Finding Dominating Induced Matchings in P_8-Free Graphs in Polynomial Time

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Abstract

Let $G = (V, E)$ be a finite undirected graph. An edge set $E' \subseteq E$ is a *dominating induced matching* (*d.i.m.*) in G if every edge in E is intersected by exactly one edge of E'. The *Dominating Induced Matching* (*DIM*) problem asks for the existence of a d.i.m. in G; this problem is also known as the *Efficient Edge Domination* problem.

The DIM problem is related to parallel resource allocation problems, encoding theory and network routing. It is $\mathbb{NP}$-complete even for very restricted graph classes such as planar bipartite graphs with maximum degree three and is solvable in linear time for P_7-free graphs. However, its complexity was open for P_k-free graphs for any $k \geq 8$; P_k denotes the chordless path with k vertices and $k-1$ edges. We show in this paper that the weighted DIM problem is solvable in polynomial time for P_8-free graphs.

Keywords: dominating induced matching; efficient edge domination; P_8-free graphs; polynomial time algorithm;

1 Introduction

Let $G = (V, E)$ be a finite undirected graph. A vertex $v \in V$ *dominates* itself and its neighbors. A vertex subset $D \subseteq V$ is an *efficient dominating set* (*e.d.s.* for short) of G if every vertex of G is dominated by exactly one vertex in D. The notion of efficient domination was introduced by Biggs [1] under the name *perfect code*. The Efficient Domination (ED) problem asks for the existence of an e.d.s. in a given graph G (note that not every graph has an e.d.s.)

If a vertex weight function $\omega : V \to \mathbb{N}$ is given, the Weighted Efficient Domination (WED) problem asks for a minimum weight e.d.s. in G, if there is one, or for determining that G has no e.d.s.

A set M of edges in a graph G is an *efficient edge dominating set* (*e.e.d.s.* for short) of G if and only if it is an e.d.s. in its line graph $L(G)$. The Efficient Edge Domination (EED) problem asks for the existence of an e.e.d.s. in a given graph G. Thus, the EED problem for a graph G corresponds to the ED problem for its line graph $L(G)$. Again, note that

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not every graph has an e.e.d.s. An efficient edge dominating set is also called *dominating induced matching* (*d.i.m.* for short) and the EED problem is called the Dominating Induced Matching (DIM) problem in some papers (see e.g. [2, 4, 6]); subsequently, we will use this notation in the manuscript. The edge-weighted version of DIM for graph G corresponds to the vertex-weighted version of ED for $L(G)$.
In [8], it was shown that the DIM problem is $\mathbb{NP}$-complete; see also [2, 6, 12, 14]. However, for various graph classes, DIM is solvable in polynomial time. For mentioning some examples, we need the following notions:
Let P_k denote the chordless path P with k vertices, say $a_1, \dots, a_k$ and $k-1$ edges $a_i a_{i+1}$, $1 \le i \le k-1$; we also denote it as $P = (a_1, \dots, a_k)$.
For indices $i, j, k \ge 0$, let $S_{i,j,k}$ denote the graph with vertices $u, x_1, \dots, x_i$, $y_1, \dots, y_j$, $z_1, \dots, z_k$ such that the subgraph induced by $u, x_1, \dots, x_i$ forms a P_{i+1} $(u, x_1, \dots, x_i)$, the subgraph induced by $u, y_1, \dots, y_j$ forms a P_{j+1} $(u, y_1, \dots, y_j)$, and the subgraph induced by $u, z_1, \dots, z_k$ forms a P_{k+1} $(u, z_1, \dots, z_k)$, and there are no other edges in $S_{i,j,k}$. Thus, *claw* is $S_{1,1,1}$, and P_k is isomorphic to e.g. $S_{0,0,k-1}$.
DIM is solvable in polynomial time for $S_{1,1,1}$-free graphs [6], for $S_{1,2,3}$-free graphs [10], and for $S_{2,2,2}$-free graphs [9]. In [9], it is conjectured that for every fixed i, j, k, DIM is solvable in polynomial time for $S_{i,j,k}$-free graphs (actually, an even stronger conjecture is mentioned in [9]); this also includes P_k-free graphs for $k \ge 8$.
In [4], DIM is solved in linear time for P_7-free graphs; recall that P_7 is isomorphic to $S_{0,0,6}$, and $S_{1,2,3}$ contains P_6 as an induced subgraph.

In this paper we show that edge-weighted DIM can be solved in polynomial time for P_8-free graphs.

2 Definitions and Basic Properties

2.1 Basic notions

Let G be a finite undirected graph without loops and multiple edges. Let V denote its vertex set and E its edge set; let $|V| = n$ and $|E| = m$. For $v \in V$, let $N(v) := \{u \in V \mid uv \in E\}$ denote the *open neighborhood of* v, and let $N[v] := N(v) \cup \{v\}$ denote the *closed neighborhood of* v. If $xy \in E$, we also say that x and y *see each other*, and if $xy \notin E$, we say that x and y *miss each other*. A vertex set S is *independent* (or *stable*) in G if for every pair of vertices $x, y \in S$, $xy \notin E$. A vertex set Q is a *clique* in G if for every pair of vertices $x, y \in Q$, $x \neq y$, $xy \in E$. For $uv \in E$ let $N(uv) := N(u) \cup N(v) \setminus \{u, v\}$ and $N[uv] := N[u] \cup N[v]$.
For $U \subseteq V$, let $G[U]$ denote the subgraph of G induced by vertex set U. Clearly $xy \in E$ is an edge in $G[U]$ exactly when $x \in U$ and $y \in U$; thus, $G[U]$ will be often denoted simply by U when that is clear in the context.
Let A and B be disjoint sets of vertices of G. If a vertex from A sees a vertex from B, we say that A *and* B *see each other*. If every vertex from A sees every vertex from B then we denote this by $A Ⓘ B$. In particular, if a vertex $u \notin B$ sees all vertices of B then we denote this by $u Ⓘ B$ (in this case, u is called *universal* for B). If every vertex from A misses every vertex from B, we say that A *and* B *miss each other* and denote this by $A ⓪ B$. If for $A' \subseteq A$, $A' ⓪ (A \setminus A')$ holds, we say that A' is *isolated* in A.

As already mentioned, a *chordless path* P_k has k vertices, say $v_1, \dots, v_k$, and edges v_iv_{i+1}, $1 \le i \le k-1$. The *length* of P_k is $k-1$. A *chordless cycle* C_k has k vertices, say $v_1, \dots, v_k$, and edges v_iv_{i+1}, $1 \le i \le k-1$, and v_kv_1. The *length* of C_k is k.
Let K_i denote the clique with i vertices. Let $K_4 - e$ or *diamond* be the graph with four vertices and five edges, say vertices a, b, c, d and edges ab, ac, bc, bd, cd; its *mid-edge* is the edge bc. A *gem* has five vertices, say, a, b, c, d, e, such that (a, b, c, d) forms a P_4 and e is universal for $\{a, b, c, d\}$. A *butterfly* has five vertices and six edges, say, a, b, c, d, e and edges ab, ac, bc, cd, ce, de. The *peripheral edges* of the butterfly are ab and de. A *star* is a graph formed by an independent set plus one vertex (the *center* of the star) which is universal for such an independent set; in particular let us say that a star is *trivial* if it is an edge, and is *non-trivial* otherwise.

We often consider an edge $e = uv$ to be a set of two vertices; then it makes sense to say, for example, $u \in e$ and $e \cap e' \neq \emptyset$ for an edge e'. For two vertices $x, y \in V$, let $dist_G(x, y)$ denote the *distance between x and y in G*, i.e., the length of a shortest path between x and y in G. The *distance between two edges* $e, e' \in E$ is the length of a shortest path between e and e', i.e., $dist_G(e, e') = \min\{dist_G(u, v) \mid u \in e, v \in e'\}$. In particular, this means that $dist_G(e, e') = 0$ if and only if $e \cap e' \neq \emptyset$.
An edge set $M \subseteq E$ is an *induced matching* if its members have pairwise distance at least 2. Obviously, if M is a d.i.m. then M is an induced matching.

For an edge xy, let $N_i(xy)$ denote the *distance levels of xy*:

$$N_i(xy) := \{z \in V \mid dist_G(z, xy) = i\}.$$

For a set $\mathcal{F}$ of graphs, a graph G is called *$\mathcal{F}$-free* if G contains no induced subgraph from $\mathcal{F}$. A graph is *hole-free* if it is C_k-free for all $k \ge 5$. A graph is *weakly chordal* if it is C_k-free and $\overline{C_k}$-free for all $k \ge 5$, i.e., the graph and its complement are hole-free.

