$$\widetilde{A}_t := \int_0^t (\widehat{\theta}^{\mathcal{F}}_u - \widehat{\theta}^{\mathcal{F}^S}_u)\mathrm{d}S^\tau_u = \int_0^t (\widehat{\theta}^{\mathcal{F}}_u - {}^{\widehat{p},\widetilde{\mathbb{G}}}\widehat{\theta}^{\mathcal{F}}_u)\mathrm{d}S^\tau_u, \quad t \in [\![0, T \wedge \tau]\!],$$

is a square integrable $(\mathbb{G}, \widehat{\mathbf{P}})$-martingale (see, e.g. [48, Lemma 2.1]). This is a consequence of the fact that S^τ is a $(\mathbb{G}, \widehat{\mathbf{P}})$-local martingale and that, by Jensen's inequality the following holds

$$\begin{aligned}
\widehat{\mathbb{E}}\left[\int_0^{T\wedge\tau}\Big(\widehat{\theta}^{\mathcal{F}^S}_u \sigma(u, S^\tau_u)S^\tau_u\Big)^2 \mathrm{d}u\right] &= \widehat{\mathbb{E}}\left[\int_0^T \Big({}^{\widehat{p},\widetilde{\mathbb{G}}}\widehat{\theta}^{\mathcal{F}}_u \sigma(u, S^\tau_u)S^\tau_u\Big)^2 \mathbf{1}_{\{\tau \geq u\}}\mathrm{d}u\right] \\
&\leq \widehat{\mathbb{E}}\left[\int_0^T {}^{\widehat{p},\widetilde{\mathbb{G}}}\left(\Big(\widehat{\theta}^{\mathcal{F}}_u \sigma(u, S^\tau_u)S^\tau_u\Big)^2 \mathbf{1}_{\{\tau \geq u\}}\right)\mathrm{d}u\right] = \widehat{\mathbb{E}}\left[\int_0^{T\wedge\tau}\Big(\widehat{\theta}^{\mathcal{F}}_u \sigma(u, S^\tau_u)S^\tau_u\Big)^2 \mathrm{d}u\right] < \infty.
\end{aligned}$$

By (4.15), conditioning (4.16) with respect to $\widetilde{\mathcal{G}}_{T\wedge\tau}$ yields

$$\begin{aligned}
N_{T\wedge\tau} &= \widehat{\mathbb{E}}\left[N_{T\wedge\tau}\right] + \int_0^T {}^{\widehat{p},\widetilde{\mathbb{G}}}\widehat{\theta}^{\mathcal{F}}_u \mathrm{d}S^\tau_u + \widehat{\mathbb{E}}\left[\widetilde{A}_{T\wedge\tau} + \widehat{A}^{\mathcal{G}}_{T\wedge\tau} | \widetilde{\mathcal{G}}_{T\wedge\tau}\right] \\
&= \widehat{\mathbb{E}}\left[N_{T\wedge\tau}\right] + \int_0^T {}^{\widehat{p},\widetilde{\mathbb{G}}}\widehat{\theta}^{\mathcal{F}}_u \mathrm{d}S^\tau_u + \widehat{A}^{\widetilde{\mathcal{G}}}_{T\wedge\tau}.
\end{aligned}$$

This provides the Galtchouk-Kunita-Watanabe decomposition of $N_{T\wedge\tau}$ with respect to $(\widetilde{\mathbb{G}}, \widehat{\mathbf{P}})$, once we verify that the square integrable $(\widetilde{\mathbb{G}}, \widehat{\mathbf{P}})$-martingale $\widehat{A}^{\widetilde{\mathcal{G}}}$ is strongly orthogonal to S^τ. Note that $\widehat{A}^{\widetilde{\mathcal{G}}}$ satisfies

$$\widehat{\mathbb{E}}\left[\widehat{A}^{\widetilde{\mathcal{G}}}_{T\wedge\tau} \int_0^{T\wedge\tau} \varphi_u \mathrm{d}S^\tau_u\right] = 0,$$

for all $\widetilde{\mathbb{G}}$-predictable processes φ such that $\widehat{\mathbb{E}}\left[\int_0^{T\wedge\tau} \varphi_u^2 \,\mathrm{d}\langle S^\tau\rangle_u\right] < \infty$, i.e. $\widehat{A}^{\widetilde{\mathcal{G}}}$ is $\widetilde{\mathbb{G}}$-weakly orthogonal to S^τ, see Definition 2.1 in Ceci et al. [17]. Indeed, since φ is $\widetilde{\mathbb{G}}$-predictable, by the tower rule

$$\widehat{\mathbb{E}}\left[\widehat{A}^{\widetilde{\mathcal{G}}}_{T\wedge\tau}\int_0^{T\wedge\tau}\varphi_u \mathrm{d}S^\tau_u\right] = \widehat{\mathbb{E}}\left[\widehat{A}^{\mathcal{G}}_{T\wedge\tau}\int_0^{T\wedge\tau}\varphi_u \mathrm{d}S^\tau_u\right] + \widehat{\mathbb{E}}\left[\widetilde{A}_{T\wedge\tau}\int_0^{T\wedge\tau}\varphi_u \mathrm{d}S^\tau_u\right].$$

Both of the terms on the right-hand side are zero: the first one because $\widehat{A}^{\mathcal{G}}$ is strongly orthogonal to S^τ, and the second one follows by the computations below,

$$\begin{aligned}\widehat{\mathbb{E}}\left[\widetilde{A}_{T\wedge\tau}\int_0^{T\wedge\tau}\varphi_u \mathrm{d}S^\tau_u\right] &= \widehat{\mathbb{E}}\left[\int_0^{T\wedge\tau}\varphi_u(\widehat{\theta}^{\mathcal{F}}_u - {}^{\widehat{p},\widetilde{\mathbb{G}}}\widehat{\theta}^{\mathcal{F}}_u)\mathrm{d}\langle S^\tau\rangle_u\right]\\ &= \widehat{\mathbb{E}}\left[\int_0^{T\wedge\tau}\varphi_u(\widehat{\theta}^{\mathcal{F}}_u - {}^{\widehat{p},\widetilde{\mathbb{G}}}\widehat{\theta}^{\mathcal{F}}_u)\sigma^2(u,S^\tau_u)(S^\tau_u)^2\mathrm{d}u\right] = 0,\end{aligned}$$

since $\{\sigma(t,S^\tau_t)S^\tau_t,\ t\in[\![0,T\wedge\tau]\!]\}$ has continuous trajectories.

Finally, the strong orthogonality between $\widehat{A}^{\widetilde{\mathcal{G}}}$ and S^τ is equivalent to $\widetilde{\mathbb{G}}$-weak orthogonality since $\widehat{A}^{\widetilde{\mathcal{G}}}$ is $\widetilde{\mathbb{G}}$-adapted (see Ceci et al. [17, Remark 2.4]). □

Theorem 4.16. *Let N be the payment stream given by* (2.9)*, associated with the endowment insurance contract (ξ, Z, τ) and let Assumptions 2.1 and 4.14 hold. Then, equation* (4.12) *coincides with the stopped Föllmer-Schweizer decomposition of $N_{T\wedge\tau}$ with respect to $\widetilde{\mathbb{G}}$, given in* (4.6)*.*

Moreover, the $(\mathbb{F}^S,\widetilde{\mathbb{G}})$-locally risk-minimizing strategy $\varphi^* = (\theta^*,\eta^*)$ for N is given by

$$\theta^*_t = \theta^{\mathcal{F}^S}_t = \frac{{}^{\widehat{p},\mathbb{F}^S}\left(\theta^{\mathcal{F}}_t e^{-\int_0^t\gamma_u \mathrm{d}u}\right)}{{}^{\widehat{p},\mathbb{F}^S}\left(e^{-\int_0^t\gamma_u \mathrm{d}u}\right)}, \quad t\in[\![0,T\wedge\tau]\!], \tag{4.17}$$

$$\eta^*_t = V_t(\varphi^*) - \theta^*_t S^\tau_t, \quad t\in[\![0,T\wedge\tau]\!],$$

and the optimal value process $V(\varphi^*)$ is given by

$$V_t(\varphi^*) = \widehat{\mathbb{E}}\left[N_{T\wedge\tau}\right] + \int_0^t \theta^*_u \mathrm{d}S^\tau_u + A^{\widetilde{\mathcal{G}}}_t - N_t, \quad t\in[\![0,T\wedge\tau]\!],$$

with

$$A^{\widetilde{\mathcal{G}}}_t = \widehat{\mathbb{E}}\left[A^{\mathcal{G}}_t\middle|\widetilde{\mathcal{G}}_t\right] + \widehat{\mathbb{E}}\left[\int_0^t \theta^{\mathcal{F}}_u \mathrm{d}S^\tau_u\middle|\widetilde{\mathcal{G}}_t\right] - \int_0^t \theta^*_u \mathrm{d}S^\tau_u, \quad t\in[\![0,T\wedge\tau]\!],$$

where $\theta^{\mathcal{F}} = \widehat{\theta}^{\mathcal{F}}$ and $A^{\mathcal{G}} = \widehat{A}^{\mathcal{G}}$ are given in decomposition (4.13) *and $\theta^{\mathcal{F}^S} = \widehat{\theta}^{\mathcal{F}^S}$ and $A^{\widetilde{\mathcal{G}}} = \widehat{A}^{\widetilde{\mathcal{G}}}$ are given in* (4.12)*.*

Proof. Under Assumption 2.1 the stopped Föllmer-Schweizer decompositions of $N_{T\wedge\tau}$ with respect to $\mathbb{G}$ and $\widetilde{\mathbb{G}}$, respectively given by equation (4.5) and (4.6), exist. Then, it follows from Biagini and Cretarola [4, Theorem 3.9] or Schweizer [48, Theorem 3.5], that decomposition (4.13) and (4.5) coincide. Analogously, by replacing the filtrations $\mathbb{G}$ and $\mathbb{F}$ with $\widetilde{\mathbb{G}}$ and $\mathbb{F}^S$, we get that also decompositions (4.12) and (4.6) coincide. Then, the result follows by Proposition 4.10 and Proposition 4.15. □

Representation (4.17) requires the knowledge of the process $\theta^{\mathcal{F}}$, that is, the first component of the $(\mathbb{F}, \mathbb{G})$-*locally risk-minimizing strategy* (see Biagini and Cretarola [4, Proposition 3.7]).

