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Understanding Bruno de Finetti's Decision Theory: A Basic Algorithm to Support Decision-Making Behaviour

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Abstract. The aim of this work is to present an algorithm inspired to Bruno de Finetti's decision theory, limited to the version proposed by him in the essay *"La probabilità: guida nel pensare e nell'agire"* released in 1965. This work is focused on decision theory within the subjective theory of probability conceived by de Finetti. It opens with a brief overview of his theory of probability, followed by a methodological analysis functional to introduce the renowned de Finetti's example model given for the solution of decision problems. Starting from this example, this work presents a mathematical generalization of the decision algorithm. Afterwards, a real decisional algorithm written in mathematical-style pseudo code is developed. Finally, some conclusive remarks are discussed along with possible future developments.

Keywords: Decision theory · Information · Earnings · Mathematical expectation · Probabilities · Uncertainty · Mathematical algorithm

JEL Classification: C44 · D81

1 Introduction

It is needless to say how great has been the fortune of the deFinettian consensus [1, 2], so it results a rather difficult task to mark out an exhaustive map of the thousands of works which are linked to de Finetti's wide scientific production embracing issues of mathematics, statistics, economics, and philosophy.

Among others, however, Kao's and Velupillai's reflections on de Finetti seem of quite sufficient weight and importance to merit a particular attention [3]. The two authors, indeed, argue that the subjective theory of probability devised by de Finetti has furnished an essential contribution to the modern behavioral economics, especially in its origins. In the wake of the deFinettian tradition, one of the leading exponents of behavioral economics, Ward Edwards [4, 5], drew from Leonard J. Savage [6] the notion of expected subjective utility. In this regard, the expected subjective utility has gradually replaced the expected utility of von Neumann-Morgenstern [7], surpassing the fundamental limits of the latter. These limits concern the extreme weakness of the behavioural axioms of the expected utility theory and, therefore, the general incapability to formalize realistic axioms regarding human decision-making under uncertainty. The possibility of exceeding this limit became possible thanks to de Finetti's contributions to the theory and the foundations of probability.

In support of the deFinettian reflection, in this work we intend to highlight the renowned example model given for the solution of decision problems and shown in the essay released in 1965, *La probabilità: guida nel pensare e nell'agire* [8] (*Probability: A guide to think and act*, authors' translation). This example effectively represents the decision problem under uncertainty, with the purpose of identifying the optimal decision, that is capable of maximizing the expected earnings of economic agents. The example considers the possibility of using any more information, evaluating *ex ante* whether it is appropriate or not to decide to make use of further information, both in terms of cost to be incurred and of benefits in decision-making. The example described by de Finetti lends itself well to generalization and to the definition of an algorithm capable of supporting the action of rational economic agents. In this work, we have therefore generalized the example of de Finetti in mathematical terms and developed a decision-making algorithm. The algorithm, called *Wish* by us and based on the decision theory of de Finetti, aims (i) to show the manner in which the decisions made by economic agents may vary, especially in the presence of additional information with respect to the probability of occurrence of an event, as well as (ii) to show if further information can improve the expected earnings of economic agents. Indeed, further information is useful in a decision-making process only if it allows to maximize the mathematical expectation placed in the earnings or utility associated with a given event by an economic agent. Instead, if the information confirm the original decision, that the economic agent would have made in the absence of it, the information represents just a cost. The algorithm provides as output both the best decision to be made in the absence of more information and the best decision in case the further information increases the mathematical expectation of earnings, as well as the earnings (in terms of mathematical expectation) for each decision, both in the presence and in the absence of further information. The methodology adopted in this work is explained in more detail in the next section: first and foremost, we choose to consider and analyze the second example shown by de Finetti in [8], in order to deepen the role of information within the decision-making process under uncertainty.

This work is composed of five sections: this first section is intended as an introductory contribution to the following sections. Section 2 offers a brief methodological note preparatory for the subsequent sections. The third section includes an extensive analysis of the process used by de Finetti to make the optimal decision in the presence of multiple alternatives decisions (six) with respect to three different events, with the possibility to use more information. Section 4 contains the mathematical generalization of the problem and the definition of the algorithm *Wish* presented in mathematical-style pseudo code. Finally, the most significant findings achieved in this work are discussed.