If M is a d.i.m. then an edge is *matched by M* if it is either in M or shares a vertex with some edge in M. Note that M is a d.i.m. in G if and only if it corresponds to a dominating set (of vertices) in the line graph $L(G)$ and an independent set of vertices in the square $L(G)^2$. The Maximum Weight Independent Set (MWIS) problem asks for a maximum weight independent set in a given graph with vertex-weight function. The DIM problem for G can be reduced to the MWIS problem for $L(G)^2$ (see [3]). For instance, in [5], it is shown that for weakly chordal graphs G, $L(G)^2$ is weakly chordal, and since MWIS can be solved in polynomial time for weakly chordal graphs [15], DIM can be solved in polynomial time for weakly chordal graphs as well. Actually, DIM can be solved in polynomial time even for hole-free graphs [2].

P_8-free graphs having a d.i.m. are C_k-free for $k \ge 9$ and $\overline{C_k}$-free for $k \ge 6$ (see Corollary 1 below) but we do not yet have a proof that, using the reduction to $L(G^2)$, DIM can be solved in polynomial time for P_8-free graphs; our approach in this paper is a direct one following the approach for P_7-free graphs given in [4].

2.2 Forbidden subgraphs and forced edges

The subsequent observations are helpful (some of them are mentioned e.g. in [2, 4]); since we deal with the larger class of P_8-free graphs instead of P_7-free graphs and in order to make this manuscript self-contained, we give all proofs where forbidding P_8 plays a role.

Observation 1 ([2, 4]). *Let M be a d.i.m. in G.*

(*i*) *M contains at least one edge of every odd cycle C_{2k+1} in G, $k \geq 1$, and exactly one edge of every odd cycle C_3, C_5, C_7 of G.*

(*ii*) *No edge of any C_4 can be in M.*

(*iii*) *For each C_6 either exactly two or none of its edges are in M.*

Proof. See Observation 2 in [4].

Since every triangle contains exactly one M-edge and no M-edge is in any C_4, and the pairwise distance of edges in any d.i.m. is at least 2, we obtain:

Corollary 1. *If a graph G has a d.i.m. then G is K_4-free, gem-free and $\overline{C_k}$-free for any $k \geq 6$.*

As a consequence of Observation 1 (*ii*), we give all edges in any C_4 of G weight ∞. Note that a d.i.m. of finite weight cannot contain any edge of a C_4.

If an edge $e \in E$ is contained in *every* d.i.m. of G, we call it a *forced* edge of G.

Observation 2. *The mid-edge of any diamond in G and the two peripheral edges of any induced butterfly are forced edges of G.*

Note that in a graph with d.i.m., the set of forced edges is an induced matching. So our algorithm solving the DIM problem on P_8-free graphs has to check whether the set of forced edges is an induced matching (and finally might be extended to a d.i.m. of G). If M is an induced matching of already collected forced edges and edge vw is a new forced edge, we can reduce the graph as follows:

Reduction-Step-(vw, M).

> If $M \cup \{vw\}$ is not an induced matching then STOP - G has no d.i.m., otherwise add vw to M, i.e., $M := M \cup \{vw\}$, delete v and w and all edges incident to v and w in G, and give all edges that were at distance 1 from vw in G weight ∞.

Obviously, the graph resulting from the reduction step is an induced subgraph of G. In particular, edges with weight ∞ are not in any d.i.m. of finite weight in G.

Observation 3 ([4]). *Let M' be an induced matching which is a set of forced edges in G. Then G has a d.i.m. M if and only if after applying the reduction step to all edges in M', the resulting graph has a d.i.m. $M \setminus M'$.*

Subsequently, this approach will often be used. Note that after applying the Reduction Step to all mid-edges of diamonds and all peripheral edges of butterflies in G, the resulting graph is (diamond, butterfly)-free. By Corollary 1, a graph G having a d.i.m. is K_4-free. Thus, from now on, we can assume that G is (P_8,K_4, diamond, butterfly)-free.

3 The Structure of P_8-Free Graphs With a Dominating Induced Matching

Throughout this section, let $G = (V, E)$ be a connected (P_8, K_4, diamond, butterfly)-free graph having a d.i.m. M. Note that if G has a d.i.m. M and $V(M)$ denotes the vertex set of M then $V \setminus V(M)$ is an independent set I, i.e.,

$$V \text{ has the partition } V = I \cup V(M). \tag{1}$$

3.1 The distance levels of an M-edge xy in a P_3

We first describe some general structure properties for the distance levels of an edge in a d.i.m. Since G is (K_4, diamond, butterfly)-free, we have:

Observation 4. *For every vertex v of G, $N(v)$ is the disjoint union of isolated vertices and at most one edge. Moreover, for every edge $xy \in E$, there is at most one common neighbor of x and y.*

Since it is trivial to check whether G has a d.i.m. with exactly one edge, from now on we can assume that $|M| \geq 2$. Since G is connected and butterfly-free, we have:

Observation 5. *If $|M| \geq 2$ then there is an edge in M which is contained in a P_3 of G.*

Let $xy \in M$ be an M-edge for which there is a vertex r such that $\{r, x, y\}$ induce a P_3 with edge $rx \in E$. We consider a partition into the distance levels $N_i = N_i(xy)$, $i \geq 1$, with respect to the edge xy. By (1) and since we assume that $xy \in M$, clearly, $N_1 \subseteq I$ and thus:

$$N_1 \text{ is an independent set.} \tag{2}$$

Since G is P_8-free and xy is contained in a P_3 $\{r, x, y\}$ of G, we obtain:

$$N_k = \emptyset \text{ for } k \geq 6. \tag{3}$$

Proof of (3): If $N_6 \neq \emptyset$ then there are vertices $v_i \in N_i$, $2 \leq i \leq 6$, such that $\{v_6, v_5, v_4, v_3, v_2\}$ induce a chordless path with $v_iv_{i+1} \in E$ for $2 \leq i \leq 5$. If $v_2r \in E$ then $\{v_6, v_5, v_4, v_3, v_2, r, x, y\}$ would induce a P_8 in G. Thus, $v_2r \notin E$; let $v_1 \in N_1$ be a neighbor of v_2. By (2), $v_1r \notin E$. Now, if $v_1x \in E$ then $\{v_6, v_5, v_4, v_3, v_2, v_1, x, r\}$ induce a P_8 in G, and if $v_1x \notin E$ then $v_1y \in E$ and thus, $\{v_6, v_5, v_4, v_3, v_2, v_1, y, x\}$ induce a P_8 in G which is a contradiction. $\square$

Subsequently, the principle of the proof of (3) will be applied in various cases whenever a P_8 has to be excluded.

Since $xy \in M$, no edge between N_1 and N_2 is in M. Since $N_1 \subseteq I$ and all neighbors of vertices in I are in $V(M)$, we have:

$$N_2 \text{ is the disjoint union of edges and isolated vertices.} \tag{4}$$

Let M_2 denote the set of edges with both ends in N_2 and let $S_2 = \{u_1, \ldots, u_k\}$ denote the set of isolated vertices in N_2; $N_2 = V(M_2) \cup S_2$ is a partition of N_2. Obviously:

$$M_2 \subseteq M \text{ and } S_2 \subseteq V(M). \tag{5}$$

If for $xy \in M$, an edge $e \in E$ is contained in *every* dominating induced matching M of G with $xy \in M$, we say that e is an *xy-forced* M-edge. The Reduction Step for forced edges can also be applied for xy-forced M-edges (then, in the unsuccessful case, G has no d.i.m. containing xy). We do this whenever an xy-forced M-edge is found. The first example is the following one; obviously, by (5), we have:

$$\text{Every edge in } M_2 \text{ is an } xy\text{-forced } M\text{-edge.} \tag{6}$$

Thus, from now on, we can assume that $M_2 = \emptyset$, i.e., $N_2 = S_2 = \{u_1, \ldots, u_k\}$. For every $i \in \{1, \ldots, k\}$, let $u'_i \in N_3$ denote the *M-mate* of u_i (i.e., $u_i u'_i \in M$). Let $M_3 = \{u_i u'_i : i \in \{1, \ldots, k\}\}$ denote the set of M-edges with one endpoint in S_2 (and the other endpoint in N_3). Obviously, by (5) and the distance condition for a d.i.m. M, the following holds:

$$\text{No edge with both ends in } N_3 \text{ and no edge between } N_3 \text{ and } N_4 \text{ is in } M. \tag{7}$$

As a consequence of (7) and the fact that every triangle contains exactly one M-edge (see Observation 1 (i)), we have:

$$\text{For every triangle } abc \text{ with } a \in N_3, \text{ and } b, c \in N_4, \ bc \in M \text{ is an } xy\text{-forced } M\text{-edge.} \tag{8}$$

This means that for the edge bc, the Reduction Step can be applied, and from now on, we can assume that there is no such triangle abc with $a \in N_3$ and $b, c \in N_4$, i.e., for every edge $uv \in E$ in N_4:

$$(N(u) \cap N_3) \cap (N(v) \cap N_3) = \emptyset. \tag{9}$$

According to (5) and the assumption that $M_2 = \emptyset$ (recall $N_2 = \{u_1, \ldots, u_k\}$), let:

$T_{one} := \{t \in N_3 : |N(t) \cap N_2| = 1\}$;

$T_i := T_{one} \cap N(u_i)$, $i \in \{1, \ldots, k\}$;

$S_3 := N_3 \setminus T_{one}$.