To characterize the process $\theta^{\mathcal{F}}$, define the process $\widehat{V} = \{\widehat{V}_t,\ t \in [\![0, T \wedge \tau]\!]\}$ by setting

$$\widehat{V}_t := \widehat{\mathbb{E}}\left[N_{T\wedge\tau}|\mathcal{G}_t\right],\ t \in [\![0, T \wedge \tau]\!]. \tag{4.18}$$

Then, by (4.13) the process $\widehat{V}$ admits the Galchouk-Kunita-Watanabe decomposition given by

$$\widehat{V}_t = \widehat{\mathbb{E}}\left[N_{T\wedge\tau}\right] + \int_0^t \theta^{\mathcal{F}}_u \mathrm{d}S^\tau_u + A^{\mathcal{G}}_t, \quad t \in [\![0, T \wedge \tau]\!],$$

where $A^{\mathcal{G}} = \widehat{A}^{\mathcal{G}}$ is a square integrable $(\mathbb{G}, \widehat{\mathbf{P}})$-martingale null at time zero, strongly orthogonal to S^τ with respect to $\widehat{\mathbf{P}}$. By taking the predictable covariation with respect to S^τ on both sides of the equality we get that

$$\theta^{\mathcal{F}}_t = \frac{\mathrm{d}\langle \widehat{V}, S^\tau\rangle^{\widehat{\mathbf{P}}}_t}{\mathrm{d}\langle S^\tau\rangle^{\widehat{\mathbf{P}}}_t}, \quad t \in [\![0, T \wedge \tau]\!], \tag{4.19}$$

where $\langle \cdot, \cdot\rangle^{\widehat{\mathbf{P}}}$ denotes the predictable covariation process under minimal martingale measure $\widehat{\mathbf{P}}$.

Now we have to face the task of computing the process $\langle \widehat{V}, S^\tau\rangle^{\widehat{\mathbf{P}}}$.

In the following section we will analyze some examples in a Markovian setting where we are able to give explicit representations of both the optimal hedging strategies $\theta^{\mathcal{F}}$ and $\theta^{\mathcal{F}^S}$ under full and partial information.

5. An application: the $\mathbb{F}$-mortality rate depending on the unobservable stochastic factor

To introduce a Markovian setting, we assume that the $\mathbb{F}$-mortality rate γ is of the form $\gamma_t = \gamma(t, X_t)$, $t \in [0, T]$, for a nonnegative measurable function γ such that $\mathbb{E}\left[\int_0^T \gamma(s, X_s)\mathrm{d}s\right] < \infty$, and the endowment insurance contract is given by the triplet (ξ, Z, τ), where $\xi = G(T, S_T)$ and $Z_t = U(t, S_t)$, for some measurable functions G and U such that $\mathbb{E}[|G(T, S_T)|^2] < \infty$ and $\mathbb{E}[|U(t, S_t)|^2] < \infty$, for every $t \in [0, T]$.

On the probability space $(\Omega, \mathcal{F}, \widehat{\mathbf{P}})$ the pair (S, X) satisfies the following system of stochastic differential equations

$$\begin{cases} \mathrm{d}S_t = S_t\sigma(t, S_t)\mathrm{d}\widehat{W}_t, \quad S_0 = s_0 \in \mathbb{R}^+, \\ \mathrm{d}X_t = \left(b(t, X_t) - a(t, X_t)\rho\, \dfrac{\mu(t, S_t, X_t)}{\sigma(t, S_t)}\right)\mathrm{d}t + a(t, X_t)\left(\rho \mathrm{d}\widehat{W}_t + \sqrt{1-\rho^2}\mathrm{d}B_t\right),\ X_0 = x_0 \in \mathbb{R}. \end{cases} \tag{5.1}$$

We assume throughout the section that

$$\widehat{\mathbb{E}}\left[\int_0^T \left\{|b(t,X_t)| + a^2(t,X_t) + S_t^2\sigma^2(t,S_t)\right\} \mathrm{d}t\right] < \infty. \tag{5.2}$$

Condition (5.2) guarantees, for instance, that S is a square integrable $(\mathbb{F}, \widehat{\mathbf{P}})$-martingale. The same holds for the martingale part of X.

The Markovianity of the pair (S, X) under $\widehat{\mathbf{P}}$ is shown in the Lemma below.

Lemma 5.1. *Under Assumption 2.1 and condition* (5.2)*, the pair* (S,X) *is an* $(\mathbb{F}, \widehat{\mathbf{P}})$*-Markov process with generator* $\widehat{\mathcal{L}}^{S,X}$ *given by*

$$\begin{aligned}\widehat{\mathcal{L}}^{S,X} f(t,s,x) = \frac{\partial f}{\partial t} &+ \left[b(t,x) - \rho\ \frac{\mu(t,s,x)a(t,x)}{\sigma(t,s)}\right]\frac{\partial f}{\partial x} + \frac{1}{2}a^2(t,x)\frac{\partial^2 f}{\partial x^2} \\ &+ \rho a(t,x)\sigma(t,s)s\frac{\partial^2 f}{\partial x \partial s} + \frac{1}{2}\sigma^2(t,s)\, s^2\frac{\partial^2 f}{\partial s^2},\end{aligned} \tag{5.3}$$

for every function $f \in \mathcal{C}_b^{1,2,2}([0,T]\times\mathbb{R}^+\times\mathbb{R})$*. Moreover, the following semimartingale decomposition holds*

$$f(t,S_t,X_t) = f(0,s_0,x_0) + \int_0^t \widehat{\mathcal{L}}^{S,X} f(u,S_u,X_u)\mathrm{d}u + M_t^f, \quad t\in[0,T],$$

where $M^f = \{M_t^f,\ t\in[0,T]\}$ *is the* $(\mathbb{F}, \widehat{\mathbf{P}})$*-martingale given by*

$$\mathrm{d}M_t^f = \frac{\partial f}{\partial x}a(t,X_t)\left[\rho \mathrm{d}\widehat{W}_t + \sqrt{1-\rho^2}\mathrm{d}B_t\right] + \frac{\partial f}{\partial s}\sigma(t,S_t)S_t\mathrm{d}\widehat{W}_t.$$

The proof is postponed to Appendix B.2.

The idea for computing the $(\mathbb{F}^S, \widetilde{\mathbb{G}})$-locally risk minimizing strategy is to derive $\theta^{\mathcal{F}}$ via (4.19) and apply equation (4.17). Therefore, we need to characterize the process $\widehat{V}$ in (4.18).

First, observe that the process M in (2.8) is a $(\mathbb{G}, \widehat{\mathbf{P}})$-martingale null at time zero that can also be written as

$$M_t = H_t - \int_0^t (1-H_r)\gamma(r,X_r)\mathrm{d}r,$$

where H is the death indicator process given in (2.5), i.e. $H_t = \mathbf{1}_{\{\tau\le t\}}$. Then we get that,

$$\begin{aligned}\widehat{V}_t &= \widehat{\mathbb{E}}\left[G(T,S_T^\tau)(1-H_T) + \int_0^T U(r,S_r^\tau)\mathrm{d}H_r|\mathcal{G}_t\right] \\ &= \widehat{\mathbb{E}}\left[G(T,S_T^\tau)(1-H_T) + \int_0^T U(r,S_r^\tau)(1-H_r)\gamma(r,X_r^\tau)\mathrm{d}r|\mathcal{G}_t\right] \\ &= \int_0^t U(r,S_r^\tau)(1-H_r)\gamma(r,X_r^\tau)\mathrm{d}r + \widehat{\mathbb{E}}\left[G(T,S_T^\tau)(1-H_T) + \int_t^T U(r,S_r^\tau)(1-H_r)\gamma(r,X_r^\tau)\mathrm{d}r|\mathcal{G}_t\right].\end{aligned}$$

In order to compute the last conditional expectation we use the Markovianity of the triplet (S^τ, X^τ, H) under $\widehat{\mathbf{P}}$, which is proved in the lemma below. Denote by $\widehat{\mathcal{C}}_b^{1,2,2}([0,T]\times\mathbb{R}^+\times\mathbb{R}\times\{0,1\})$

the set of measurable and bounded functions $f : [0,T]\times\mathbb{R}^+\times\mathbb{R}\times\{0,1\}\to\mathbb{R}$ which are continuous and differentiable with respect to t, continuous and twice differentiable with respect to (s,x) with bounded derivatives (of all necessary orders).

Lemma 5.2. *Under Assumption 2.1 and condition* (5.2)*, the triplet* (S^τ, X^τ, H) *is a* $(\mathbb{G},\widehat{\mathbf{P}})$*-Markov process with generator* $\widehat{\mathcal{L}}^{S,X,H}$ *given by*

$$\widehat{\mathcal{L}}^{S,X,H} f(t,s,x,z) = \widehat{\mathcal{L}}^{S,X} f(t,s,x,z)(1-z) + \{f(t,s,x,z+1) - f(t,s,x,z)\}\gamma(t,x)(1-z) \quad (5.4)$$

for every function $f\in\widehat{\mathcal{C}}_b^{1,2,2}([0,T]\times\mathbb{R}^+\times\mathbb{R}\times\{0,1\})$*, where* $\widehat{\mathcal{L}}^{S,X}$ *is given in* (5.3)*.*

Moreover, the following $(\mathbb{G},\widehat{\mathbf{P}})$*-semimartingale decomposition holds*

$$f(t,S^\tau_t,X^\tau_t,H_t) = f(0,s_0,x_0,0) + \int_0^t \widehat{\mathcal{L}}^{S,X,H} f(u,S^\tau_u,X^\tau_u,H_u)\mathrm{d}u + M^f_t, \quad t\in[\![0,T\wedge\tau]\!],$$

where $M^f=\{M^f_t,\ t\in[0,T]\}$ *is the* $(\mathbb{G},\widehat{\mathbf{P}})$*-martingale given by*

$$\begin{aligned}\mathrm{d}M^f_t =& \frac{\partial f}{\partial x}(1-H_t)a(t,X^\tau_t)\left[\rho\mathrm{d}\widehat{W}^\tau_t + \sqrt{1-\rho^2}\mathrm{d}B^\tau_t\right] + \frac{\partial f}{\partial s}(1-H_t)\sigma(t,S^\tau_t)S^\tau_t\mathrm{d}\widehat{W}^\tau_t \\ &+ \{f(t,S^\tau_t,X^\tau_t,H_{t^-}+1) - f(t,S^\tau_t,X^\tau_t,H_{t^-})\}\mathrm{d}M_t.\end{aligned} \quad (5.5)$$

The proof is postponed to Appendix B.2.

Then the following result provides a characterization of the locally risk-minimizing strategy for the insurance claim under full information.

