ROOT POLYTOPES AND BOREL SUBALGEBRAS

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ABSTRACT. Let Φ be a finite crystallographic irreducible root system and $\mathcal{P}_\Phi$ be the convex hull of the roots in Φ. We give a uniform explicit description of the polytope $\mathcal{P}_\Phi$, analyze the algebraic-combinatorial structure of its faces, and provide connections with the Borel subalgebra of the associated Lie algebra. We also give several enumerative results.

Keywords: Root system; Root polytope; Weyl group; Borel subalgebra; Abelian ideal

1. INTRODUCTION

Let Φ be a finite crystallographic root system in an n-dimensional Euclidean space $\mathcal{E}$, with scalar product $(\,\cdot\,,\cdot\,)$. We denote by $\mathcal{P}_\Phi$ the convex hull of all the roots in Φ, and we call it the *root polytope* of Φ. The aim of this paper is to give a uniform explicit description of the root polytope $\mathcal{P}_\Phi$.

The root polytopes, or some strictly related objects, are studied in some recents papers. Some authors (see in particular [16] and [17]), intend by *root polytope* the convex hull of the positive roots together with the origin, first introduced in [9] for the root system of type A_n. We call this the *positive* root polytope and, if confusion may arise, we call $\mathcal{P}_\Phi$ the *complete* root polytope. In this paper we only consider the complete root polytope; our results have direct applications to the study of the positive root polytopes of types A_n and C_n (see [4]). In [1], some properties of the complete root polytopes are provided for the classical types through a case by case analysis, using the usual coordinate descriptions of the root systems. In our paper, we give case free statements and proofs for all types.

We provide both a simple global description of the root polytope, and a clear analysis of the combinatorial structure of its faces. Our results have also a direct interesting application in the study of partition functions. More precisely, for all γ in the root lattice, let $|\gamma|$ be the minimum number of roots needed to express γ as a sum of roots. Chirivì uses the results in Sections 3 through 5 to prove several properties of the map $\gamma \mapsto |\gamma|$; in particular, the map is piecewise quasi-linear with the cones over the facets of $\mathcal{P}_\Phi$ as quasi-linearity domains (see [7]). Finally, our results give information about $\bigcup_{w \in W} w(A)$, where A is the fundamental alcove of the affine Weyl group of Φ, since this union set is the polar polytope of $\mathcal{P}_\Phi$ (see [5]).

The Weyl group W of Φ acts on $\mathcal{P}_\Phi$ thus, in order to describe $\mathcal{P}_\Phi$, it is natural to describe the orbits of its faces, of all dimensions, under the action of W. And,

in order to describe the faces, we may describe a special representative in each W-orbit.