By definition, T_i is the set of *private* neighbors of u_i in N_3 (note that $u'_i \in T_i$), and $T_1 \cup \ldots \cup T_k$ is a partition of T_{one}, and $T_{one} \cup S_3$ is a partition of N_3.

Lemma 1. *The following statements hold:*

(i) *For all $i \in \{1, \ldots, k\}$, $T_i \cap V(M) = \{u'_i\}$.*

(ii) *For all $i \in \{1, \ldots, k\}$, T_i is the disjoint union of vertices and at most one edge.*

(iii) *$G[N_3]$ is bipartite.*

(iv) *$S_3 \subseteq I$, i.e., S_3 is an independent vertex set.*

(v) *If a vertex $t_i \in T_i$ sees two vertices in T_j, $i \neq j$, $i, j \in \{1, \ldots, k\}$, then $u_i t_i \in M$ is an xy-forced M-edge.*

Proof. (i): Holds by definition of T_i and by the distance condition of a d.i.m. M.
(ii): Holds by Observation 4.
(iii): Follows by Observation 1 (i) since every odd cycle in G must contain at least one M-edge, and by (7).
(iv): If $v \in S_3 := N_3 \setminus T_{one}$, i.e., v sees at least two M-vertices then clearly, $v \in I$, and thus, $S_3 \subseteq I$ is an independent vertex set (recall that I is an independent vertex set).
(v): Suppose that $t_1 \in T_1$ sees a and b in T_2. Then, if $ab \in E$, u_2, a, b, t_1 induce a diamond in G. Thus, $ab \notin E$ and now, u_2, a, b, t_1 induce a C_4 in G; the only possible M-edge for dominating t_1a, t_1b is u_1t_1, i.e., $t_1 = u'_1$. $\square$

Thus, by (v), from now on, we can assume that for every $i, j \in \{1, \ldots, k\}$, $i \neq j$, any vertex $t_i \in T_i$ sees at most one vertex in T_j.

Then let us split the problem of checking if a d.i.m. M with xy exists into two cases: The case $N_4 = \emptyset$ and the case $N_4 \neq \emptyset$.

4 The case $N_4 = \emptyset$

Throughout this section, we assume that $N_4 = \emptyset$.

Lemma 2. *The following statements hold:*

(i) *For every edge $vw \in E$, $v, w \in N_3$, with $vu_i \in E$ and $wu_j \in E$, $|\{v, w\} \cap \{u'_i, u'_j\}| = 1$.*

(ii) *For every edge $st \in E$ with $s \in S_3$ and $t \in T_i$, $t = u'_i$ holds, and thus u_it is an xy-forced M-edge.*

Proof. (i): Since $N_4 = \emptyset$ and $vw \notin M$ (by (7), N_3 does not contain any M-edge), vw has to be dominated by exactly one of the M-edges $u_iu'_i$, $u_ju'_j$.

(ii): By Lemma 1, $S_3 \subseteq I$ and thus, by (i), for the edge st with $s \in S_3$, $t = u'_i$ holds. $\square$

From now on, we can assume that S_3 is isolated in N_3. This means that every edge between N_2 and N_3 containing a vertex of S_3 is dominated; thus, we can assume that $S_3 = \emptyset$. This means that for every $t \in N_3$, there is exactly one $i \in \{1, \ldots, k\}$ such that $u_it \in E$. Recall that $N_2 = S_2 = \{u_1, \ldots, u_k\}$.

Let us observe that to check if a vertex set $W \subseteq T_{one}$ may be such that $W \subset V(M)$ (i.e., formed by the M-mates of some vertices of S_2) and to check the implications of this choice can be done by repeatedly applying forcing rules; the details are given in the following procedure which is correct by the above and which can be executed in polynomial time.

Procedure Extend[W**-in-**M]
Given: A vertex set $W \subseteq T_{one}$ and the vertex set $W' \subseteq S_2 \cup T_{one}$ formed by the vertices of those connected components of $G[S_2 \cup T_{one}]$ containing W.
Task: Return a proof that G has no d.i.m. M with $W \subset V(M)$, or return a partition of $T_{one} \cap W'$, into the set $T_{one,Col}$ of *colored* vertices (by black or white) and the set $T_{one,Uncol}$ of *uncolored* vertices, such that:

(i) $T_{one,Col} \textcircled{0} T_{one,Uncol}$

(ii) the set of black vertices of $T_{one,Col}$ and the set of their respective neighbors in S_2, say the set $S_{2,Col}$ (with $S_{2,Col} \subseteq S_2$), induce a d.i.m. of $G[S_{2,Col} \cup T_{one,Col}]$, and

(iii) the set of white vertices of $T_{one,Col}$ is that of vertices of $G[S_{2,Col} \cup T_{one,Col}]$ which are not in such a d.i.m.

Comment: Once assumed that $W \subset V(M)$, the procedure colors vertices of $T_{one} \cap W'$ which should be in $V(M)$ black, and vertices of $T_{one} \cap W'$ which should be in I white.

Step 1. Color all vertices of W black.
Step 2. Color some vertices of $T_{one} \cap W'$ either black or white by repeatedly applying the following forcing rules:

(a) set $X := \emptyset$;

(b) Repeat

  (b.1) take a colored vertex of $(T_{one} \cap W') \setminus X$, say $v \in T_i \cap W'$, and set $X := X \cup \{v\}$;

  (b.2) if v is black, then color all neighbors of v in $T_{one} \cap W'$ white, and color all vertices of $T_i \setminus \{v\}$ white;

  (b.3) if v is white, then color all neighbors of v in $T_{one} \cap W'$ black.

  until there is no colored vertex in $(T_{one} \cap W') \setminus X$.

Step 3. If referring to Step 2, a vertex of $T_{one} \cap W'$ should change its color, i.e., it is colored white (black, respectively) while being black (white, respectively), then return a proof that G has no d.i.m. M with $t_1 \in V(M)$. Otherwise, return a partition of $T_{one} \cap W'$ according to the Task.

For convenience, let us say that Procedure Extend[W-in-M] is *complete* if it either returns a proof that G has no d.i.m. M with $W \subset V(M)$, or returns $T_{one,Uncol} = \emptyset$, and is *incomplete* otherwise. Note that Procedure Extend[W-in-M] may be incomplete. Furthermore note that a white vertex of $T_{one,Col}$ may have a neighbor in $S_2 \setminus S_{2,Col}$.

Then let us focus on $G[S_2 \cup T_{one}]$. Only two cases are possible according to the following subsections 4.1 and 4.2:

4.1 T_i ⓪T_j

4.2 T_i sees T_j for some $1 \leq i < j \leq k$

4.1 There is no edge between T_i and T_j for $1 \leq i < j \leq k$

In this case the problem of checking if M exists can be solved in polynomial time as follows: For each vertex $t_i \in T_i$, for $i = 1, \ldots, k$, run Procedure Extend[W-in-M] with $W = \{t_i\}$ and choose a minimum finite weight solution (if such a solution exists) over $t \in T_i$. Note that Procedure Extend[W-in-M] with $W = \{t_i\}$ is complete (that can be easily checked since the connected component of $G[S_2 \cup T_{one}]$ containing t_i is $G[\{u_i\} \cup T_i]$). Finally either return that G has no d.i.m. M or return M.

4.2 There is an edge between T_i and T_j for some $1 \le i < j \le k$

Assume that there is an edge $t_i t_j \in E$ between $t_i \in T_i$ and $t_j \in T_j$, for some $i, j \in \{1, \ldots, k\}$, $i \neq j$; without loss of generality, let $i = 1$ and $j = 2$ and $t_1 t_2 \in E$. Let G' be the subgraph of G induced by the non-neighborhood of t_1, t_2.