Proposition 5.3 (The full information case)**.** *Let Assumption 2.1 and condition* (5.2) *hold and assume that* $N_{T\wedge\tau}$ *is* $\widehat{\mathbf{P}}$*-square integrable. Let* $g\in\mathcal{C}_b^{1,2,2}([0,T]\times\mathbb{R}^+\times\mathbb{R})$ *be a solution of the problem*

$$\begin{cases}\widehat{\mathcal{L}}^{S,X} g(t,s,x) - \gamma(t,x)g(t,s,x) + U(t,s)\gamma(t,x) = 0, & (t,s,x)\in[0,T)\times\mathbb{R}^+\times\mathbb{R},\\ g(T,s,x) = G(T,s), \quad (s,x)\in\mathbb{R}^+\times\mathbb{R}.\end{cases} \quad (5.6)$$

Then, the $(\mathbb{F},\mathbb{G})$*-locally risk minimizing strategy is given by*

$$\theta^{\mathcal{F}}_t = \frac{\partial g}{\partial s}(t,S_t,X_t) + \rho\frac{a(t,X_t)}{S_t\sigma(t,S_t)}\frac{\partial g}{\partial x}(t,S_t,X_t), \quad t\in[\![0,T\wedge\tau]\!]. \quad (5.7)$$

Proof. First, note that if $g\in\mathcal{C}_b^{1,2,2}([0,T]\times\mathbb{R}^+\times\mathbb{R})$ is a solution of the problem (5.6) then the function $\widehat{g}\in\widehat{\mathcal{C}}_b^{1,2,2}([0,T]\times\mathbb{R}^+\times\mathbb{R}\times\{0,1\})$, defined as $\widehat{g}(t,s,x,0):=g(t,s,x)$ and $\widehat{g}(t,s,x,1):=0$ solves the backward Cauchy problem

$$\begin{cases}\widehat{\mathcal{L}}^{S,X,H}\widehat{g}(t,s,x,z) + U(t,s)(1-z)\gamma(t,x) = 0, & (t,s,x,z)\in[0,T)\times\mathbb{R}^+\times\mathbb{R}\times\{0,1\},\\ \widehat{g}(T,s,x,z) = (1-z)G(T,s), & (s,x,z)\in\mathbb{R}^+\times\mathbb{R}\times\{0,1\}.\end{cases}$$

By Lemma 5.2 and the Feynman-Kac formula we have that

$$\widehat{g}(t,S^\tau_t,X^\tau_t,H_t) = \widehat{\mathbb{E}}\left[G(T,S^\tau_T)(1-H_T) + \int_t^T U(r,S^\tau_r)(1-H_r)\gamma(r,X^\tau_r)\mathrm{d}r\middle|\mathcal{G}_t\right]$$

and the following $(\mathbb{G}, \widehat{\mathbf{P}})$-martingale decomposition of $\widehat{V}$ holds,

$$\mathrm{d}\widehat{V}_t = \frac{\partial \widehat{g}}{\partial s}\mathrm{d}S^\tau_t + \frac{\partial \widehat{g}}{\partial x}\mathrm{d}\widetilde{X}^\tau_t + \{\widehat{g}(t, S^\tau_t, X^\tau_t, H_{t^-} + 1) - \widehat{g}(t, S^\tau_t, X^\tau_t, H_{t^-})\}\mathrm{d}M_t,$$

where $\widetilde{X}^\tau$ denotes the martingale part of X^τ, that is

$$\widetilde{X}^\tau_t = \rho \int_0^{t\wedge\tau} a(u, X^\tau_u)\mathrm{d}\widehat{W}^\tau_u + \sqrt{1-\rho^2}\int_0^{t\wedge\tau} a(u, X^\tau_u)\mathrm{d}B^\tau_u, \quad t \in [0, T].$$

Then, taking the predictable covariation with respect to S^τ one immediately obtains

$$\mathrm{d}\langle \widehat{V}, S^\tau\rangle^{\widehat{\mathbf{P}}}_t = \frac{\partial \widehat{g}}{\partial s}(t, S^\tau_t, X^\tau_t, H_{t^-})\mathrm{d}\langle S^\tau\rangle^{\widehat{\mathbf{P}}}_t + \frac{\partial \widehat{g}}{\partial x}(t, S^\tau_t, X^\tau_t, H_{t^-})\mathrm{d}\langle \widetilde{X}^\tau, S^\tau\rangle^{\widehat{\mathbf{P}}}_t,$$

with $\mathrm{d}\langle \widetilde{X}^\tau, S^\tau\rangle^{\widehat{\mathbf{P}}}_t = \rho \dfrac{a(t, X^\tau_t)}{S^\tau_t \sigma(t, S^\tau_t)}\mathrm{d}\langle S^\tau\rangle^{\widehat{\mathbf{P}}}_t$. The expression of $\theta^{\mathcal{F}}_t$ easily follows from (4.19), observing that $\widehat{g}(t, S^\tau_t, X^\tau_t, H_{t^-}) = \widehat{g}(t, S^\tau_t, X^\tau_t, 0) = g(t, S_t, X_t)$ for any $t \in [\![0, T \wedge \tau]\!]$. □

Remark 5.4. *Note that if* $\rho = 0$ *in* (5.7)*, then* $\theta^{\mathcal{F}}$ *reduces to one single term of a delta hedge type, as in the classical Black & Scholes model. The additional term is a correction term due to correlation. Such a representation is similar to that obtained in stochastic volatility models in a Brownian motion setting, see e.g. [43, Proposition 1] or [36, Equation 7].*

Remark 5.5. *Existence and uniqueness of classical solutions to* (5.6) *can be obtained under suitable assumptions by applying the results in Heath and Schweizer [30].*

Remark 5.6. *By the Feynmann-Kac formula the process* $\{g(t, S_t, X_t),\ t \in [0, T]\}$ *has the following stochastic representation*

$$g(t, S_t, X_t) = \widehat{\mathbb{E}}\left[e^{-\int_t^T \gamma(r, X_r)\mathrm{d}r} G(T, S_T) + \int_t^T e^{-\int_t^r \gamma(u, X_u)\mathrm{d}u} U(r, S_r)\gamma(r, X_r)\mathrm{d}r \middle| \mathcal{F}_t\right]. \quad (5.8)$$

5.1. **A filtering approach to local risk-minimization under partial information.** In this section we wish to apply some results from filtering theory to compute the locally risk-minimizing hedging strategy under partial information. Precisely, this requires to compute conditional expectations of processes that depend on the trajectories of X. To apply the classical methodology, we introduce as an additional state process, the $\mathbb{F}$-survival process of τ given by $\mathbf{P}(\tau > t|\mathcal{F}_t) = 1 - F_t = e^{-\int_0^t \gamma(u, X_u)\mathrm{d}u}$, for each $t \in [0, T]$, and denote it by Y_t. The dynamics of $Y = \{Y_t,\ t \in [0, T]\}$, is given by

$$\mathrm{d}Y_t = -\gamma(t, X_t)Y_t\mathrm{d}t, \quad Y_0 = 1. \quad (5.9)$$

Remark 5.7. *Following the same lines of the proof of Lemma 5.1 for the triplet* (S, X, Y)*, it is easy to verify that the vector process* (S, X, Y) *is an* $(\mathbb{F}, \widehat{\mathbf{P}})$*-Markov process. Then, considering the dynamics of the processes* S*,* X *and* Y *in system* (5.1) *and equation* (5.9)*, and applying Itô's formula, we get that for every function* $f \in \mathcal{C}_b^{1,2,2,1}([0, T] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+)$*, the process*

$\{f(t,S_t,X_t,Y_t),\ t\in[0,T]\}$ *has the following semimartingale decomposition*

$$f(t,S_t,X_t,Y_t)=f(0,s_0,x_0,1)+\int_0^t \widehat{\mathcal{L}}^{S,X,Y} f(u,S_u,X_u,Y_u)\mathrm{d}u+M^f_t,\quad t\in[0,T], \tag{5.10}$$

where $M^f=\{M^f_t,\ t\in[0,T]\}$ *is the* $(\mathbb{F},\widehat{\mathbf{P}})$*-martingale given by*

$$\mathrm{d}M^f_t=\frac{\partial f}{\partial x}a(t,X_t)\left[\rho\mathrm{d}\widehat{W}_t+\sqrt{1-\rho^2}\mathrm{d}B_t\right]+\frac{\partial f}{\partial s}\sigma(t,S_t)S_t\mathrm{d}\widehat{W}_t,$$

and $\widehat{\mathcal{L}}^{S,X,Y}$ *given by*

$$\widehat{\mathcal{L}}^{S,X,Y} f(t,s,x,y)=\widehat{\mathcal{L}}^{S,X} f(t,s,x,y)-y\gamma(t,x)\frac{\partial f}{\partial y}(t,s,x,y)$$

provides the $(\mathbb{F},\widehat{\mathbf{P}})$*-Markov generator of* (S,X,Y).

For every measurable function f such that $\widehat{\mathbb{E}}\left[|f(t,S_t,X_t,Y_t)|\right]<\infty$, for each $t\in[0,T]$, we define the *filter* $\pi(f)=\{\pi_t(f),\ t\in[0,T]\}$ with respect to $\widehat{\mathbf{P}}$, by setting

$$\pi_t(f):=\widehat{\mathbb{E}}\left[f(t,S_t,X_t,Y_t)|\mathcal{F}^S_t\right],\quad t\in[0,T].$$

It is well known that π is a probability measure-valued process with càdlàg trajectories (see, e.g. Kurtz and Ocone [34]), and provides the $\widehat{\mathbf{P}}$-conditional law of the stochastic factor X given the filtration generated by the risky asset prices process. The filter dynamics is given in Proposition 5.9 below.

Assumption 5.8. *The functions* b*,* a*,* γ*,* μ*, and* σ *are jointly continuous and satisfy the following growth and locally Lipschitz conditions:*

(G) *for some nonnegative constant* C*, and for every* $(t,s,x)\in[0,T]\times\mathbb{R}^+\times\mathbb{R}$*,*

$$|b(t,x)|^2+|a(t,x)|^2+|\gamma(t,x)|^2\le C(1+|x|^2),$$
$$|\mu(t,s,x)|^2\le C(1+|s|^2+|x|^2)\ \text{and}\ |\sigma(t,s)|^2\le C(1+|s|^2);$$

(LL) *for all* $r>0$ *there exists a constant* L *such that for every* $t\in[0,T]$*,* $s,s',x,x'\in B_r(0):=\{z\in\mathbb{R}:\quad |z|\le r\}$*,*

$$|b(t,x)-b(t,x')|+|a(t,x)-a(t,x')|+|\gamma(t,x)-\gamma(t,x')|\le L|x-x'|,$$
$$|\mu(t,s,x)-\mu(t,s',x')|\le L(|s-s'|+|x-x'|)\ \text{and}\ |\sigma(t,s)-\sigma(t,s')|\le L|s-s'|.$$

Proposition 5.9. *Under Assumptions 2.1 and 5.8 and condition* (5.2)*, for every function* $f\in\mathcal{C}^{1,2,2,1}_b([0,T]\times\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^+)$ *and* $t\in[0,T]$*, the filter* π *is the unique strong solution of the following equation*

$$\pi_t(f)=f(0,s_0,x_0,1)+\int_0^t \pi_u(\widehat{\mathcal{L}}^{S,X,Y} f)\mathrm{d}u+\int_0^t\left[\rho\pi_u\left(a\ \frac{\partial f}{\partial x}\right)+S_u\sigma(t,S_u)\pi_u\left(\frac{\partial f}{\partial s}\right)\right]\mathrm{d}\widehat{W}_u. \tag{5.11}$$

The proof is postponed to Appendix B.2.