2 A Methodological Note

The route followed in this work is the same adopted by de Finetti [8] in order to address the decision problem under uncertainty. This type of decisions do not aim at utility maximization, but rather to the maximization of mathematical expectation of earnings associated to a given event in monetary terms. We consider only the second of the two examples shown by de Finetti in [8], as the first example is restricted to addressing the decision relating to or not to proceed in performing a certain action.

The second example is different from the first one as it includes the opportunity to make use of additional information before choosing the optimal decision. The optimality of the decision is given by the maximum of the mathematical expectation calculated starting from the earnings corresponding to the decision D which in turn corresponds to the single events E.

With regard to the additional role of information in decision-making, it should be noted that it can be extremely useful if it allows to obtain different probabilities associated with events respect to those already known, thus modifying also the mathematical expectation to earn in each decision. Consequently, the optimal decision to be made will be different, as compared to that taken in the absence of information. If the information confirms the previous decision, it can only represent a cost.

3 A Method for Decision-Making Under Uncertainty

Bruno de Finetti seeks to reduce the uncertainty problem using known probabilities with respect to the occurrence of an event E, as well as with respect to any information H which can modify the choices that economic agents would have made in absence of H.

Here, we bring back up the second example addressed by de Finetti in [8], in order to establish a possible method to address decisions under uncertainty.

The example addressed by de Finetti concerns six decisions associated with three different events $E = \{E_1,\ E_2,\ E_3\}$, to which probabilities $\boldsymbol{p} = [p_1, p_2, p_3]$ are associated respectively, whereas there are six possible decisions that can be made $D = \{D_1, \ldots, D_6\}$. The matrix of earnings is also introduced $\boldsymbol{A} = [[a_{11}, \ldots, a_{16}], \ldots, [a_{31}, \ldots, a_{36}]]$ where each element represents the consequent earnings related to a certain decision D with respect to the probability p of a certain event E. The author calculates the mathematical expectations to earn associated to each decision in average values (for instance, for the decision D_1 we have the mathematical expectation $a_1 = p_1a_{11} + p_2a_{21} + p_3a_{31}$).

In doing so, the mathematical expectations obtained $a_1, \ldots, a_6$ corresponding to each of the six possible decisions, of which the greater one corresponds to the best decision in the absence of additional information.

De Finetti also suggests the possibility of acquiring additional information, which we denote by H and they may have three possible outcomes $H^{'}$, $H^{''}$, $H^{'''}$, each of which may change the probabilities of the three events E_1, E_2, E_3: therefore, we may have for $H^{'}$ the new probabilities $p_1^{'}$, $p_2^{'}$, $p_3^{'}$ and so on. Furthermore, the three possible outcomes will each have their own probability of occurrence, respectively $c^{'}$, $c^{''}$, $c^{'''}$ (see Table 1).

By means of the example model just rebuilt, de Finetti points out how it is possible to decide *ex ante* whether or not to use the additional information, calculating the mathematical expectations of earnings for each of the three possible outcomes of H and verifying if the mathematical expectation in the presence of H is greater than the mathematical expectation in the absence of more information.

Table 1. Bruno de Finetti's example model.

$$
\begin{array}{c}
\\ \\ \\
\begin{array}{ccccccc}
 & D_1 & D_2 & D_3 & D_4 & D_5 & D_6 \\
\mathrm{E}_1 & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
\mathrm{E}_2 & a_{21} & \mathrm{a}_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
\mathrm{E}_3 & a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\
 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\
 & a_1' & a_2' & a_3' & a_4' & a_5' & a_6' \\
 & a_1'' & a_2'' & a_3'' & a_4'' & a_5'' & a_6'' \\
 & a_1''' & a_2''' & a_3''' & a_4''' & a_5''' & a_6'''
\end{array}
\end{array}
\qquad
\begin{array}{cccc}
 & H' & H'' & H''' \\
 & c' & c'' & c''' \\
p_1 & p_1' & p_1'' & p_1''' \\
p_2 & p_2' & p_2'' & p_2''' \\
p_3 & p_3' & p_3'' & p_3'''
\end{array}
$$