The choice of a root basis provides a special set of faces of $\mathcal{P}_\Phi$. We study these faces and then prove that they are representatives of the W-orbits. Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be a root basis and let $\breve{\Omega} = \{\breve{\omega}_1, \ldots, \breve{\omega}_n\}$ be the dual basis of Π in $\mathcal{E}$ i.e., the corresponding set of fundamental coweights. Moreover, let Φ^+ be the set of positive roots with respect to Π. If θ is the highest root of Φ with respect to Π, and $\theta = \sum_{i=1}^{n} m_i \alpha_i$, then each hyperplane $(x, \breve{\omega}_i) = m_i$, for $i = 1, \ldots, n$, supports a face F_i of $\mathcal{P}_\Phi$ of some dimension, that contains θ. Thus for each $I \subseteq \{1, \ldots, n\}$, the intersection F_I of the faces F_i with i in I is a face of $\mathcal{P}_\Phi$. We also set $F_\emptyset = \mathcal{P}_\Phi$. We call the F_I, for all $I \subseteq \{1, \ldots, n\}$, the *standard parabolic faces*. Let V_I be the set of roots in F_I: $V_I = F_I \cap \Phi$. For all nonempty I, the sets V_I are readily seen to be filters, or *dual order ideals*, in the poset Φ^+ with the usual order: $\alpha \leq \beta$ if and only if $\beta - \alpha$ is a sum of positive roots. Each filter of Φ^+ has a natural algebraic interpretation. Let $\mathfrak{g}$ be a complex simple Lie algebra having root system Φ with respect to a Cartan subalgebra $\mathfrak{h}$. For all $\alpha \in \Phi$, let $\mathfrak{g}_\alpha$ be the root space corresponding to α, and let $\mathfrak{b}$ be the standard Borel subalgebra of $\mathfrak{g}$ corresponding to Φ^+: $\mathfrak{b} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \oplus \mathfrak{h}$. For any filter V of Φ^+, $\bigoplus_{\alpha \in V} \mathfrak{g}_\alpha$ is an ideal of $\mathfrak{b}$. Conversely, each ideal of $\mathfrak{b}$ included in $\bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ is obtained in such a way. We say that V is an *abelian* dual order ideal if the corresponding $\mathfrak{b}$-ideal is abelian. Moreover, we say that V is *principal* if the corresponding ideal is: this amounts to say that V has a minimum. A useful technique for dealing with the abelian or with the *ad*-nilpotent ideals of a Borel subalgebra is to see them as special subsets of the affine root system associated to Φ (see [6]). The idea, for the abelian ideals, is due to D. Peterson, and was first described in [15]. We analyze the sets V_I in this way. We denote by $\widehat{\Phi}$ the affine root system associated to Φ and by $\widehat{\Pi}$ an extension of Π to a simple system of $\widehat{\Phi}$: $\widehat{\Pi} = \Pi \cup \{\alpha_0\}$. For each $\Gamma \subseteq \Pi$, we denote by $\Phi(\Gamma)$ the standard parabolic subsystem of Φ generated by Γ, and by $W\langle\Gamma\rangle$ the standard parabolic subgroup of the Weyl group generated by the reflections with respect to the roots in Γ. Similarly, for $\Gamma \subseteq \widehat{\Pi}$, we denote by $\widehat{\Phi}(\Gamma)$ the standard parabolic subsystem of $\widehat{\Phi}$ generated by Γ. For each $I \subseteq \{1, \ldots, n\}$, we set $\Pi_I = \{\alpha_i \mid i \in I\}$. Our first results are the following (see Lemma 3.1, Proposition 3.2, and Corollary 3.8).

Theorem. *Let* $I \subseteq \{1, \ldots, n\}$, $F_I = \{x \in \mathcal{P}_\Phi \mid (x, \breve{\omega}_i) = m_i, \ \forall i \in I\}$, *and* $V_I = F_I \cap \Phi$. *Then* F_I *is a face of* $\mathcal{P}_\Phi$, *and* V_I *is a principal abelian dual order ideal of* Φ^+. *Moreover,*

  (1) *if* $I \neq \emptyset$, V_I *is in bijection with the positive roots in* $\widehat{\Phi}(\widehat{\Pi} \setminus \Pi_I) \setminus \Phi(\Pi \setminus \Pi_I)$, *and through this bijection, the vertices of* F_I *correspond to the positive long roots in this set;*

(2) *the number of vertices of F_I is $[W\langle\Pi\setminus\Pi_I\rangle : W\langle(\Pi\setminus\Pi_I)\cap\theta^\perp\rangle]$.*

In particular, the face F_I depends only on the irreducible component of α_0 in the Dynkin diagram of $\widehat{\Pi}\setminus\Pi_I$. Hence it is clear that different sets of indices may yield the same face. For each $I \subseteq \{1,\ldots,n\}$, the set of all $J \subseteq \{1,\ldots,n\}$ such that $F_J = F_I$ is an interval in the Boolean algebra of the subsets of $\{1,\ldots,n\}$. In particular, it has a maximum and a minimum, that we describe explicitly. The dimension and the stabilizer of F_I can be directly computed from the maximum and the minimum, respectively. Moreover, the minimum provides information about the barycenter of F_I. We sum up these results in the following statement (see Propositions 3.3, 3.5, 3.7, and 4.4).