Now, we can characterize the optimal hedging strategy for the given endowment insurance contract (ξ, Z, τ) under partial information as follows.

Theorem 5.10. *Assume that the hypotheses of Proposition 5.9 hold and that $N_{T\wedge\tau}$ is $\widehat{\mathbf{P}}$-square integrable. Let g be a solution to problem (5.6). Then, the first component θ^* of the $(\mathbb{F}^S, \widetilde{\mathbb{G}})$-locally risk-minimizing strategy for the payment stream N associated with the unit-linked life insurance contract (ξ, Z, τ) is given by*

$$\theta^*_t = \frac{\pi_t\left(id_y \frac{\partial g}{\partial s}\right) + \frac{\rho}{\sigma(t,S_t)S_t}\pi_t\left(a\ id_y \frac{\partial g}{\partial x}\right)}{\pi_t(id_y)}, \tag{5.12}$$

for every $t \in [\![0, T\wedge\tau]\!]$, where $id_y(t,s,x,y) := y$.

Proof. By equation (4.17) in Theorem 4.16 and (5.7) we get

$$\begin{aligned}\theta^*_t &= \frac{{}^{\widehat{p},\mathbb{F}^S}\left(\theta^{\mathcal{F}}_t e^{-\int_0^t \gamma(u,X_u)\mathrm{d}u}\right)}{{}^{\widehat{p},\mathbb{F}^S}\left(e^{-\int_0^t \gamma_u \mathrm{d}u}\right)}\\ &= \frac{{}^{\widehat{p},\mathbb{F}^S}\left(e^{-\int_0^t \gamma(u,X_u)\mathrm{d}u}\frac{\partial g}{\partial s}(t,S_t,X_t)\right)}{{}^{\widehat{p},\mathbb{F}^S}\left(e^{-\int_0^t \gamma_u \mathrm{d}u}\right)} + \frac{{}^{\widehat{p},\mathbb{F}^S}\left(e^{-\int_0^t \gamma(u,X_u)\mathrm{d}u}\frac{\rho a(t,X_t)}{S_t\sigma(t,S_t)}\frac{\partial g}{\partial x}(t,S_t,X_t)\right)}{{}^{\widehat{p},\mathbb{F}^S}\left(e^{-\int_0^t \gamma_u \mathrm{d}u}\right)},\end{aligned}$$

for every $t \in [\![0, T\wedge\tau]\!]$. Finally, (5.12) follows by the definition of the filter. ☐

5.2. **An example with uncorrelated Brownian motions.** Throughout this section we choose $\rho = 0$, which corresponds to the case where W and B are $\mathbf{P}$-independent, and therefore $\widehat{W}$ and B are $\widehat{\mathbf{P}}$-independent. In this case, a simpler expression for the first component of the optimal hedging strategy θ^* under partial information is provided.

On the probability space $(\Omega, \mathcal{F}, \widehat{\mathbf{P}})$ the dynamics of the vector process (S, X, Y) is given by

$$\begin{cases} \mathrm{d}S_t = S_t\sigma(t,S_t)\mathrm{d}\widehat{W}_t, & S_0 = s_0 \in \mathbb{R}^+,\\ \mathrm{d}X_t = b(t,X_t)\mathrm{d}t + a(t,X_t)\mathrm{d}B_t, & X_0 = x_0 \in \mathbb{R},\\ \mathrm{d}Y_t = -Y_t\gamma(t,X_t)\mathrm{d}t, & Y_0 = 1.\end{cases}$$

Moreover, we choose a recovery function of the form $U(t,s) = \delta s$, for every $(t,s) \in [0,T]\times\mathbb{R}^+$, where δ is a given positive constant. Then, the payment stream N is given by $N_t = \delta\int_0^t S_u \mathrm{d}H_u$ if $t \in [0,T)$ and $N_T = G(T, S_T)\mathbf{1}_{\{\tau > T\}}$.

In the sequel we wish to characterize the optimal hedging strategy under full information, given in (5.7), and under partial information via (4.17), in this simpler example. This requires to compute g in equation (5.8).

The independence between X and S under $\widehat{\mathbf{P}}$ (that also holds when conditioning on $\mathcal{F}_t$, for each t), implies that

$$\begin{aligned}\widehat{\mathbb{E}}\left[G(T,S_T)e^{-\int_t^T \gamma(u,X_u)\mathrm{d}u}\middle|\mathcal{F}_t\right] &= \widehat{\mathbb{E}}\left[G(T,S_T)|\mathcal{F}_t\right]\widehat{\mathbb{E}}\left[e^{-\int_t^T \gamma(u,X_u)\mathrm{d}u}\middle|\mathcal{F}_t\right]\\ &= \widetilde{g}(t,S_t)\frac{\widehat{\mathbb{E}}\left[Y_T|\mathcal{F}_t\right]}{Y_t} = \widetilde{g}(t,S_t)\widehat{\mathbb{E}}\left[Y_T|\mathcal{F}_t\right]e^{\int_0^t \gamma(r,X_r)\mathrm{d}r},\end{aligned}$$

for every $t \in [0,T]$, where by the Feynman-Kac theorem the function $\widetilde{g}$ can be characterized as the solution of the problem

$$\begin{cases}\dfrac{\partial \widetilde{g}}{\partial t}(t,s) + \dfrac{\partial^2 \widetilde{g}}{\partial s^2}(t,s)\sigma^2(t,s)s^2 = 0, & (t,s) \in [0,T) \times \mathbb{R}^+,\\ \widetilde{g}(T,s) = G(T,s), & s \in \mathbb{R}^+.\end{cases}$$

Then, for the remaining part of the conditional expectation in (5.8), using the $\widehat{\mathbf{P}}$-independence between (X,Y) and S and the fact that S is an $(\mathbb{F},\widehat{\mathbf{P}})$-martingale, we have

$$\begin{aligned}\delta\widehat{\mathbb{E}}\left[\int_t^T e^{-\int_t^r \gamma(u,X_u)\mathrm{d}u}S_r\gamma(r,X_r)\mathrm{d}r\middle|\mathcal{F}_t\right] &= \delta\widehat{\mathbb{E}}\left[\int_t^T \frac{Y_r}{Y_t}S_r\gamma(r,X_r)\mathrm{d}r\middle|\mathcal{F}_t\right]\\ = -\frac{\delta}{Y_t}\widehat{\mathbb{E}}\left[\int_t^T S_r\mathrm{d}Y_r\middle|\mathcal{F}_t\right] &= -\frac{\delta}{Y_t}\widehat{\mathbb{E}}\left[\int_t^T \mathrm{d}(S_rY_r)\middle|\mathcal{F}_t\right] + \frac{\delta}{Y_t}\widehat{\mathbb{E}}\left[\int_t^T Y_r\mathrm{d}S_r\middle|\mathcal{F}_t\right]\\ = -\frac{\delta}{Y_t}\widehat{\mathbb{E}}\left[S_TY_T - S_tY_t\middle|\mathcal{F}_t\right] &= \frac{\delta S_t}{Y_t}\left(Y_t - \widehat{\mathbb{E}}\left[Y_T|\mathcal{F}_t\right]\right).\end{aligned}$$

This implies that

$$\begin{aligned}g(t,X_t,S_t) &= \widetilde{g}(t,S_t)\widehat{\mathbb{E}}\left[Y_T|\mathcal{F}_t\right]e^{\int_0^t \gamma(r,X_r)\mathrm{d}r} + \frac{\delta S_t}{Y_t}\left(Y_t - \widehat{\mathbb{E}}\left[Y_T|\mathcal{F}_t\right]\right)\\ &= \left(\widetilde{g}(t,S_t) - \delta S_t\right)e^{\int_0^t \gamma(r,X_r)\mathrm{d}r}\widehat{\mathbb{E}}\left[Y_T|\mathcal{F}_t\right] + \delta S_t.\end{aligned}$$

Remark 5.11. *Note that for every $t \in [0,T]$,*

$$\widehat{\mathbb{E}}\left[Y_T|\mathcal{F}_t\right] = e^{-\int_0^t \gamma(u,X_u)\mathrm{d}u}\widehat{\mathbb{E}}\left[e^{-\int_t^T \gamma(u,X_u)\mathrm{d}u}\middle|\mathcal{F}_t\right] = e^{-\int_0^t \gamma(u,X_u)\mathrm{d}u}\widehat{\mathbb{E}}\left[e^{-\int_t^T \gamma(u,X_u)\mathrm{d}u}\middle|\mathcal{F}_t^B\right],$$

where the last equality follows by the independence of X and W under $\widehat{\mathbf{P}}$. By the Feynman-Kac theorem, if there exists a function $\Phi \in \mathcal{C}_b^{1,2}([0,T] \times \mathbb{R})$ which solves the problem

$$\begin{cases}\dfrac{\partial \Phi}{\partial t}(t,x) + \dfrac{\partial \Phi}{\partial x}(t,x)b(t,x) + \dfrac{1}{2}\dfrac{\partial^2 \Phi}{\partial x^2}(t,x)a^2(t,x) - \Phi(t,x)\gamma(t,x) = 0, & (t,x) \in [0,T) \times \mathbb{R},\\ \Phi(T,x) = 1, & x \in \mathbb{R},\end{cases}$$

then, $\Phi(t,X_t) = \widehat{\mathbb{E}}\left[e^{-\int_t^T \gamma(u,X_u)\mathrm{d}u}\middle|\mathcal{F}_t^B\right]$ and the process $\left\{e^{-\int_0^t \gamma(u,X_u)\mathrm{d}u}\Phi(t,X_t),\ t \in [0,T]\right\}$ is an $(\mathbb{F}^B,\widehat{\mathbf{P}})$-martingale.