Lemma 3. *The following statements hold for every $i \in \{3, \ldots, k\}$ in G':*

(*i*) *Each edge e_i in T_i misses each vertex in $\{T_3, \ldots, T_k\} \setminus \{T_i\}$.*

(*ii*) *Each vertex $t_i \in T_i$ sees at most one vertex in $\{T_3, \ldots, T_k\} \setminus \{T_i\}$.*

Proof. (*i*): Without loss of generality, suppose to the contrary that for an edge $t_i t'_i \in E$ with $t_i, t'_i \in T_i$, there is a vertex $t_j \in T_j$ with $t_i t_j \in E$. Then by Lemma 1 (*iii*), $t'_i t_j \notin E$ but now, the subgraph of G induced by $t_2, t_1, u_1, N_1, x, y, u_j, t_j, t_i, t'_i$ contains a P_8.
(*ii*): By Lemma 1 (*v*), we can assume that no vertex in T_i sees two vertices in T_j. Without loss of generality, suppose to the contrary that there is a vertex $t_i \in T_i$ which sees $t_j \in T_j$ and $t_q \in T_q$, $j \neq q$. Then again by Lemma 1 (*iii*), $t_j t_q \notin E$ but now, the subgraph of G induced by $t_2, t_1, u_1, N_1, x, y, u_q, t_q, t_i, t_j$ contains a P_8. □

Let Z be the graph with nodes $\{z_3, \ldots, z_k\}$, where z_i corresponds to T_i for $i \in \{3, \ldots, k\}$, such that for $i \neq j$, $z_i z_j$ is an edge in Z if and only if T_i sees T_j in G. Let us say that:

(*i*) T_i forms a *singleton-type* in $G[H]$ if the node of Z corresponding to T_i is an isolated node of Z.

(*ii*) T_i and T_j form an *edge-type* in $G[H]$ if $z_i z_j$ is an isolated edge of Z.

(*iii*) $T_i, T_{j_1}, \ldots, T_{j_h}$ form a *star-type* in $G[H]$ if the nodes of Z corresponding to $T_i, T_{j_1}, \ldots, T_{j_h}$ form an isolated non-trivial star of Z with center T_i, for $i, j_1, \ldots, j_h \in \{3, \ldots, k\}$. Let

$$T'_i := \{t_i \in T_i : t_i \text{ sees an element of } \{T_{j_1}, \ldots, T_{j_h}\}\} \text{ and}$$
$$T'_{i,j} := \{t_i \in T_i : t_i \text{ sees an element of } T_j\} \text{ for } j \in \{j_1, \ldots, j_h\}.$$

Lemma 4. *Each component of Z in G' is either a singleton or an edge or a non-trivial star.*

Proof. If for all $i \in \{3, \ldots, k\}$, T_i sees at most one element of $\{T_3, \ldots, T_k\} \setminus \{T_i\}$, then the components of Z are either singletons or edges, and Lemma 4 follows. Thus assume that there is an $i \in \{3, \ldots, k\}$ such that T_i sees more than one element of $\{T_3, \ldots, T_k\} \setminus \{T_i\}$, say T_i sees $T_{j_1}, \ldots, T_{j_h}$, for some $\{j_1, \ldots, j_h\} \subseteq \{3, \ldots, k\} \setminus \{i\}$ with $h \ge 2$. Let us prove that the nodes of Z corresponding to $T_i, T_{j_1}, \ldots, T_{j_h}$ induce in Z an isolated non-trivial star with center T_i; that will imply Lemma 4.
Let T'_i and $T'_{i,j}$ be as defined in (*iii*) above. Then $T'_i = T'_{i,j_1} \cup \ldots \cup T'_{i,j_h}$ is a partition of T'_i by Lemma 3 (*ii*). Moreover T'_i misses $T_i \setminus T'_i$ by Lemma 3 (*i*).

Notation: For a clear reading let us write $j_1 = \xi$ and $j_2 = \eta$.

Claim 1. *$T'_i \subset I$.*

Proof. By contradiction assume that a vertex from T'_i is in $V(M)$, say a vertex $t_{i,\xi} \in T'_{i,\xi}$ without loss of generality, i.e., $t_{i,\xi}$ is the M-mate of u_i. Then $T'_{i,j} \subset I$ for all $j \in \{j_2, \ldots, j_h\}$ by Lemma 1 (i). By definition of $T'_{i,\xi}$, $t_{i,\xi}$ sees a vertex $t'_\xi \in T_\xi$. Then, since $t_{i,\xi} \in V(M)$, we have $t'_\xi \in I$. Then by Lemma 1 (i) there is a vertex $t_\xi \in T_\xi$ such that $t_\xi \in V(M)$, namely the M-mate u'_ξ of u_ξ: In particular by Lemma 3 (i) we derive that t'_ξ misses t_ξ. On the other hand by definition of $T'_{i,\eta}$, a vertex $t_{i,\eta} \in T'_{i,\eta}$ sees a vertex $t'_\eta \in T_\eta$. Then since $t_{i,\eta} \in I$, one has $t'_\eta \in V(M)$, i.e., t'_η is the M-mate u'_η of u_η: In particular by Lemma 3 (i) we derive that $t_{i,\eta}$ misses $t_{i,\xi}$ but then, by Lemma 3 (ii) and by the above, $u_\eta, t'_\eta, t_{i,\eta}, u_i, t_{i,\xi}, t'_\xi, u_\xi, t_\xi$ induce a P_8. This shows Claim 1. $\diamond$

Claim 1 implies: $T_i \setminus T'_i \neq \emptyset$ and contains the M-mate of u_i by Lemma 1 (i); each vertex of $T'_{i,j}$, for $j \in \{j_1, \ldots, j_h\}$, sees exactly one vertex of T_j, namely the M-mate u'_j of u_j (in particular all vertices of $T'_{i,j}$ have the same neighborhood in T_j).

Claim 2. *The elements of $\{T_{j_1}, \ldots, T_{j_h}\}$ miss each other.*

Proof. Without loss of generality, by symmetry let us only show that T'_ξ misses T'_η. By contradiction assume that there is an edge $t'_\xi t'_\eta$ between T'_ξ and T'_η. Let $t_{i,\eta} \in T'_{i,\eta}$ and let $t_\eta \in T'_\eta$ be the M-mate of u_η. Then $t_{i,\eta}$ sees t_η (by the above) and consequently: $t_\eta \neq t'_\eta$ by Lemma 3 (ii), any $t_{i,\xi} \in T'_{i,\xi}$ misses t'_η since they are both in I, t_η misses t'_η by Lemma 3 (i), and finally $t_{i,\xi}$ and t_η miss t'_ξ by Lemma 3 (ii). Then $u_\xi, t'_\xi, t'_\eta, u_\eta, t_\eta, t_{i,\eta}, u_i$ and any vertex of $T_i \setminus T'_i$ induce a P_8. This completes the proof of Claim 2. $\diamond$

Claim 3. *No element of $\{T_i, T_{j_1}, \ldots, T_{j_h}\}$ sees any element of $\{T_3, \ldots, T_k\} \setminus \{T_i, T_{j_1}, \ldots, T_{j_h}\}$.*

Proof. The fact holds true for T_i by construction. Without loss of generality by symmetry we only need to show that T_η misses T_ζ, where $\zeta \in \{3, \ldots, k\} \setminus \{i, j_1, \ldots, j_h\}$. Suppose to the contrary that there is an edge $t'_\eta t'_\zeta$ between T_η and T_ζ. Let $t_{i,\eta} \in T_{i,\eta}$ and let $t_\eta \in T_\eta$ be the M-mate of u_η. Then $t_{i,\eta}$ sees t_η (by the above) and consequently: $t_\eta \neq t'_\eta$ by Lemma 3 (ii), $t_{i,\eta}$ misses t'_η since they are both in I, t_η misses t'_η by Lemma 3 (i) , and finally $t_{i,\eta}$ and t_η miss t'_ζ by Lemma 3 (ii). Then $u_\zeta, t'_\zeta, t'_\eta, u_\eta, t_\eta, t_{i,\eta}, u_i$ and any vertex of $T_i \setminus T'_i$ induce a P_8. This completes the proof of Claim 3. $\diamond$

Now Claims 1, 2, and 3 imply that the nodes of Z corresponding to $T_i, T_{j_1}, \ldots, T_{j_h}$ induce an isolated non-trivial star in Z. Thus Lemma 4 follows. $\square$

According to Lemma 4, let us focus on a connected component of $G[\{u_3, \ldots, u_k\} \cup T_3 \cup \ldots \cup T_k]$, say $Q = G[\{u_i, u_{j_1}, \ldots, u_{j_h}\} \cup T_i \cup T_{j_1} \cup \ldots \cup T_{j_h}]$, with $T_i, T_{j_1}, \ldots, T_{j_h}$ inducing a (trivial or non-trivial) star in Z with center T_i i.e., let us consider the general case in which the cardinality of the family $\{T_{j_1}, \ldots, T_{j_h}\}$ may be even equal to 0 or to 1.

Then let us observe that, to compute a minimum weight d.i.m. of Q (if it exists), say M', with $\{u_i, u_{j_1}, \ldots, u_{j_h}\} \in V(M')$, and with a fixed vertex $t_i \in T_i$ being in $V(M')$ (i.e., being the M-mate of u_i), can be done by the following procedure which is correct by the above and which can be executed in polynomial time.