Theorem. *Let $I \subseteq \{1,\ldots,n\}$. There exist two subsets ∂I and $\overline{I}$ of $\{1,\ldots,n\}$ such that $\{J \subseteq \{1,\ldots,n\} \mid F_J = F_I\} = \{J \subseteq \{1,\ldots,n\} \mid \partial I \subseteq J \subseteq \overline{I}\}$. Moreover,*

(1) *the dimension of F_I is $|\Pi| - |\overline{I}|$,*
(2) *the stabilizer of F_I in W is $W\langle\Pi\setminus\Pi_{\partial I}\rangle$,*
(3) *the barycenter of F_I lies in the cone generated by $\{\breve{\omega}_i \mid i \in \partial I\}$,*
(4) *$\{\overline{I} \mid I \subseteq \{1,\ldots,n\}\} = \{I \subseteq \{1,\ldots,n\} \mid \widehat{\Phi}(\widehat{\Pi}\setminus\Pi_I)$ is irreducible$\}$.*

In particular, the set of the standard parabolic faces is in bijection with the irreducible standard parabolic subsystems of the affine root system that contain the affine root α_0.

One can easily check that the sets ∂I have at most three roots. Thus each standard parabolic face F_I, of any dimension, can always be obtained as an intersection of one, two, or three faces of type F_i, $i \in \{1,\ldots,n\}$. We call these the *coordinate faces*. By item (3) of the above theorem, the barycenter of any F_I is a positive linear combination of at most three fundamental coweights.

Associating to each F_I its minimal root, we obtain an injective map from the set of standard parabolic faces into the set of positive roots. We characterize the image of this map (Proposition 3.9).

The analysis of the standard parabolic and coordinate faces is made in Sections 3 and 4.

In Section 5 we deal with general faces and prove that each face belongs to the W-orbit of a standard parabolic face. Thus, each face of $\mathcal{P}_\Phi$ corresponds to an abelian ideal of a Borel subalgebra of $\mathfrak{g}$.

Part of the results of this section hold in the context of weight polytopes and follows by Vinberg's results of [18] (see also [14] for a generalization). A weight polytope $P(\lambda)$ is the convex hull of the W-orbit of the weight λ of the Lie algebra $\mathfrak{g}$. Since it is easy to prove that $\mathcal{P}_\Phi$ is the convex hull of the long roots, we have that $\mathcal{P}_\Phi = P(\theta)$, hence Vinberg's results specialize to the root polytope. More precisely, the fact that the orbits of the faces are in bijection with the irreducible subsystems

of the affine root system that contain the affine root can be directly deduced from
Vinberg's results. Since we obtain this fact as an easy consequence of the results
of the previous sections and of a further result that does not hold in the general
context of weight polytopes (Theorem 5.2), we take our proofs independent from
Vinberg's results. We state the main results of Section 5 in the next theorem and
corollary. The theorem sums up Proposition 5.1, Theorem 5.5, and Corollary 5.7;
the corollary corresponds to Corollary 5.6.

Theorem. *The faces F_I form a complete set of representatives of the W-orbits.
Moreover, the f-polynomial of $\mathcal{P}_\Phi$ is*

$$\sum_{I \in \mathcal{I}} [W : W\langle \Pi \setminus \Pi_{\partial I}\rangle]\, t^{n-|I|},$$

where $\mathcal{I} = \{I \subseteq \{1, \ldots, n\} \mid \widehat{\Phi}(\widehat{\Pi} \setminus \Pi_I) \text{ is irreducible}\}$.

In particular, the orbits of the facets correspond to the simple roots of Φ that do
not disconnect the extended Dynkin graph. Thus we obtain the following explicit
representation of $\mathcal{P}_\Phi$ as an intersection of a minimal set of half-spaces.