Hence, $g(t, S_t, X_t) = \widetilde{g}(t, S_t)\Phi(t, X_t) + \delta S_t(1 - \Phi(t, X_t))$ and by using (5.7) the optimal hedging strategy under full information is given by

$$\theta_t^{\mathcal{F}} = \left(\frac{\partial \widetilde{g}}{\partial s}(t, S_t) - \delta\right)\Phi(t, X_t) + \delta, \quad t \in [\![0, T \wedge \tau]\!].$$

Finally, by (4.17) we get that the $(\mathbb{F}^S, \widetilde{\mathbb{G}})$-locally risk-minimizing strategy can be written as

$$\theta_t^* = \frac{\left(\frac{\partial \widetilde{g}}{\partial s}(t, S_t) - \delta\right)\pi_t\left(id_y\ \Phi\right)}{\pi_t\left(id_y\right)} + \delta, \quad t \in [\![0, T \wedge \tau]\!]. \tag{5.13}$$

Note that, by the $\widehat{\mathbf{P}}$-independence of (X, Y) and S, and the fact that the change of probability measure from $\mathbf{P}$ to $\widehat{\mathbf{P}}$ does not affect the law of X, we have that the computation of the filter reduces to ordinary expectations with respect to $\mathbf{P}$

$$\pi_t(\Phi\ id_y) = \widehat{\mathbb{E}}\left[\Phi(t, X_t)e^{-\int_0^t \gamma(u, X_u)\mathrm{d}u}\middle|\mathcal{F}_t^S\right] = \widehat{\mathbb{E}}\left[\Phi(t, X_t)e^{-\int_0^t \gamma(u, X_u)\mathrm{d}u}\right] = \Phi(0, x_0) = \mathbb{E}\left[Y_T\right],$$
$$\pi_t(id_y) = \widehat{\mathbb{E}}\left[e^{-\int_0^t \gamma(u, X_u)\mathrm{d}u}\middle|\mathcal{F}_t^S\right] = \widehat{\mathbb{E}}\left[e^{-\int_0^t \gamma(u, X_u)\mathrm{d}u}\right] = \widehat{\mathbb{E}}\left[Y_t\right] = \mathbb{E}\left[Y_t\right],$$

for every $t \in [0, T]$. Then, we can write (5.13) as

$$\theta_t^* = \frac{\left(\frac{\partial \widetilde{g}}{\partial s}(t, S_t) - \delta\right)\mathbb{E}\left[Y_T\right] + \delta\mathbb{E}\left[Y_t\right]}{\mathbb{E}\left[Y_t\right]}, \quad t \in [\![0, T \wedge \tau]\!],$$

where $\mathbb{E}\left[Y_t\right] = \mathbb{E}\left[e^{-\int_0^t \gamma(u, X_u)\mathrm{d}u}\right]$, $t \in [0, T]$.

Acknowledgements

The authors wish to acknowledge Professor Monique Jeanblanc for valuable comments and suggestions. The authors are also grateful to the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) for the financial support through project number 2016/000371.

Appendix A. The hazard process and the martingale hazard process of τ under partial information

We define the conditional distribution of τ with respect to $\mathcal{F}_t^S$, for every $t \in [0, T]$, as

$$F_t^S = \mathbf{P}(\tau \leq t|\mathcal{F}_t^S), \quad t \in [0, T].$$

By the tower rule it is easy to check that $F_t^S = \mathbb{E}\left[F_t|\mathcal{F}_t^S\right]$, for each $t \in [0, T]$. Hence, the assumption $F_t < 1$, for every $t \in [0, T]$, also implies that $F_t^S < 1$ for every $t \in [0, T]$.

We now introduce the $\mathbb{F}^S$-hazard process of τ under $\mathbf{P}$, $\Gamma^S = \{\Gamma_t^S,\ t \in [0, T]\}$, by setting

$$\Gamma_t^S = -\ln(1 - F_t^S), \quad t \in [0, T]. \tag{A.1}$$

Remark A.1. *Notice that the relation between the $\mathbb{F}$-hazard process Γ, see* (2.7)*, and the $\mathbb{F}^S$-hazard process Γ^S, see* (A.1)*, is given by*

$$e^{-\Gamma_t^S} = \mathbb{E}\left[e^{-\Gamma_t}|\mathcal{F}_t^S\right], \quad t \in [0,T].$$

If Γ^S is continuous and increasing, by Bielecki and Rutkowski [7, Proposition 5.1.3] the process $\{H_t - \Gamma^S_{t\wedge\tau},\ t \in [0,T]\}$ is a $(\widetilde{\mathbb{G}}, \mathbf{P})$-martingale. However, without these assumptions, we will prove in Proposition A.6 the existence of an $(\mathbb{F}^S, \widetilde{\mathbb{G}})$-martingale hazard process.

For the sake of clarity, we recall the definition of martingale hazard process in our setting.

Definition A.2. *An $\mathbb{F}^S$-predictable, increasing process $\Lambda = \{\Lambda_t,\ t \in [0,T]\}$ is called an $(\mathbb{F}^S, \widetilde{\mathbb{G}})$-martingale hazard process of the random time τ if and only if the process $\{H_t - \Lambda_{t\wedge\tau},\ t \in [0,T]\}$ follows a $(\widetilde{\mathbb{G}}, \mathbf{P})$-martingale.*

In general, the $(\mathbb{F}^S, \widetilde{\mathbb{G}})$-martingale hazard process does not coincide with the $\mathbb{F}^S$-hazard process Γ^S. This property is fulfilled if the martingale invariance property holds, that is, any $(\mathbb{F}^S, \mathbf{P})$-martingale turns out to be a $(\widetilde{\mathbb{G}}, \mathbf{P})$-martingale. In such a case, the $(\mathbb{F}^S, \widetilde{\mathbb{G}})$-martingale hazard process uniquely specifies the $\mathbb{F}^S$-survival probabilities of τ. Nevertheless, we do not make this assumption in the paper.

In order to derive the $(\mathbb{F}^S, \widetilde{\mathbb{G}})$-martingale hazard process of τ we need some preliminary results.

Recall that given any subfiltration $\mathbb{H} = \{\mathcal{H}_t,\ t \in [0,T]\}$ of $\mathbb{G}$, ${}^{o,\mathbb{H}}Y$ (respectively ${}^{p,\mathbb{H}}Y$) denotes the optional (respectively predictable) projection of a given $\mathbf{P}$-integrable, $\mathbb{G}$-adapted process Y with respect to $\mathbb{H}$ and $\mathbf{P}$.

Lemma A.3. *Given a $\mathbf{P}$-integrable, $\mathbb{G}$-adapted process Y, we have*

$$\mathbf{1}_{\{\tau>t\}}{}^{o,\widetilde{\mathbb{G}}}Y_t = \mathbf{1}_{\{\tau>t\}} \frac{{}^{o,\mathbb{F}^S}\left(Y_t \mathbf{1}_{\{\tau>t\}}\right)}{{}^{o,\mathbb{F}^S}\mathbf{1}_{\{\tau>t\}}}, \tag{A.2}$$

$$\mathbf{1}_{\{\tau\geq t\}}{}^{p,\widetilde{\mathbb{G}}}Y_t = \mathbf{1}_{\{\tau\geq t\}} \frac{{}^{p,\mathbb{F}^S}\left(Y_t \mathbf{1}_{\{\tau\geq t\}}\right)}{{}^{p,\mathbb{F}^S}\mathbf{1}_{\{\tau\geq t\}}}, \tag{A.3}$$

for each $t \in [0,T]$. Moreover, if Y is $\mathbb{F}$-predictable then

$$\mathbf{1}_{\{\tau\geq t\}}{}^{p,\widetilde{\mathbb{G}}}Y_t = \mathbf{1}_{\{\tau\geq t\}} \frac{{}^{p,\mathbb{F}^S}\left(Y_t e^{-\int_0^t \gamma_u \mathrm{d}u}\right)}{{}^{p,\mathbb{F}^S}\left(e^{-\int_0^t \gamma_u \mathrm{d}u}\right)}, \quad t \in [0,T]. \tag{A.4}$$

Proof. How to get formula (A.2) is shown in Bielecki and Rutkowski [7, Lemma 5.1.2].

To prove (A.3), recall that, since $F_t < 1$ for all $t \in [0,T]$, there exists an $\mathbb{F}^S$-predictable process $\widetilde{Y} = \{\widetilde{Y}_t,\ t \in [0,T]\}$ such that $\widetilde{Y}_t\, \mathbf{1}_{\{\tau\geq t\}} = {}^{p,\widetilde{\mathbb{G}}}Y_t\, \mathbf{1}_{\{\tau\geq t\}}$, $\mathbf{P}$-a.s. for each $t \in [0,T]$. By the predictable projection properties, for any $\mathbb{F}^S$-predictable process $\varphi = \{\varphi_t,\ t \in [0,T]\}$ and for each $t \in [0,T]$, we get

$$\mathbb{E}\left[\int_0^t \varphi_s \widetilde{Y}_s \ {}^{p,\mathbb{F}^S}\mathbf{1}_{\{\tau \geq s\}} \mathrm{d}s\right] = \mathbb{E}\left[\int_0^t \varphi_s \widetilde{Y}_s \mathbf{1}_{\{\tau \geq s\}} \mathrm{d}s\right] = \mathbb{E}\left[\int_0^t \varphi_s \mathbf{1}_{\{\tau \geq s\}} \ {}^{p,\widetilde{\mathbb{G}}}Y_s \mathrm{d}s\right]$$
$$= \mathbb{E}\left[\int_0^t \varphi_s \mathbf{1}_{\{\tau \geq s\}} Y_s \mathrm{d}s\right] = \mathbb{E}\left[\int_0^t \varphi_s \ {}^{p,\mathbb{F}^S}\left(\mathbf{1}_{\{\tau \geq s\}} Y_s\right) \mathrm{d}s\right],$$

since the process $\{\varphi_t \mathbf{1}_{\{\tau \geq t\}},\ t \in [0,T]\}$ is $\widetilde{\mathbb{G}}$-predictable.

Now consider the case where Y is $\mathbb{F}$-predictable. Since $\{{}^{o,\mathbb{F}}\mathbf{1}_{\{\tau > t\}} = e^{-\int_0^t \gamma_u \mathrm{d}u},\ t \in [0,T]\}$ is a continuous process, we get

$${}^{o,\mathbb{F}}\mathbf{1}_{\{\tau > t\}} = {}^{o,\mathbb{F}}\mathbf{1}_{\{\tau \geq t\}} = {}^{p,\mathbb{F}}\mathbf{1}_{\{\tau \geq t\}}, \quad t \in [0,T].$$

Finally, equation (A.4) is consequence of the following chains of equalities

$${}^{p,\mathbb{F}^S}\mathbf{1}_{\{\tau \geq t\}} = {}^{p,\mathbb{F}^S}\left({}^{p,\mathbb{F}}\mathbf{1}_{\{\tau \geq t\}}\right) = {}^{p,\mathbb{F}^S}\left(e^{-\int_0^t \gamma_u \mathrm{d}u}\right),$$

and

$${}^{p,\mathbb{F}^S}\left(Y_t \mathbf{1}_{\{\tau \geq t\}}\right) = {}^{p,\mathbb{F}^S}\left(Y_t \ {}^{p,\mathbb{F}}\mathbf{1}_{\{\tau \geq t\}}\right) = {}^{p,\mathbb{F}^S}\left(Y_t e^{-\int_0^t \gamma_u \mathrm{d}u}\right),$$

for every $t \in [0,T]$. □

Remark A.4. *Note that the $\mathbb{F}^S$-hazard process $\Gamma^S = \{\Gamma^S_t,\ t \in [0,T]\}$, can be written as*

$$\Gamma^S_t = -\ln\left({}^{o,\mathbb{F}^S}\left(e^{-\int_0^t \gamma_u \mathrm{d}u}\right)\right), \quad t \in [0,T].$$

Remark A.5. *Given a $(\mathbb{G}, \mathbf{P})$-martingale $m = \{m_t,\ t \in [0,T]\}$ and a $\mathbb{G}$-progressively measurable process $\psi = \{\psi_t,\ t \in [0,T]\}$ such that $\mathbb{E}\left[\int_0^T |\psi_u| \mathrm{d}u\right] < \infty$, the processes $\left\{{}^{o,\widetilde{\mathbb{G}}}m_t,\ t \in [0,T]\right\}$ and $\left\{{}^{o,\widetilde{\mathbb{G}}}\left(\int_0^t \psi_u \mathrm{d}u\right) - \int_0^t {}^{o,\widetilde{\mathbb{G}}}\psi_u \mathrm{d}u,\ t \in [0,T]\right\}$ are $(\widetilde{\mathbb{G}}, \mathbf{P})$-martingales, see for instance Ceci and Colaneri [13, Remark 2.1].*

Finally, we give the $(\mathbb{F}^S, \widetilde{\mathbb{G}})$-martingale hazard process of τ.