Step 1. Run Procedure Extend[W-in-M] with $W = \{t_i\}$.
Step 2. If it returns $T_{one,Uncol} = \emptyset$ (i.e., if it is complete), then we are done.
Step 3. If it is incomplete and returns a partition of $T_{one} \cap W'$, namely $\{T_{one,Col}, T_{one,Uncol}\}$, with $T_{one,Uncol} \neq \emptyset$ then we can easily color the vertices of $T_{one,Uncol}$ such that black vertices are finally the M-mates of $\{u_i, u_{j_1}, \ldots, u_{j_h}\}$: in fact by construction and by the

above, we have $T_{one,Uncol} \subseteq T_{j_1} \cup \ldots \cup T_{j_h}$, and in particular, for each $j \in \{j_1, \ldots, j_h\}$, $T_{one,Uncol} \cap T_j$ has a co-join to $T_{one,Col} \cup (T_{one,Uncol} \setminus T_j)$ and induces a graph with at most one isolated edge $e_j = ab$ (say with $w(au_j) \leq w(bu_j)$) and isolated vertices; then: if ab exists, then we color vertex a black; if ab does not exist, then we color exactly one vertex $t_j \in T_{one,Uncol} \cap T_j$ black such that $w(t_j u_j) \leq w(tu_j)$ for $t \in T_{one,Uncol} \cap T_j$.

Then let us summarize the above: In this case the problem of checking if a d.i.m. M exists can be solved in polynomial time by Lemma 4 as follows:

(a) For each vertex $t_1 \in T_1$ such that t_1 has a neighbor in T_2, and for each vertex $t'_2 \in T_2$ such that t'_2 is a non-neighbor of t_1 in T_2 (such a non-neighbor may not exist), do as follows:

(a.1) Run Procedure Extend[W-in-M] with $W = \{t_1, t'_2\}$. If it returns a partition of $T_{one} \cap W'$, namely $\{T_{one,Col}, T_{one,Uncol}\}$, then go to Step (a.2). Note that $T_{one,Uncol} \subseteq T_3 \cup \ldots \cup T_k$, and that more generally $G[(S_2 \setminus S_{2,Col}) \cup T_{one,Uncol}]$ is a subgraph of $G[\{u_3, \ldots, u_k\} \cup T_3 \cup \ldots \cup T_k]$.

(a.2) For each connected component Q of $G[(S_2 \setminus S_{2,Col}) \cup T_{one,Uncol}]$ do as follows: for each $q \in Q$, compute a minimum finite weight d.i.m. of Q (if it exists), say M', with $\{u_i, u_{j_1}, \ldots, u_{j_h}\} \in V(M)$, and with q being in $V(M')$, as shown above, and choose a minimum weight solution (if a solution exists) over $q \in Q$.

(a.3) Obtain a minimum finite weight d.i.m. containing t_1 and t'_2 by collecting those solutions found in steps (a.1)-(a.2) (if those solutions exist).

(b) Analogously, for each vertex $t_2 \in T_2$ such that t_2 has a neighbor in T_1, and for each $t'_1 \in T_1$ such that t'_1 is a non-neighbor of t_2 in T_1 (such a non-neighbor may not exist), proceed as in steps (a.1), (a.2), (a.3), by symmetry.

(c) Choose a minimum finite weight solution (if such a solution exists) among those found in steps (a)-(b) respectively for $(t_1, t'_2) \in T_1 \times T_2$ and for $(t'_1, t_2) \in T_1 \times T_2$ as defined above and return M, or return that G has no d.i.m. M with xy.

5 The case $N_4 \neq \emptyset$

The aim of this section is to reduce the graph step by step so that finally $N_4 = \emptyset$.

5.1 Components of N_4

The aim of this subsection is to reduce the graph so that N_4 becomes an independent set. For showing this, we need several lemmas:

Lemma 5. *N_4 is P_3-free.*

Proof. Suppose to the contrary that there is a P_3 in G with vertices $a, b, c \in N_4$ and edges ab and bc. Let a' be a neighbor of a in N_3. This proof follows the principle of the proof of (3). Let us recall that $\{r, x, y\}$ induces a P_3 with edge rx. Then, to avoid a P_8 in the subgraph induced by $c, b, a, a', N_2 \cup N_1, x, y$ (in detail, denoted as a'' a neighbor of $a' \in N_2$, and denoted as r'' a neighbor of a'' in N_1, the P_8 would be induced by c, b, a, a', a'', and:

either r'', x, y if $r'' = r$, or r'', x, r if $r'' \neq r$), a' sees either b or c but not both since G is diamond-free.

Case 1. a' sees c (and misses b).

Then a', a, b, c induce a C_4 in G, and thus, by Observation 1 (ii), either $a', b \in V(M)$ (and $a, c \in I$), or $a, c \in V(M)$ (and $a', b \in I$).
Assume first that $a', b \in V(M)$ (and $a, c \in I$). Let b^* be the M-mate of b. Since by (7), no edge between N_3 and N_4 is in M, it follows that $b^* \in N_4 \cup N_5$ but then to avoid a P_8 (in the subgraph induced by b^*, b, a, a', $N_2 \cup N_1, x, y$), a sees b^*, and to avoid a P_8 (in the subgraph induced by b^*, b, c, a',$N_2 \cup N_1, x, y$), c sees b^* but now a, b, b^*, c induce a diamond which is a contradiction.
Thus, assume that $a, c \in V(M)$ (and $a', b \in I$). Let a^*, c^* respectively be the M-mates of a and c. Since by (7), no edge between N_3 and N_4 is in M, it follows that $a^*, c^* \in N_4 \cup N_5$. Let b' be a neighbor of b in N_3; clearly, $b' \neq a'$. Then $b' \in V(M)$ (since $b \in I$). Then b' misses c, c^*, and thus a P_8 arises (in the subgraph induced by c^*, c, b, b', $N_2 \cup N_1, x, y$ if $bc^* \notin E$ or in the subgraph induced by a^*, a, b, b', $N_2 \cup N_1, x, y$ if $bc^* \in E$; in that case, $ba^* \notin E$ since G is butterfly-free). Thus, Case 1 is impossible.

Case 2. a' sees b (and misses c).

Let c' be a neighbor of c in N_3. By symmetry with respect to Case 1, c' sees b (and misses a). Then the subgraph induced by a', a, b, c, c' contains a butterfly or a diamond. Thus, also Case 2 is impossible which completes the proof of Lemma 5. $\square$

Recall that a graph is P_3-free if and only if it is the disjoint union of complete graphs. Since we can assume that G is K_4-free, we have:

Corollary 2. *The components of N_4 are triangles, edges or isolated vertices.*

5.1.1 Triangles in N_4

Lemma 6. *Let H be a triangle component of N_4 with vertices a, b, c, edges ab, ac, bc, and let $A := N(a) \cap N_3$, $B := N(b) \cap N_3$, and $C := N(c) \cap N_3$. Then the following statements hold:*

- (i) *A, B, C are pairwise disjoint independent sets.*
- (ii) *$H \textcircled{0} N_5$.*
- (iii) *$(A \cup B \cup C) \cap S_3 = \emptyset$.*
- (iv) *There exists $j, 1 \leq j \leq k$, such that $A \cup B \cup C \subseteq T_j$.*

Proof. (i): Holds by Observation 4 since G is (K_4, diamond, butterfly)-free.

(ii): Without loss of generality, suppose to the contrary that there is a neighbor of c in N_5, say z. Then z misses b, otherwise a diamond or a K_4 arises. Let b' be a neighbor of b in N_3. Then by (i), b' misses c but now, a P_8 arises (with z, c, b, b', $N_2 \cup N_1$ and a P_3 containing x, y).

(iii): Without loss of generality, suppose to the contrary that there is a vertex $a' \in A \cap S_3$, say $a'u_1 \in E$ and $a'u_2 \in E$. Let $b' \in B$ and $c' \in C$. If $b' \in S_3$ and $c' \in S_3$ as well, then

$a, b, c \in V(M)$ (recall that by Lemma 1 (iv), $S_3 \subseteq I$). Thus, assume that $b' \notin S_3$, i.e., b' has only one neighbor in $u_1, \ldots, u_k$ and thus, b' misses u_1 or u_2, say $b'u_1 \notin E$. Then if $a'b' \notin E$, the subgraph induced by $b', b, a, a', u_1, N_1, x, y$ contains a P_8, and if $a'b' \in E$, the subgraph induced by $c, b, b', a', u_1, N_1, x, y$ contains a P_8 which is a contradiction.

(iv): The proof is similar to that of (iii); without loss of generality, let $a' \in A$ see u_1 and suppose to the contrary that there is a vertex $b' \in B$ missing u_1. Then if $a'b' \notin E$, the subgraph induced by $b', b, a, a', u_1, N_1, x, y$ contains a P_8, and if $a'b' \in E$, the subgraph induced by $c, b, b', a', u_1, N_1, x, y$ contains a P_8 which is a contradiction. $\square$

As in Lemma 6, for a triangle $a_ib_ic_i$ in N_4 let A_i $(B_i, C_i$, respectively) denote the neighborhood of a_i (of b_i, c_i, respectively) in N_3.