Corollary. *Let $\Pi_{\mathcal{I}} = \left\{\alpha \in \Pi \mid \widehat{\Phi}\left(\widehat{\Pi} \setminus \{\alpha\}\right) \text{ is irreducible}\right\}$ and let $\mathcal{L}(W^\alpha)$ be a set
of representatives of the left cosets of W modulo the subgroup $W\langle \Pi \setminus \{\alpha\}\rangle$. Then*

$$\mathcal{P}_\Phi = \left\{x \mid (x, w\breve{\omega}_\alpha) \le m_\alpha \text{ for all } \alpha \in \Pi_{\mathcal{I}} \text{ and } w \in \mathcal{L}(W^\alpha)\right\}.$$

*Moreover, the above one is the minimal set of linear inequalities that defines $\mathcal{P}_\Phi$ as
an intersection of half-spaces.*

Finally, in Section 6, we find the minimal faces that contain the short roots, and
prove that they form a single W-orbit. Moreover, we study the 1-skeleton of $\mathcal{P}_\Phi$
and find the special property that either all edges are long roots, or all edges are
the double of short roots.

2. Preliminaries

In this section, we fix the notation and recall the basic results that we most
frequently use in the paper. For basic facts about root systems, Weyl groups, Lie
algebras, and convex polytopes, we refer the reader, respectively, to [3], [2] and [12],
[11], and [10].

Given $n, m \in \mathbb{Z}$, with $n \le m$, we let $[n, m] = \{n, n+1, \ldots, m\}$ and, for $n \in \mathbb{N}\setminus\{0\}$,
we let $[n] = [1, n]$. For every set I, we denote its cardinality by $|I|$. We write :=
when the term at its left is defined by the expression at its right. We denote by
$Span_{\mathbb{R}}X$ the real vector space generated by X.

Let Φ be a finite irreducible (reduced) crystallographic root system in the real
vector space $Span_{\mathbb{R}}\Phi$ endowed with the positive definite bilinear form $(\,\cdot\,,\cdot\,)$. We fix
our further notation on the root system and its Weyl group in the following list:

n	the rank of Φ,
Φ_ℓ	the set of long roots ($\Phi_\ell = \Phi$ in simply laced cases),
Γ_ℓ	$= \Gamma \cap \Phi_\ell$, for all $\Gamma \subseteq \Phi$,
Φ_s	the set of short roots,
Γ_s	$= \Gamma \cap \Phi_s$, for all $\Gamma \subseteq \Phi$,
$\alpha\text{-}(\beta)$	$= (\beta + \mathbb{Z}\alpha) \cap \Phi$, the α-string through β,
$\Pi = \{\alpha_1, \ldots, \alpha_n\}$	the set of simple roots,
$\breve{\Omega} = \{\breve{\omega}_1, \ldots, \breve{\omega}_n\}$	the set of fundamental coweights (the dual basis of Π),
Φ^+	the set of positive roots w.r.t. Π,
$\widehat{\Phi}$	the affine root system associated with Φ,
α_0	the affine simple root of $\widehat{\Phi}$,
$\widehat{\Pi}$	$= \{\alpha_0\} \cup \Pi$,
$\widehat{\Phi}^+$	the set of positive roots of $\widehat{\Phi}$ w.r.t. $\widehat{\Pi}$,
$\Phi(\Gamma),\ \widehat{\Phi}(\Gamma)$	the root subsystem generated by Γ in Φ, in $\widehat{\Phi}$, resp. (for $\Gamma \subseteq \Phi$, $\Gamma \subseteq \widehat{\Phi}$, resp.),
$\Phi^+(\Gamma),\ \widehat{\Phi}^+(\Gamma)$	$= \Phi(\Gamma) \cap \Phi^+,\ \widehat{\Phi}(\Gamma) \cap \widehat{\Phi}^+$, resp.
$c_i(\alpha)$	the i-th coordinate of α w.r.t. $\widehat{\Pi}$: $\alpha = \sum_{i=0}^n c_i(\alpha)\alpha_i$,
$\operatorname{supp}(\alpha)$	$= \{\alpha_i \in \widehat{\Pi} \mid c_i(\alpha) \neq 0\}$, the support of α,
$ht(\alpha)$	$= \sum_{i=0}^n c_i(\alpha)$, the height of the root α,
θ	the highest root in Φ,
θ_s	the highest short root in Φ,
m_i	$= c_i(\theta)$
W	the Weyl group of Φ,
s_α	the reflection with respect to α,
ℓ	the length function of W w.r.t. Π,
$D_r(w)$	$= \{i \in [n] \mid \ell(ws_{\alpha_i}) < \ell(w)\}$, the right descent set of w,
w_0	the longest element of W w.r.t. Π,
$W\langle\Gamma\rangle$	the subgroup of W generated by $\{s_\alpha \mid \alpha \in \Gamma\}$ ($\Gamma \subseteq \Phi$),
$\widehat{W}$	the Weyl group of $\widehat{\Phi}$.