Proposition A.6. *The death time τ admits an $(\mathbb{F}^S, \widetilde{\mathbb{G}})$-martingale hazard process $\Lambda = \{\Lambda_t,\ t \in [0,T]\}$, where $\Lambda_t := \int_0^t \gamma^S_u \mathrm{d}u$, with $\gamma^S = \{\gamma^S_t,\ t \in [0,T]\}$ being a nonnegative, $\mathbb{F}^S$-predictable process. Moreover, for every $t \in [0,T]$,*

$$\gamma^S_t \mathbf{1}_{\{\tau \geq t\}} = {}^{p,\widetilde{\mathbb{G}}}\gamma_t \mathbf{1}_{\{\tau \geq t\}} \quad \mathbf{P}-a.s. \tag{A.5}$$

and

$$\gamma^S_t = \frac{{}^{p,\mathbb{F}^S}\left(\gamma_t e^{-\int_0^t \gamma_u \mathrm{d}u}\right)}{{}^{p,\mathbb{F}^S}\left(e^{-\int_0^t \gamma_u \mathrm{d}u}\right)}, \quad t \in [\![0, T \wedge \tau]\!].$$

Proof. By applying Remark A.5 to the $(\mathbb{G}, \mathbf{P})$-martingale M, see (2.8), we have that

$$\left\{ H_t - \int_0^t {}^{o,\widetilde{\mathbb{G}}}\lambda_u \mathrm{d}u,\ t \in [0,T] \right\}$$

is a $(\widetilde{\mathbb{G}}, \mathbf{P})$-martingale, which implies, taking Lemma B.1 into account, that also

$$\left\{ H_t - \int_0^t {}^{p,\widetilde{\mathbb{G}}}\lambda_u \mathrm{d}u = H_t - \int_0^{t\wedge\tau} {}^{p,\widetilde{\mathbb{G}}}\gamma_u \mathrm{d}u,\ t \in [0,T] \right\}$$

is a $(\widetilde{\mathbb{G}}, \mathbf{P})$-martingale.

Since $F_t < 1$ for all $t \in [0,T]$, for any $\widetilde{\mathbb{G}}$-predictable process $h = \{h_t,\ t \in [0,T]\}$ there exists an $\mathbb{F}^S$-predictable process $\widetilde{h} = \{\widetilde{h}_t,\ t \in [0,T]\}$ such that $\widetilde{h}_t \mathbf{1}_{\{\tau \geq t\}} = h_t \mathbf{1}_{\{\tau \geq t\}}$, $\mathbf{P}$-a.s. for each $t \in [0,T]$. This implies the existence of an $\mathbb{F}^S$-predictable process γ^S such that (A.5) is satisfied.

Hence, the process $\{\Lambda_t = \int_0^t \gamma_u^S \mathrm{d}u,\ t \in [0,T]\}$ is an $(\mathbb{F}^S, \widetilde{\mathbb{G}})$-martingale hazard process of τ since $H_t - \Lambda_{t\wedge\tau} = H_t - \int_0^{t\wedge\tau} \gamma_u^S \mathrm{d}u$, for each $t \in [0,T]$, is a $(\widetilde{\mathbb{G}}, \mathbf{P})$-martingale. To complete the proof is sufficient to apply the relation (A.4) in Lemma A.3. ☐

Note that Proposition A.6 ensures that τ turns out to be a totally inaccessible $\widetilde{\mathbb{G}}$-stopping time thanks to Dellacherie and Meyer [23, Chapter 6.78].

Appendix B. Technical results

B.1. **On optional and predictable projections under partial information.**

Lemma B.1. *Given a* $\mathbb{G}$*-progressively measurable process* $\psi = \{\psi_t,\ t \in [0,T]\}$ *such that* $\mathbb{E}\left[\int_0^T |\psi_u| \mathrm{d}u\right] < \infty$*, then*

$$\int_0^t {}^{o,\widetilde{\mathbb{G}}}\psi_u \mathrm{d}u = \int_0^t {}^{p,\widetilde{\mathbb{G}}}\psi_u \mathrm{d}u \quad \mathbf{P}-a.s. \quad t \in [0,T].$$

Proof. First, we prove that the process $U = \{U_t,\ t \in [0,T]\}$ given by $U_t := \int_0^t ({}^{o,\widetilde{\mathbb{G}}}\psi_u - {}^{p,\widetilde{\mathbb{G}}}\psi_u)\mathrm{d}u$, $t \in [0,T]$, is a $(\widetilde{\mathbb{G}}, \mathbf{P})$-martingale.

By the properties of predictable and optional projections, for any $\widetilde{\mathbb{G}}$-predictable process $\varphi = \{\varphi_t,\ t \in [0,T]\}$ we get

$$\mathbb{E}\left[\int_0^T \varphi_u {}^{p,\widetilde{\mathbb{G}}}\psi_u \mathrm{d}u\right] = \mathbb{E}\left[\int_0^T \varphi_u \psi_u \mathrm{d}u\right] = \mathbb{E}\left[\int_0^T \varphi_u {}^{o,\widetilde{\mathbb{G}}}\psi_u \mathrm{d}u\right].$$

By choosing $\varphi_u = \mathbf{1}_A \mathbf{1}_{(s,t]}(u)$, $s < t$, $A \in \widetilde{\mathcal{G}}_s$, we obtain that

$$\mathbb{E}\left[\mathbf{1}_A \int_s^t ({}^{p,\widetilde{\mathbb{G}}}\psi_u - {}^{o,\widetilde{\mathbb{G}}}\psi_u)\mathrm{d}u\right] = 0.$$

Finally, since U is a process of finite variation by Revuz and Yor [45, Chapter IV, Proposition 1.2], U is necessarily constant and equal to $U_0 = 0$, which concludes the proof. ☐

For reader's convenience, we provide a version of the Kallianpur-Striebel formula holding for predictable projections.

Lemma B.2. *If* $G = \{G_t,\ t \in [0,T]\}$ *is an* $\mathbb{F}$*-adapted process, such that* $\mathbb{E}[G_t L_t] < \infty$*, for any* $t \in [0,T]$*, then*

$$^{\widehat{p},\mathbb{F}^S}G_t = \frac{^{p,\mathbb{F}^S}(G_t L_t)}{^{p,\mathbb{F}^S}L_t}, \quad t \in [0,T],$$

where L *is the density process given in* (2.3)*.*

Proof. To prove the result, we need to check that for every $\mathbb{F}^S$-predictable process φ, the following equality holds

$$\widehat{\mathbb{E}}\left[\int_0^t \varphi_s\ ^{\widehat{p},\mathbb{F}^S}G_s\ ^{p,\mathbb{F}^S}L_s \mathrm{d}s\right] = \widehat{\mathbb{E}}\left[\int_0^t \varphi_s\ ^{p,\mathbb{F}^S}(G_s L_s)\,\mathrm{d}s\right],$$

for every $t \in [0,T]$. By applying Fubini's theorem twice, and the property of the predictable projection, for every $\mathbb{F}^S$-predictable process φ and for every $t \in [0,T]$, we get

$$\begin{aligned}\widehat{\mathbb{E}}\left[\int_0^t \varphi_s\ ^{\widehat{p},\mathbb{F}^S}G_s\ ^{p,\mathbb{F}^S}L_s \mathrm{d}s\right] &= \widehat{\mathbb{E}}\left[\int_0^t \varphi_s G_s\ ^{p,\mathbb{F}^S}L_s \mathrm{d}s\right] = \int_0^t \widehat{\mathbb{E}}\left[\varphi_s G_s\ ^{p,\mathbb{F}^S}L_s\right]\mathrm{d}s\\ &= \int_0^t \mathbb{E}\left[\varphi_s G_s L_s\ ^{p,\mathbb{F}^S}L_s\right]\mathrm{d}s = \int_0^t \mathbb{E}\left[\varphi_s\ ^{p,\mathbb{F}^S}(G_s L_s)\ ^{p,\mathbb{F}^S}L_s\right]\mathrm{d}s\\ &= \int_0^t \mathbb{E}\left[\varphi_s\ ^{p,\mathbb{F}^S}(G_s L_s)\,L_s\right]\mathrm{d}s = \widehat{\mathbb{E}}\left[\int_0^t \varphi_s\ ^{p,\mathbb{F}^S}(G_s L_s)\,\mathrm{d}s\right],\end{aligned}$$

which concludes the proof. □

If the process G is $\mathbb{G}$-adapted but not necessarily $\mathbb{F}$-adapted, then a similar result is showed in the following lemma.