4 Algorithm Generalization

The decision algorithm proposed by de Finetti can be generalized as follows. We consider a decision problem $P = \{D, E, \boldsymbol{p}, \boldsymbol{A}\}$ where we have to choose the best decision from a set of n possible decisions $D = \{D_1, \ldots, D_n\}$. Whichever decision will be taken, we can have one out of m possible outcomes (events) $E = \{E_1, \ldots, E_m\}$, where each event E_i has probability p_i, and $\boldsymbol{p} = [p_1, \ldots, p_m]$ is the vector of all probabilities. For each possible combination of decision and event we have an earning a_{ij} where i indicates the event and j indicates the decision, where $1 \le i \le m$, $1 \le j \le n$ and $\boldsymbol{A}$ is the matrix of earnings.

For each decision D_j we can calculate the mathematical expectation (that is, the expected earning) $a_j = p_1 a_{1j} + \cdots + p_m a_{mj}$. The best decision corresponds to the maximum earning given by $a_h^* = \max(a_i)$, $1 \le i \le n$. We indicate the best decision with the symbol D_h^*.

Using matrix notation, we can obtain the vector of all expected earnings $\boldsymbol{a}$ by multiplying the vector of the probabilities with the matrix of the earnings: $\boldsymbol{a} = \boldsymbol{pA}$, where $a_h^* = \max(\boldsymbol{a})$ and D_h^* is the best decision. The Table 2 shows the schema in graphical form.

Table 2. The mathematical generalization of the decision algorithm.

$$
\begin{array}{cc}
 & \boldsymbol{p} \\
E_1 & p_1 \\
\vdots & \vdots \\
E_m & p_m \\
 &
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccc} D_1 & \cdots & D_n \end{array} \\
\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \boldsymbol{A} \\
\begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} = \boldsymbol{a}
\end{array}
$$

Now we consider the case where we can ask for further information before taking any decision: let's call H the further research we can ask (and pay for) to add useful information before taking the decision, and with $H = \{H^1 \ldots H^r\}$ the possible outcomes of H. Before asking for H we should estimate what are the probabilities $\boldsymbol{c} = c^1 \ldots c^r$ of each one of the possible outcomes and, for each outcome H_i, we should estimate the probability vector $\boldsymbol{p^i} = p_1^i \ldots p_m^i$ of new probabilities for the events $E_1 \ldots E_m$ related to the outcome H_i.

This extended decision problem can be defined as $P^H = \{D, E, \boldsymbol{p}, \boldsymbol{A}, H, \boldsymbol{c}, \boldsymbol{p^i}\}$.

As de Finetti shows in [6], we will now have $p_i = p_i^1 c^1 + \ldots + p_i^r c^r$ that is the probability for each event now is the average of the probability we expect for each of the possible outcomes of the research, weighted with the probability which we expect for each H_i to happen.

We can calculate, for each of the H^i, the optimal value a_*^i as in the previous case as $a_*^i = \max(\boldsymbol{p^i A})$: if, for each of the H^i, we will have that $a_*^i = a_h^*$ this will mean that asking for H will not give any improvement so the extra information given by H will be not useful. If we have at least one different value, then we will have an improvement; the new expected earning is $a_H^* = c^1 a_*^1 + \cdots + c^r a_*^r \geq a_h^*$ and the best decision is the corresponding D_H^*.

The Table 3 shows graphically the extension of the decision algorithm.

Table 3. The generalization of the extended decision algorithm.

$$
\begin{array}{cc c ccc c ccc}
 & & & & & & & H^1 & & H^r \\
 & \boldsymbol{p} & & D_1 & \cdots & D_n & & c^1 & \cdots & c^r \\
E_1 & p_1 & \left[\right. & a_{11} & \cdots & a_{1n} & \left.\right] & p_1^1 & \cdots & p_1^r \\
\vdots & \vdots & & \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\
E_m & p_m & \left[\right. & a_{m1} & \cdots & a_{mn} & \left.\right] & p_m^1 & \cdots & p_m^r \\
 & & [& a_1 & \cdots & a_n &] & & & \\
 & & [& a_1^1 & \cdots & a_n^1 &] & & & \\
 & & & & \vdots & & & & & \\
 & & [& a_1^r & \cdots & a_n^r &] & & &
\end{array}
$$

The new expected (and improved) earning is not free, but it comes at a cost (the cost of H) so, before asking for further information, we can not only estimate if we will have an improvement in the expected earning but also calculate if the cost of obtaining the improved decision information is excessive due to a too small improvement on the expected earnings.