Corollary 3. *There exists $j, 1 \leq j \leq k$, such that for all triangles $a_ib_ic_i$ in N_4, $A_i \cup B_i \cup C_i \subseteq T_j$.*

Proof. Let $a_1b_1c_1$ and $a_2b_2c_2$ be two triangles in N_4 such that, without loss of generality, $A_1 \cup B_1 \cup C_1 \subseteq T_1$. If there is a vertex in $A_2 \cup B_2 \cup C_2 \setminus T_1$, say $a'_2 \in A_2$ with $a'_2u_1 \notin E$ then by Lemma 6, a P_8 arises. Thus, $A_2 \cup B_2 \cup C_2 \subseteq T_1$ holds as well. $\square$

From now on, without loss of generality, suppose that for every triangle $a_ib_ic_i$ in N_4, $A_i \cup B_i \cup C_i \subseteq T_1$. Assume that for the triangle $a_1b_1c_1$, the M-edge is $b_1c_1 \in M$. Then $A_1 = \{u'_1\}$ since otherwise, if there is $a' \in A_1$ with $a' \neq u'_1$ then the edge $aa' \in E$ is not dominated by M. Since every triangle contains exactly one M-edge, this implies that one of the sets A_2, B_2, C_2 is equal to $\{u'_1\}$, say $A_2 = \{u'_1\}$ which forces the M-edge $b_2c_2 \in M$ and similarly for every triangle $a_ib_ic_i$ in N_4.

Thus, if there is a triangle in N_4, we have to consider three possible cases according to the M-edges in the triangles (which in each of the cases can be considered as xy-forced).

5.1.2 Edges in triangle-free N_4

From now on, we can assume that N_4 is triangle-free. If component H in N_4 is not a triangle then by Lemma 5, H is either a vertex or an edge.

Lemma 7. *Let H be a component of N_4 and assume that $H \circledcirc N_5$. Then we have:*

(i) If $H = \{h\}$ then $h \in I$.

(ii) If $H = \{a, b\}$ with $ab \in E$ then $ab \in M$ and thus, ab is an xy-forced M-edge.

Proof. The lemma follows by (7) - none of the edges in N_3 and between N_3 and N_4 is in M. $\square$

From now on, we can assume that N_4 is triangle-free and every edge in N_4 has a neighbor in N_5. If uv is an edge in N_4 then by (9), we can assume that u and v do not have a common neighbor in N_3; let $u' \in N_3$ ($v' \in N_3$, respectively) be a neighbor of u (of v, respectively).

Lemma 8. *Let edge $ab \in E$ be a component H in N_4 (i.e., $\{a, b\} \circledcirc (N_4 \setminus \{a, b\})$) and let $c \in N_5$ be a neighbor of ab. Let $A := N(a) \cap N_3$ and $B := N(b) \cap N_3$. Then the following statements hold:*

(*i*) *Any neighbor* $c \in N_5$ *of* ab *must see both of* a *and* b.

(*ii*) $A \cap B = \emptyset$ *and* A, B *are independent sets.*

(*iii*) *For all* $a' \in A$ *and* $b' \in B$, $N(a') \cap N_2 = N(b') \cap N_2$.

(*iv*) *If there is* $a' \in A$ *with* $|N(a') \cap N_2| \geq 2$ (*there is* $b' \in B$ *with* $|N(b') \cap N_2| \geq 2$, *respectively*), *then* $A⓪B$ *and* ab *is an* xy*-forced* M*-edge.*

(*v*) *Otherwise, if for all* $a' \in A$, $|N(a') \cap N_2| = 1$ *and for all* $b' \in B$, $|N(b') \cap N_2| = 1$ *then there is an index* $i, 1 \leq i \leq k$ *such that* $A \cup B \subseteq T_i$.

Proof. (*i*): If a neighbor $c \in N_5$ of ab sees only one of a and b, say $bc \in E$ and $ac \notin E$, then there is a P_8 in the subgraph induced by c, b, a, a', $N_2 \cup N_1$ and a P_3 containing x, y. Thus, we can assume that each edge component in N_4 is contained in such a triangle with a common neighbor in N_5.

(*ii*): By (9), we can assume that a and b do not have a common neighbor in N_3. Moreover, since a and b have the common neighbor $c \in N_5$, a common neighbor of a and b in N_3 would lead to a diamond. Thus, $A \cap B = \emptyset$. Moreover, A and B are independent sets since otherwise, there is a butterfly in G.

(*iii*): Without loss of generality, suppose to the contrary that $a' \in A$ sees u_1 and $b' \in B$ misses u_1. Then if $a'b' \in E$, a P_8 arises in the subgraph induced by $c, b, b', a', u_1, N_1, x, y$, and if $a'b' \notin E$, a P_8 arises in the subgraph induced by $b', b, a, a', u_1, N_1, x, y$.

(*iv*): Without loss of generality, assume that $a' \in A$ sees u_1 and u_2. Then by (*iii*) each vertex of $A \cup B$ sees u_1 and u_2. Then $A⓪B$, since otherwise a diamond arises. Moreover, since $\{u_1, a', u_2, b'\}$ induce a C_4, $a' \neq u_1'$ and $a' \neq u_2'$, and thus, for the C_5 induced by $\{u_1, a', b', a, b\}$ (with $b' \in B$), exactly one edge is in M (recall Observation 1 (*i*) for C_5). Then, since $a', b' \in I$ (as they are in S_3), the only possible way is that $ab \in M$.

(*v*): It follows by statement (*iii*). □

According to Lemma 8 (*iv*)-(*v*), in what follows let us assume that, for any triangle abc with an edge ab in N_4 and $c \in N_5$, $A \cup B \subseteq T_j$ for some index j, $1 \leq j \leq k$.

Lemma 9. *Let* a_1b_1 *and* a_2b_2 *be distinct edge components in* N_4 *such that* $a_1b_1c_1$ *and* $a_2b_2c_2$ *are triangles with* $c_1, c_2 \in N_5$, *and denote by* A_i (B_i, *respectively*) *the neighborhood of* a_i (b_i, *respectively*), $i = 1, 2$, *in* N_3. *Then there is an index* $j, 1 \leq j \leq k$ *such that* $A_1 \cup B_1 \cup A_2 \cup B_2 \subseteq T_j$.

Proof. Clearly, $c_1 \neq c_2$ since otherwise there is a butterfly in G. Now, if there are two such triangles, say $a_1b_1c_1$ and $a_2b_2c_2$ such that without loss of generality, there are $a_1' \in A_1$ with $u_1a_1' \in E$ and $a_2' \in A_2$ with $u_2a_2' \in E$ then a P_8 arises. □

Let $\{a_1b_1c_1, \dots, a_\ell b_\ell c_\ell\}$, $\ell \leq m$, be the set of all triangles with an edge a_ib_i in N_4 and $c_i \in N_5$. As above, denote by A_i (B_i, respectively) the neighborhood of a_i (b_i, respectively), in N_3.

Without loss of generality, assume that u_1 is a common N_2-neighbor of A_i and B_i, $i \in \{1, \dots, \ell\}$. Now there are at most n (where $n = |V|$) possible cases for $u_1u_1' \in M$ and the M-edges in the triangles according to the property whether the M-mate u_1' of u_1 is in $A_1 \cup B_1 \cup \dots \cup A_\ell \cup B_\ell$ or not (which implies the other M-edges in the triangles):

Corollary 4.

(*i*) *If for* $i \in \{1, \ldots, \ell\}$ *and for* $a'_i \in A_i$, $u_1 a'_i \in M$, *then:*

- *for all* j *such that* $a'_i \notin A_j$ *and* $a'_i \notin B_j$, *it follows that* $a_j b_j \in M$;
- *for all* j *such that* $a'_i \in A_j$ *and* $a'_i \notin B_j$, *it follows that* $b_j c_j \in M$;
- *for all* j *such that* $a'_i \notin A_j$ *and* $a'_i \in B_j$, *it follows that* $a_j c_j \in M$.

Likewise, by symmetry, if for $i \in \{1, \ldots, \ell\}$ *and for* $b'_i \in B_i$, $u_1 b'_i \in M$, *the corresponding implications follow.*

(*ii*) *If for all* $i \in \{1, \ldots, \ell\}$ *and for all* $(a'_i, b'_i) \in A_i \times B_i$, *neither* $u_1 a'_i \in M$ *nor* $u_1 b'_i \in M$ *then for all* $i \in \{1, \ldots, \ell\}$, $a_i b_i \in M$.