Lemma B.3. *If* $G = \{G_t,\ t \in [0,T]\}$ *is a* $\mathbb{G}$*-adapted process, such that* $\mathbb{E}[G_t L_t] < \infty$*, for any* $t \in [0,T]$*, then*

$$\mathbf{1}_{\{\tau \geq t\}}{}^{\widehat{p},\widetilde{\mathbb{G}}}G_t = \mathbf{1}_{\{\tau \geq t\}}\frac{^{p,\widetilde{\mathbb{G}}}(G_t L_t)}{^{p,\widetilde{\mathbb{G}}}L_t}, \quad t \in [0,T].$$

Proof. Similarly to the proof of Lemma B.2, for every $\mathbb{G}$-adapted process G and every $\widetilde{\mathbb{G}}$-predictable process φ we have

$$\begin{aligned}\widehat{\mathbb{E}}\left[\int_0^t \mathbf{1}_{\{\tau\geq s\}}\varphi_s\ ^{\widehat{p},\widetilde{\mathbb{G}}}G_s\ ^{p,\widetilde{\mathbb{G}}}L_s \mathrm{d}s\right] &= \widehat{\mathbb{E}}\left[\int_0^t \mathbf{1}_{\{\tau\geq s\}}\varphi_s G_s\ ^{p,\widetilde{\mathbb{G}}}L_s \mathrm{d}s\right] = \int_0^t \widehat{\mathbb{E}}\left[\mathbf{1}_{\{\tau\geq s\}}\varphi_s G_s\ ^{p,\widetilde{\mathbb{G}}}L_s\right]\mathrm{d}s\\ &= \int_0^t \mathbb{E}\left[L_s^\tau \mathbf{1}_{\{\tau\geq s\}}\varphi_s G_s\ ^{p,\widetilde{\mathbb{G}}}L_s\right]\mathrm{d}s = \int_0^t \mathbb{E}\left[\mathbf{1}_{\{\tau\geq s\}}\varphi_s\ ^{p,\widetilde{\mathbb{G}}}(G_s L_s)\ ^{p,\widetilde{\mathbb{G}}}L_s\right]\mathrm{d}s\\ &= \int_0^t \mathbb{E}\left[\mathbf{1}_{\{\tau\geq s\}}\varphi_s\ ^{p,\widetilde{\mathbb{G}}}(G_s L_s)\,L_s\right]\mathrm{d}s = \widehat{\mathbb{E}}\left[\int_0^t \mathbf{1}_{\{\tau\geq s\}}\varphi_s\ ^{p,\widetilde{\mathbb{G}}}(G_s L_s)\,\mathrm{d}s\right],\end{aligned}$$

for every $t \in [0,T]$. Note that, in the second line, we use the fact that $L^\tau_t = L_t$ for every $t \in [\![0, T \wedge \tau]\!]$, where L^τ is the density process given in (4.10). □

Corollary B.4. *Let $\theta = \{\theta_t,\ t \in [0,T]\}$ be an $\mathbb{F}$-predictable process. Then,*

$$\mathbf{1}_{\{\tau \geq t\}}\ {}^{\widehat{p},\widetilde{\mathbb{G}}}\theta_t = \mathbf{1}_{\{\tau \geq t\}} \frac{{}^{\widehat{p},\mathbb{F}^S}\big(\theta_t e^{-\int_0^t \gamma_u \mathrm{d}u}\big)}{{}^{\widehat{p},\mathbb{F}^S}\big(e^{-\int_0^t \gamma_u \mathrm{d}u}\big)}, \quad t \in [0,T].$$

Proof. By Lemma B.3 we get

$$\begin{aligned}
\mathbf{1}_{\{\tau \geq t\}}{}^{\widehat{p},\widetilde{\mathbb{G}}}\theta_t &= \mathbf{1}_{\{\tau \geq t\}} \frac{{}^{p,\widetilde{\mathbb{G}}}\left(\theta_t L_t\right)}{{}^{p,\widetilde{\mathbb{G}}}L_t} \\
&= \mathbf{1}_{\{\tau \geq t\}} \frac{{}^{p,\mathbb{F}^S}\left(\theta_t L_t e^{-\int_0^t \gamma_u \mathrm{d}u}\right)}{{}^{p,\mathbb{F}^S}\left(e^{-\int_0^t \gamma_u \mathrm{d}u}\right)} \cdot \frac{{}^{p,\mathbb{F}^S}\left(e^{-\int_0^t \gamma_u \mathrm{d}u}\right)}{{}^{p,\mathbb{F}^S}\left(e^{-\int_0^t \gamma_u \mathrm{d}u} L_t\right)} && \text{(B.1)} \\
&= \mathbf{1}_{\{\tau \geq t\}} \frac{{}^{\widehat{p},\mathbb{F}^S}\big(\theta_t e^{-\int_0^t \gamma_u \mathrm{d}u}\big)}{{}^{\widehat{p},\mathbb{F}^S}\big(e^{-\int_0^t \gamma_u \mathrm{d}u}\big)} && \text{(B.2)}
\end{aligned}$$

where in line (B.1) we use Lemma A.3 and in line (B.2) we apply Lemma B.2. □

B.2. **Some proofs.**

Proof of Lemma 3.2. To prove that the process I^τ is a $(\widetilde{\mathbb{G}}, \mathbf{P})$-Brownian motion on $[\![0, \tau \wedge T]\!]$, we wish to apply the Lévy theorem. First, note that I^τ is a square integrable process with continuous trajectories, and since the following equality is fulfilled

$$I^\tau_t = \int_0^t \frac{1}{\sigma(u, S^\tau_u) S^\tau_u}\, \mathrm{d}S^\tau_u - \int_0^t \frac{{}^{p,\widetilde{\mathbb{G}}}\mu_u}{\sigma(u, S^\tau_u)}\, \mathrm{d}u, \quad t \in [0,T],$$

it turns out to be $\widetilde{\mathbb{G}}$-adapted. We now prove that I^τ is a $(\widetilde{\mathbb{G}}, \mathbf{P})$-martingale. As a consequence of Lemma B.1 in Appendix B, we can work with the $(\widetilde{\mathbb{G}}, \mathbf{P})$-optional projection of μ, that is ${}^{o,\widetilde{\mathbb{G}}}\mu$, instead of the $(\widetilde{\mathbb{G}}, \mathbf{P})$-predictable projection ${}^{p,\widetilde{\mathbb{G}}}\mu$. Hence, for every $0 \leq s \leq t \leq T$, we have

$$\mathbb{E}\left[I^\tau_t - I^\tau_s \middle| \widetilde{\mathcal{G}}_s\right] = \mathbb{E}\left[\int_{s\wedge\tau}^{t\wedge\tau} \frac{\mu(u, S^\tau_u, X^\tau_u) - {}^{o,\widetilde{\mathbb{G}}}\mu_u}{\sigma(u, S^\tau_u)}\, \mathrm{d}u \middle| \widetilde{\mathcal{G}}_s\right] + \mathbb{E}\left[W^\tau_t - W^\tau_s \middle| \widetilde{\mathcal{G}}_s\right].$$

By the properties of the conditional expectation we obtain that

$$\begin{aligned}
\mathbb{E}\left[I^\tau_t - I^\tau_s \middle| \widetilde{\mathcal{G}}_s\right] = &\int_s^t \mathbb{E}\left[\mathbb{E}\left[\frac{\mu(u, S^\tau_u, X^\tau_u)}{\sigma(u, S^\tau_u)}\mathbf{1}_{\{\tau > u\}} - {}^{o,\widetilde{\mathbb{G}}}\left(\frac{\mu_u}{\sigma_u}\mathbf{1}_{\{\tau > u\}}\right) \middle| \widetilde{\mathcal{G}}_u\right] \middle| \widetilde{\mathcal{G}}_s\right] \mathrm{d}u \\
&+ \mathbb{E}\left[\mathbb{E}\left[W^\tau_t - W^\tau_s \middle| \mathcal{G}_s\right] \middle| \widetilde{\mathcal{G}}_s\right].
\end{aligned}$$

Since $\mathbb{E}[W^\tau_t - W^\tau_s | \mathcal{G}_s] = 0$, finally we get

$$\mathbb{E}\left[I^\tau_t - I^\tau_s \middle| \widetilde{\mathcal{G}}_s\right] = \int_s^t \mathbb{E}\left[{}^{o,\widetilde{\mathbb{G}}}\left(\frac{\mu_u}{\sigma_u}\mathbf{1}_{\{\tau > u\}}\right) - {}^{o,\widetilde{\mathbb{G}}}\left(\frac{\mu_u}{\sigma_u}\mathbf{1}_{\{\tau > u\}}\right) \middle| \widetilde{\mathcal{G}}_s\right] \mathrm{d}u = 0.$$

To conclude, we apply the Lévy theorem taking into account that $\langle I^\tau \rangle = \langle W^\tau \rangle$. □

Proof of Lemma 5.1. Recall that the process $\widehat{W}$ given in (2.4) and B are independent $(\mathbb{F}, \widehat{\mathbf{P}})$-Brownian motions. Since the change of probability measure from $\mathbf{P}$ to $\widehat{\mathbf{P}}$ is Markovian, the pair (S, X) is still an $(\mathbb{F}, \widehat{\mathbf{P}})$-Markov process, see Ceci and Gerardi [15, Proposition 3.4]. Then, the Markov generator $\widehat{\mathcal{L}}^{S,X}$ of the pair (S, X) can be easily computed considering the semimartingale decompositions of the processes S and X with respect to filtration $\mathbb{F}$ and the measure $\widehat{\mathbf{P}}$ in system (5.1) and applying Itô's formula to any function $f \in C_b^{1,2,2}([0,T] \times \mathbb{R}^+ \times \mathbb{R})$. □

Proof of Lemma 5.2. In order to compute the $(\mathbb{G}, \widehat{\mathbf{P}})$-Markov generator of the process (S^τ, X^τ, H) we recall that the death indicator process H, is given by

$$H_t = \int_0^t (1 - H_r)\gamma(r, X_r)\mathrm{d}r + M_t, \quad t \in [0,T],$$

and that on the stochastic interval $[\![0, T \wedge \tau]\!]$, the dynamics of the stopped processes S^τ and X^τ are given by

$$\begin{aligned}
\mathrm{d}S_t^\tau &= (1 - H_{t-})S_t^\tau \sigma(t, S_t^\tau)\mathrm{d}\widehat{W}_t^\tau, \\
\mathrm{d}X_t^\tau &= (1 - H_{t-})\left\{ \left(b(t, X_t^\tau) - a(t, X_t^\tau)\rho \, \frac{\mu(t, S_t^\tau, X_t^\tau)}{\sigma(t, S_t^\tau)} \right)\mathrm{d}t + a(t, X_t^\tau)\left(\rho \mathrm{d}\widehat{W}_t^\tau + \sqrt{1-\rho^2}\mathrm{d}B_t^\tau \right) \right\}.
\end{aligned}$$

Finally, by applying Itô's formula to any function $f \in \widehat{C}_b^{1,2,2}([0,T] \times \mathbb{R}^+ \times \mathbb{R} \times \{0,1\})$, we have that, on the stochastic interval $[\![0, T \wedge \tau]\!]$,

$$\mathrm{d}f(t, S_t^\tau, X_t^\tau, H_t) = \widehat{\mathcal{L}}^{S,X,H} f(t, S_t^\tau, X_t^\tau, H_t)\mathrm{d}t + \mathrm{d}M_t^f,$$

where $\widehat{\mathcal{L}}^{S,X,H}$ is the operator given in (5.4) and M^f is the $(\mathbb{G}, \widehat{\mathbf{P}})$-martingale in (5.5). □

Proof of Proposition 5.9. First, observe that $\widehat{W}$ is an $(\mathbb{F}^S, \widehat{\mathbf{P}})$-Brownian motion since the following equality holds

$$\widehat{W}_t = \widetilde{I}_t + \int_0^t \frac{{}^{p,\mathbb{F}^S}\mu_u}{\sigma(u, S_u)}\mathrm{d}u, \quad t \in [0,T],$$

where $\{\widetilde{I}_t := W_t + \int_0^t \frac{\mu(u,S_u,X_u) - {}^{p,\mathbb{F}^S}\mu_u}{\sigma(u,S_u)}\mathrm{d}u, \ t \in [0,T]\}$ is the so-called innovation process which is known to be an $(\mathbb{F}^S, \mathbf{P})$-Brownian motion (see, for instance Liptser and Shiryaev [38]).