4.1 Pseudo Code for the Algorithm

Starting from the mathematical generalization given in the section above, we can write an algorithm capable to calculate what is the best decision to make, with or without the additional information given by H, and what are the expected earnings for each decision.

The algorithm needs for input (i) all the estimated information about earnings and (ii) estimated probabilities for each event, (iii) the estimated probabilities for each outcome of H and (iv) the modified probabilities for each event corresponding to each outcome of H. The outputs of the algorithm are (i) the expected earnings with and without doing the additional research H, (ii) the best decision to take without doing the additional research H and, (iii) the best decision to take according to the possible outcome of the research H.

Note that the algorithm, as introduced in a mathematical formulation in the section above, accepts only one possible extra decision H but this is not a limitation because, if we have to choose one among several possible extra decisions we can run the algorithm for each one of the possible extra decisions to consider, to find which is the most convenient to make.

The algorithm deriving from the generalization can be described in mathematical-style pseudo code as follows:

algorithm wish **is**
    input: matrix A of earnings,
        vector p of probabilities,
        vector $\boldsymbol{c} = c^1 \dots c^r$ of probabilities for $H^1 \dots H^r$,
        vectors $\boldsymbol{p^1} \dots \boldsymbol{p^r}$ of probabilities for $H^1 \dots H^r$.
    output: expected earnings a_h^* and a_H^*,
        best decision D_h^* to take before asking for H,
        vector $\boldsymbol{D}^* = d_*^1 \dots d_*^r$ of the best decision to take
        for each of the possible results of H: $H^1 \dots H^r$.

    $a_h^* \leftarrow \max(\boldsymbol{pA})$
    $D_h^* \leftarrow D_h$

    for each i **in** $\{1 \dots r\}$ **do**
        $a_*^i \leftarrow \max(\boldsymbol{p^i A})$
        $d_*^i \leftarrow$ decision corresponding to a_*^i

    $\boldsymbol{D}_* \leftarrow [d_*^1 \dots d_*^r]$
    $a_H^* \leftarrow c^1 a_*^1 + \cdots + c^r a_*^r$
    return $a_h^*, a_H^*, D_h^*, \boldsymbol{D}^*$.

5 Discussions and Conclusions

Within the framework of the wide deFinettian scientific production, the example model taken from *La probabilità: guida nel pensare e nell'agire* [8] has been chosen for its effectiveness to express the gain or loss related to decisions in monetary terms, instead of utility functions. It should be recalled that according to de Finetti there are no *a priori* absolute certainties, therefore any decision is made on condition of uncertainty and the latter may be reduced, but never completely removed.

With respect to the example revived in this work, the key role of information in decision theory has emerged: if the decision maker decides to make use of information (although that means that she supports a cost), this may allow new perspectives of earnings related to the different decision made, due to the fact of obtaining different probabilities of those known. By contrast, if the information only allows the decision maker to confirm the decision she would have already liked to make, it turns out to be merely a cost. Thus, the element of uncertainty is also present in this example and consists in the fact that we do not know in advance whether the information will increase our wished gain or less. Bruno de Finetti adopted this process because it allows to relate to the probability in terms of earnings associated with the degree of uncertainty. In fact, de Finetti argues that "*being able to reduce uncertainty of decision theory to a mere accounting is therefore not a decrease, but the greatest success; this success does not preclude, of course, the use of complex mathematical tools and high mathematical abilities to study in this same spirit complex and sensitive issues*." de Finetti [8] p. 55 (authors' translation).

Possible future developments of the algorithm proposed in this work concern further generalizations of it, identifying the opportunity to have not one but N possible extra information to be acquired: in this way, we will be able to identify the best subset of information that allows to maximize the earnings, considering further constraints on the costs of the additional information.

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