Subsequently, we can assume that N_4 is an independent set.

5.2 Components of N_5

Throughout this subsection, let H be a component in N_5. Recall that we can assume that N_4 is an independent set.

Lemma 10. *The following statements hold:*

(*i*) *For every neighbor* $u \in N_4$ *of any vertex of* H, $u \textcircled{1} H$ *holds.*

(*ii*) H *is either a single vertex or an edge.*

Proof. (*i*): It follows since otherwise a P_8 arises (with a P_3 containing x, y).
(*ii*): It follows by statement (*i*) and since G is (diamond, K_4)-free. □

Now we have two cases which will be examined in the following subsections.

5.2.1 H is an edge, say $h_1 h_2$

Lemma 11. *Let* $h_1 h_2 \in E$ *be an edge in* N_5, *let* $c \in N_4$ *be a common neighbor of* h_1, h_2 *and let* $N(c) \cap N_5$ *contain another vertex* $h \notin \{h_1, h_2\}$. *Then:*

(*i*) $N(c) \cap N_5$ *is formed by the disjoint union of vertices and edge* $h_1 h_2 \in E$, *and is isolated in* N_5.

(*ii*) *If without loss of generality,* $w(h_1 c) \leq w(h_2 c)$ *then* $h_1 c \in M$ *is an xy-forced* M*-edge.*

Proof. (*i*): By Observation 4, $N(c) \cap N_5$ is formed by the disjoint union of vertices and at most one edge, namely $h_1 h_2 \in E$.
For showing that $N(c) \cap N_5$ is isolated in N_5, suppose to the contrary that there is an edge between $N(c) \cap N_5$ and $N(d) \cap N_5$ for some $d \in N_4$, $d \neq c$. Then there are $h \in N(c) \cap N_5$ and $h' \in (N(d) \setminus N(c)) \cap N_5$ such that $hh' \in E$. Then, by Lemma 10 (*i*), $ch' \in E$ which is a contradiction.

(*ii*): By Observation 1 (*i*), any triangle contains exactly one M-edge. We claim that the M-edge in the triangle $h_1 h_2 c$ must be either $h_1 c$ or $h_2 c$: Suppose to the contrary that $h_1 h_2 \in M$. Then in order to dominate the edge hc, we need another neighbor $c' \in N_4$ of

h such that $c'h \in M$ (clearly, $cc' \notin E$). Now for any neighbor $d \in N_3$ of c, d sees c', since otherwise a P_8 arises (with c', h, c, d, $N_2 \cup N_1$ and a P_3 containing x, y) but then d, c, h, c' induce a C_4 with $hc' \in M$ which is a contradiction to Observation 1 (ii). Thus, either $h_1c \in M$ or $h_2c \in M$ and by the weight condition we can assume that $h_1c \in M$ is an xy-forced M-edge. □

From now on, we can assume that for every $v \in N_4$, $N(v) \cap N_5$ is either an edge or an independent set. Subsequently, we first consider the case when $N(v) \cap N_5$ is an edge.

Lemma 12. *The following statements hold:*

(i) $|N(H) \cap N_4| = 1$, *say* $N(H) \cap N_4 = \{c\}$.

(ii) $N(c) \cap N_3$ *is an independent set.*

Proof. (i): By Lemma 10 (i), $N(h_1) \cap N_4 = N(h_2) \cap N_4$. Let $c \in N(h_1) \cap N_4$. If h_1 has another neighbor $c' \neq c$ in N_4 then by Lemma 10 (i) (and by the assumption that N_4 is an independent set), $c'h_2 \in E$, and thus h_1, h_2, c, c' induce a diamond which is a contradiction.

(ii): It follows by Observation 4 since otherwise, there is a butterfly in G. □

Without loss of generality assume that $w(h_1c) \leq w(h_2c)$. Then let:

$D := N(c) \cap N_3$ (then by Lemma 12 (ii), D is an independent set);

$D_i := T_i \cap D$, for $i \in \{1, \ldots, k\}$.

Lemma 13. *If $D \cap S_3 \neq \emptyset$ or $|D_i| \geq 2$ for some $i \in \{1, \ldots, k\}$, then $h_1c \in M$ is an xy-forced M-edge.*

Proof. First assume that $D \cap S_3 \neq \emptyset$: Since $S_3 \subseteq I$ by Lemma 1 (iv), it follows that $c \in V(M)$, and then since h_1h_2c is a triangle the assertion follows.
If $|D_i| \geq 2$ for some $i \in \{1, \ldots, k\}$ then for every $d \in D_i$, the edges u_id belong to a C_4; then, since $u_i \in V(M)$, by Observation 1 (ii) it follows that $D_i \subseteq I$, and then $c \in V(M)$, and since h_1h_2c is a triangle, Lemma 13 has been shown. □

According to Lemma 13, in what follows let us assume that $D \cap S_3 = \emptyset$ (i.e., $D \subseteq T_{one}$), and that $|D_i| \leq 1$ for all $i \in \{1, \ldots, k\}$.
Let $\{a_1b_1c_1, \ldots, a_\ell b_\ell c_\ell\}$, be the set of all triangles with $a_i \in N_4$ and $b_i, c_i \in N_5$. Without loss of generality, let $w(a_ib_i) \leq w(a_ic_i)$. Clearly, $a_i \neq a_j$ for $i \neq j$ since otherwise there is a butterfly in G, and $a_ia_j \notin E$ since we can assume that N_4 is an independent set.
Similarly as for triangles in N_4 and for triangles with an edge in N_4, we are going to show that there are only polynomially many possible cases for M-edges in these triangles.
Clearly, either $a_ib_i \in M$ or $b_ic_i \in M$ since $a_ib_ic_i$ is a triangle, b_ic_i is a component in N_5 having exactly one neighbor in N_4, namely a_i, and $w(a_ib_i) \leq w(a_ic_i)$.
Let d_i denote a neighbor of a_i in N_3. By Lemma 13, we can assume that every d_i sees only one of $u_1, \ldots, u_k$.

Lemma 14. *Let $a_1b_1c_1$ and $a_2b_2c_2$ be triangles as above with $b_1, b_2, c_1, c_2 \in N_5$, and denote by d_i a neighbor of a_i, $i = 1, 2$, in N_3. If $d_1 \in T_1$ and $d_2 \in T_2$ then d_1, d_2, a_1, a_2 induce a C_4 in G.*

Proof. First let us show that $d_1 d_2 \notin E$. Assume to the contrary that $d_1 d_2 \in E$. Then d_2 misses a_1, since otherwise a butterfly arises. Let us recall that $\{r, x, y\}$ induces a P_3 with edge rx. Then there is a P_8 with b_1, a_1, d_1, d_2, u_2, N_1 and x, y which is a contradiction. Thus $d_1 d_2 \notin E$.
Since there is no P_8 in the subgraph induced by $b_1, a_1, d_1, u_1, N_1, u_2, d_2, a_2, b_2$, it follows that either $d_1 a_2 \in E$ or $d_2 a_1 \in E$. We claim that $d_1 a_2 \in E$ if and only if $d_2 a_1 \in E$: In fact, if $d_1 a_2 \in E$ and $d_2 a_1 \notin E$, then a P_8 is induced by $b_1, a_1, d_1, a_2, d_2, u_2$, a vertex of N_1, and x or y; the other implication can be shown similarly by symmetry. Then a C_4 is induced by d_1, d_2, a_1, a_2. □

Now the C_4 leads to the fact that $a_1 b_1 \in M$ if and only if $a_2 b_2 \in M$. We say that two triangles $a_1 b_1 c_1$ and $a_2 b_2 c_2$ are *C_4-connected* if there are d_1, d_2 as above such that d_1, d_2, a_1, a_2 induce a C_4 in G, and we say that a set of such triangles is a *C_4-connected component* if there is a sequence of such C_4-connected pairs reaching all of them. Obviously, for such a component, there are only two possibilities for M-edges.

Then let us focus on triangles which are not in such a C_4-connected component. Similarly as for Lemma 9, we claim:

Lemma 15. *Let $a_1 b_1 c_1$ and $a_2 b_2 c_2$ be triangles as above with $b_1, b_2, c_1, c_2 \in N_5$, and denote by d_i a neighbor of a_i, $i = 1, 2$, in N_3. Assume that $a_1 b_1 c_1$ and $a_2 b_2 c_2$ are not C_4-connected. Then there is an index $j, 1 \leq j \leq k$ such that $d_1, d_2 \in T_j$.*

Proof. If there are two such triangles $a_1 b_1 c_1$ and $a_2 b_2 c_2$ such that d_1, d_2 do not have a common neighbor in N_2, say without loss of generality, $u_1 d_1 \in E$ and $u_2 d_2 \in E$ but $u_1 d_2 \notin E$ and $u_2 d_1 \notin E$ then a P_8 arises. □

Let $\{a_1 b_1 c_1, \ldots, a_\ell b_\ell c_\ell\}$, be the set of all triangles, which are not in a C_4-connected component, with an edge $b_i c_i$ in N_5, and let A_i be the neighborhood of a_i in N_3. Assume without loss of generality that $w(a_i b_i) \leq w(a_i c_i)$. Without loss of generality, assume that u_1 is the only N_2-neighbor of A_i, $i \in \{1, \ldots, \ell\}$. Now there are at most n (where $n = |V|$) possible cases for $u_1 u_1' \in M$ and the M-edges in the triangles:

Corollary 5.