Recalling the semimartingale decomposition of $f(t, S_t, X_t, Y_t)$, given in (5.10), we can proceed as in the proof of Ceci et al. [19, Proposition A.2] and prove that the filter π solves equation (5.11).

Strong uniqueness for the solution to the filtering equation follows by uniqueness of the *filtered martingale problem* for the operator $\widehat{\mathcal{L}}^{S,X,Y}$ (see, e.g. Kurtz and Ocone [34], Ceci and Colaneri [13], Ceci and Colaneri [14]). Precisely, by applying Kurtz and Ocone [34, Theorem 3.3] we get that the filtered martingale problem for the operator $\widehat{\mathcal{L}}^{S,X,Y}$ has a unique solution, and this implies uniqueness of equation (5.11). □

References

[1] J. Ansel and C. Stricker. Unicité et existence de la loi minimale. In *Séminaire de Probabilités XXVII*, pages 22–29. Springer, 1993.

[2] J. Barbarin. Risk minimizing strategies for life insurance contracts with surrender option. *Available at SSRN 1334580*, 2007.

[3] J. Barbarin. *Valuation and Hedging in Insurance: Applications to Life and Non-Life Insurance Contracts*. VDM Publishing, 2009.

[4] F. Biagini and A. Cretarola. Local risk-minimization for defaultable claims with recovery process. *Applied Mathematics & Optimization*, 65(3):293–314, 2012.

[5] F. Biagini, T. Rheinländer, and I. Schreiber. Risk-minimization for life insurance liabilities with basis risk. *Mathematics and Financial Economics*, 10(2):151–178, 2016.

[6] F. Biagini, C. Botero, and I. Schreiber. Risk-minimization for life insurance liabilities with dependent mortality risk. *Mathematical Finance*, 27(2):505–533, 2017.

[7] T. R. Bielecki and M. Rutkowski. *Credit Risk: Modeling, Valuation and Hedging*. Springer Finance. Springer-Verlag Berlin, Heidelberg, New York, 2004.

[8] T. R. Bielecki, M. Jeanblanc, and M. Rutkowski. Stochastic methods in credit risk modelling, valuation and hedging. *CIME-EMS Summer School on Stochastic Methods in Finance, Bressanone, Lecture Notes in Mathematics. Springer*, 2004.

[9] T. R. Bielecki, M. Jeanblanc, and M. Rutkowski. Completeness of a general semimartingale market under constrained trading. In *Stochastic Finance*, pages 83–106. Springer, 2006.

[10] T. R. Bielecki, M. Jeanblanc, and M. Rutkowski. Hedging of credit derivatives in models with totally unexpected default. In J. Akahori, S. Ogawa, and S. Watanabe, editors, *Stochastic Processes and Applications to Mathematical Finance. Proceedings of the 5-th Ritsumeikan International Symposium, Ritsumeikan University, Japan, 3-6 March 2005*, pages 35–100. World Scientific, Singapore, 2006.

[11] T. R. Bielecki, M. Jeanblanc, and M. Rutkowski. Replication of contingent claims in a reduced-form credit risk model with discontinuous asset prices. *Stochastic Models*, 22(4):661–687, 2006.

[12] C. Blanchet-Scalliet and M. Jeanblanc. Hazard rate for credit risk and hedging defaultable contingent claims. *Finance and Stochastics*, 8(1):145–159, 2004.

[13] C. Ceci and K. Colaneri. Nonlinear filtering for jump diffusion observations. *Advances in Applied Probability*, 44(03):678–701, 2012.

[14] C. Ceci and K. Colaneri. The Zakai equation of nonlinear filtering for jump-diffusion observations: existence and uniqueness. *Applied Mathematics & Optimization*, 69(1):47–82, 2014.

[15] C. Ceci and A. Gerardi. Nonlinear filtering equation of a jump process with counting observations. *Acta Applicandae Mathematica*, 66(2):139–154, 2001.

[16] C. Ceci, A. Cretarola, and F. Russo. BSDEs under partial information and financial applications. *Stochastic Processes and their Applications*, 124(8):2628–2653, 2014.

[17] C. Ceci, A. Cretarola, and F. Russo. GKW representation theorem under restricted information: An application to risk-minimization. *Stochastics & Dynamics*, 14(02):1350019, 2014.

[18] C. Ceci, K. Colaneri, and A. Cretarola. Hedging of unit-linked life insurance contracts with unobservable mortality hazard rate via local risk-minimization. *Insurance: Mathematics and Economics*, 60:47–60, 2015.

[19] C. Ceci, K. Colaneri, and A. Cretarola. Local risk-minimization under restricted information to asset prices. *Electronic Journal of Probability*, 20(96):1–30, 2015.

[20] T. Choulli, N. Vandaele, and M. Vanmaele. The Föllmer-Schweizer decomposition: comparison and description. *Stochastic Processes and their Applications*, 120(6):853–872, 2010.

[21] T. Choulli, C. Daveloose, and M. Vanmaele. Hedging mortality risk and optional martingale representation theorem for enlarged filtration. *Preprint*, 2015.

[22] M. Dahl and T. Møller. Valuation and hedging of life insurance liabilities with systematic mortality risk. *Insurance: Mathematics and Economics*, 39(2):193–217, 2006.

[23] C. Dellacherie and P. Meyer. *Probabilities and Potential B*, volume 72 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1982.

[24] C. Dellacherie, B. Maisonneuve, and P.A. Meyer. *Probabilités et potentiel. Chapitres XVII à XXIV. Processus de Markov (fin), compléments de calcul stochastique*. Hermann, 1992.

[25] R. J. Elliott, M. Jeanblanc, and M. Yor. On models of default risk. *Mathematical Finance*, 10(2):179–195, 2000.

[26] H. Föllmer and M. Schweizer. Minimal martingale measure. *Encyclopedia of Quantitative Finance*, 2010.

[27] R. Frey and W. Runggaldier. Pricing credit derivatives under incomplete information: a nonlinear-filtering approach. *Finance and Stochastics*, 14(4):495–526, 2010.

[28] R. Frey and T. Schmidt. Pricing and hedging of credit derivatives via the innovations approach to nonlinear filtering. *Finance and Stochastics*, 16(1):105–133, 2012.

[29] M. Gantenbein and M. A. Mata. *Swiss Annuities and Life Insurance: Secure Returns, Asset Protection, and Privacy*, volume 400. John Wiley & Sons, 2008.

[30] D. Heath and M. Schweizer. Martingales versus PDEs in finance: an equivalence result with examples. *Journal of Applied Probability*, 37(4):947–957, 2000.

[31] T. Jeulin. *Semi-martingales et grossissement d'une filtration*, volume 833 of *Lecture Notes in Mathematics*. Springer, 1980.

[32] T. Jeulin and M. Yor. Grossissement d'une filtration et semi-martingales: formules explicites. In *Séminaire de Probabilités XII*, pages 78–97. Springer, 1978.

[33] T. Jeulin and M. Yor. *Grossissements de filtrations: exemples et applications: Séminaire de Calcul Stochastique 1982/83 Université Paris VI*, volume 1118 of *Lecture Notes in Mathematics*. Springer, 1985.

[34] T. G. Kurtz and D. L. Ocone. Unique characterization of conditional distributions in nonlinear filtering. *The Annals of Probability*, pages 80–107, 1988.

[35] S. Kusuoka. A remark on default risk models. In *Advances in Mathematical Economics*, pages 69–82. Springer, 1999.

[36] P. Leoni, N. Vandaele, and M. Vanmaele. Hedging strategies for energy derivatives. *Quantitative Finance*, 14(10):1725–1737, 2014.

[37] J. Li and A. Szimayer. The uncertain mortality intensity framework: Pricing and hedging unit-linked life insurance contracts. *Insurance: Mathematics and Economics*, 49(3):471–486, 2011.

[38] R. Liptser and A. N. Shiryaev. *Statistics of Random Processes: I. General Theory*, volume 5 of *Applications of Mathematics*. Springer Science & Business Media, 2013.

[39] T. Møller. Risk-minimizing hedging strategies for unit-linked life insurance contracts. *Astin Bulletin*, 28(01):17–47, 1998.

[40] T. Møller. Risk-minimizing hedging strategies for insurance payment processes. *Finance and Stochastics*, 5(4):419–446, 2001.

[41] P. Monat and C. Stricker. Föllmer-Schweizer decomposition and mean-variance hedging for general claims. *The Annals of Probability*, 23(2):605–628, 1995.

[42] B. Øksendal. *Stochastic Differential Equations: an Introduction with Applications*. Springer Science & Business Media, fifth edition, 2013.

[43] R. Poulsen, K. R. Schenk-Hoppé, and C.-O. Ewald. Risk minimization in stochastic volatility models: Model risk and empirical performance. *Quantitative Finance*, 9(6):693–704, 2009.

[44] Y. V. Prokhorov and A. N. Shiryaev, editors. *Probability Theory III*, volume 45 of *Encyclopaedia of Mathematical Sciences*. Springer Verlag Berlin, Heidelberg, New York, 1998.

[45] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293. Springer Science & Business Media, 2013.

[46] M. Schweizer. Approximating random variables by stochastic integrals. *The Annals of Probability*, 22(3):1536–1575, 1994.

[47] M. Schweizer. On the minimal martingale measure and the Föllmer-Schweizer decomposition. *Stochastic Analysis and Applications*, 13(5):573–599, 1995.

[48] M. Schweizer. A guided tour through quadratic hedging approaches. In E. Jouini, J. Cvitanic, and M. Musiela, editors, *Option Pricing, Interest Rates and Risk Management*, pages 538–574. Cambridge University Press, Cambridge, 2001.

[49] M. Schweizer. Local risk-minimization for multidimensional assets and payment streams. *Banach Center Publications*, 83:213–229, 2008.

[50] P. Tardelli. Partially informed investors: hedging in an incomplete market with default. *Journal of Applied Probability*, 52(3):718–735, 2015.

[51] N. Vandaele and M. Vanmaele. A locally risk-minimizing hedging strategy for unit-linked life insurance contracts in a Lévy process financial market. *Insurance: Mathematics and Economics*, 42(3):1128–1137, 2008.