(i) *If for $i \in \{1, \ldots, l\}$ and for $d_i \in A_i$, $u_1 d_i \in M$ then for all j such that $d_i \in A_j$ it follows that $b_j c_j \in M$, and for all j such that $d_j \notin A_j$ it follows that $a_j b_j \in M$.*

(ii) *If for all $i \in \{1, \ldots, \ell\}$ and for all $d_i \in A_i$, $u_1 d_i \notin M$ then for all $i \in \{1, \ldots, \ell\}$, $a_j b_j \in M$.*

Subsequently, we can assume that N_5 is an independent set.

5.2.2 H is a single vertex, say h

Lemma 16. *If $|N(h) \cap N_4| \geq 2$ then $h \in I$.*

Proof. Let us recall that $N(h) \cap N_4$ is an independent set. Let $a, b \in N(h) \cap N_4$, $a \neq b$, and let $c \in N_3$ be a neighbor of a. Then $bc \in E$ since otherwise a P_8 with b, h, a, c, $N_2 \cup N_1$ and x, y arises. This holds for every pair of neighbors $a, b \in N(h) \cap N_4$ of h. Thus every edge incident to h is in a C_4, i.e., $h \in I$. □

Lemma 17. *Assume that $|N(h) \cap N_4| = 1$, say $N(h) \cap N_4 = \{v_4\}$. Then $v_4v_5 \in M$ is an xy-forced M-edge for some $v_5 \in N(v_4) \cap N_5$ having exactly one neighbor in N_4, depending on the best alternative.*

Proof. Since we can assume now that N_5 is an independent set, since by (7) no edge between N_3 and N_4 is in M, since by Lemma 16, $v_4u \notin M$ for every $u \in N_5$ having more than one neighbor in N_4, and since v_4 is the only neighbor of h in N_4, it follows that $v_4v_5 \in M$ for some $v_5 \in N(v_4) \cap N_5$ having exactly one neighbor in N_4 (depending on the best alternative; possibly $h = v_5$) since otherwise, the edge v_4h is not dominated. $\square$

Thus, from now on, we can assume that every vertex of N_5 has more than one neighbor in N_4, i.e., $N_5 \subset I$ by Lemma 16.

Lemma 18. *No vertex of N_5 has more than one neighbor in N_4, i.e., $N_5 = \emptyset$.*

Proof. Suppose to the contrary that $|N(h) \cap N_4| \geq 2$ for $h \in N_5$. As shown in the proof of Lemma 16, there is a vertex $c \in N_3$ such that c sees every vertex of $N(h) \cap N_4$. Thus every edge incident onto h is in a C_4 (and thus not in M). Then, since $N_5 \subset I$ and since by (7) no edge between N_3 and N_4 is in M, the edges of such C_4's are not dominated which is a contradiction. $\square$

Thus, from now on, we can assume that $N_5 = \emptyset$ and N_4 is an independent set.

Lemma 19. *If $w \in N_4$ and $w' \in N_3$ is a neighbor of w then w' is an M-mate u'_i of some u_i, and thus, every $w \in N_4$ leads to xy-forced M-edges.*

Proof. Since we can assume that $N_5 = \emptyset$, N_4 is an independent set and there is no M-edge in N_3, edges between N_3 and N_4 must be dominated by M-edges $u_iu'_i$. The only possible way is that every neighbor $w' \in N_3$ of $w \in N_4$ is an M-mate u'_i of some u_i. $\square$

From now on, we can assume that $N_4 = \emptyset$.

6 A polynomial-time algorithm for DIM on P_8-free graphs

In this section let us describe a polynomial-time algorithm to solve DIM on P_8-free graphs. The main part of the algorithm is simple: For every edge xy in a P_3 of G apply the subsequent procedure DIM-with-xy, which either returns a proof that G has no d.i.m. with xy or returns a minimum (finite) weight d.i.m. of G with xy (by the results introduced above). Note that every possible d.i.m. M has to be checked whether it is really a d.i.m.; this can be done in linear time for each candidate M (see [4]).

Procedure DIM-with-xy

Given: A connected (P_8, K_4,diamond,butterfly)-free $G = (V, E)$ with edge weights, and an edge $xy \in E$ of finite weight which is part of a P_3 in G.
Task: Return a proof that G has no d.i.m. M with $xy \in M$ (STOP with failure), or return a d.i.m. M with $xy \in M$ of finite minimum weight (STOP with success).

1. Set $M := \{\{x, y\}\}$. Determine the distance levels $N_i = N_i(xy)$, $1 \leq i \leq 5$, with respect to xy.

2. Check if N_1 is an independent set (see condition (2)) and N_2 is the disjoint union of edges and isolated vertices (see condition (4)). If not, then STOP with failure.

3. For the set M_2 of edges in N_2, apply the Reduction Step for every edge in M_2 correspondingly. Moreover, apply the Reduction Step for each edge bc according to condition (8) and then for each edge $u_i t_i$ according to Lemma 1 (v).

4. If $N_4 \neq \emptyset$ then, using the results of Subsections 5.1 and 5.2 according to the xy-forced M-edges and the polynomially many cases described in Corollaries 3, 4, and 5, split the problem into polynomially many such cases. Then, since each such case allows us to finally reduce the problem to the case in which $N_4 = \emptyset$, solve each such case according to the next step and choose a minimum finite weight solution (if such a solution exists).

5. {Now $N_4 = \emptyset$.} Apply the approach described in Section 4. Then either return that G has no d.i.m. M with $xy \in M$ or return M as a d.i.m. of smallest finite weight with $xy \in M$.

Theorem 1. *Procedure DIM-with-xy is correct and runs in polynomial time.*

Proof. The correctness of the procedure follows from the structural analysis of P_8-free graphs with a d.i.m.
The polynomial time bound follows from the fact that Steps 1, 2 can clearly be done in polynomial time, Step 3 can be done in polynomial time since the Reduction Step can be done in polynomial time, Step 4 can be done in polynomial time by the results in Section 5, and Step 5 can be done in polynomial time as shown in Section 4. □

Since a graph G with a d.i.m. is K_4-free, we can assume that the input graph is K_4-free.

Algorithm DIM-P_8

Given: A connected (P_8, K_4)-free graph $G = (V, E)$ with edge weights.
Task: Determine a d.i.m. of G of finite minimum weight if one exists or find out that G has no d.i.m. of finite weight.

(a) Determine the set F_1 of all mid-edges of diamonds in G, and the set F_2 of all peripheral edges of butterflies in G. Let $M := F_1 \cup F_2$. Check whether M is an induced matching in G. If not then STOP - G has no d.i.m. Otherwise, check whether M is a dominating edge set of G. If yes, we are done. Otherwise apply the Reduction Step for every edge in $F_1 \cup F_2$; without loss of generality, assume that the resulting graph $G' = (V', E')$ is connected (if not, do the next steps for each connected component of G'). Let $G := G'$.

    {From now on, G is (P_8, K_4, diamond, butterfly)-free.}

(b) Check whether G has a single edge $uv \in E$ of finite weight which is a d.i.m. of G. If yes then select such an edge with smallest weight as output and STOP - this is a d.i.m. of G of finite minimum weight.

    {Otherwise, every d.i.m. of G would have at least two edges.}

(c) For each edge $xy \in E$ of finite weight in a P_3 of G carry out procedure DIM-with-xy. If DIM-with-xy stops with failure for all edges xy in a P_3 of G, then STOP - G has no d.i.m. Otherwise, select the best result from all successful applications of the procedure DIM-with-xy. If the result does not have finite weight then STOP - G has no d.i.m. of finite weight. Otherwise, STOP and return the best result as solution.

Theorem 2. *Algorithm DIM-P_8 is correct and runs in polynomial time.*

Proof. The correctness of the procedure follows from the structural analysis of P_8-free graphs with a d.i.m. In particular: concerning Step (b), one can easily verify that if G has a d.i.m. of one edge, then G has no d.i.m. with more than one edge; concerning Step (c), one can refer to Observation 5. The time bound follows from the fact that Step (a) can be done in polynomial time (in particular the Reduction Step can be done in polynomial time), Step (b) can be done in polynomial time, and Step (c) can be done in polynomial time by Theorem 1. □

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