By *the root poset of* Φ (w.r.t. the basis Π) we intend the partially ordered set whose underlying set is Φ^+, with the standard order, $\alpha \leq \beta$ if and only if $\beta - \alpha$ is a nonnegative linear combination of roots in Φ^+. The root poset could be equivalently defined as the transitive closure of the relation $\alpha \lhd \beta$ if and only if $\beta - \alpha$ is a simple root. The root poset hence is ranked by the height function and has the highest root θ as maximum. A dual order ideal is, as usual, a subset I of Φ^+ such that, if $\alpha \in I$ and $\beta \geq \alpha$, then $\beta \in I$.

For the reader convenience, we collect in the following propositions the standard results on root systems that are frequently used in the paper, often without explicit mention.

Proposition 2.1. *Let Φ be any root system. If $\alpha, \beta \in \Phi$, $\alpha \neq -\beta$, $(\alpha, \beta) < 0$, then $\alpha + \beta \in \Phi$. Moreover,*

(1) *if L is a subset of Π which is connected in the Dynkin diagram of Φ, then $\sum_{\alpha \in L} \alpha$ is a positive root,*

(2) *the support of a root is connected in the Dynkin diagram of Φ.*

Proposition 2.2 ([3], Ch. VI, §1)**.** *Let Φ be any root system, and let α and β be non-proportional roots of Φ. Then the set $\{j \in \mathbb{Z} \mid \beta + j\alpha \in \Phi\}$ is an interval $[-q, p]$ containing 0. The α-string through β, α-(β), has exactly $-\frac{2(\gamma, \alpha)}{(\alpha, \alpha)} + 1$ roots, where $\gamma = \beta - q\alpha$ is the origin of the string.*

Proposition 2.3 ([3], Ch. VI, §1, Proposition 2.4)**.** *Let Φ be any root system and let Φ' be the intersection of Φ with a subspace of $Span_{\mathbb{R}}\Phi$. Then*

(1) *Φ' is a root system in the subspace it spans;*

(2) *given any basis Π' of Φ', there exists a basis of Φ containing Π'.*

The following result might be less known than the previous ones and we give a proof of it.

Proposition 2.4. *Let Φ be an irreducible root system such that the i-th coordinate m_i of θ w.r.t. Π is 1. Then $\{-\theta\} \cup \Pi \setminus \{\alpha_i\}$ is a basis of Φ. Moreover, if w_i is the longest element in $W\langle \Pi \setminus \{\alpha_i\}\rangle$, then w_i is an involution, and*

$$w_i(\Pi \setminus \{\alpha_i\}) = -(\Pi \setminus \{\alpha_i\}), \qquad w_i(\alpha_i) = \theta.$$

Proof. The first statement follows from the next ones, since then $\{-\theta\} \cup \Pi \setminus \{\alpha_i\}$ would be the opposite of $w_i(\Pi)$, which is a basis. The fact that w_i is an involution that maps $\Pi \setminus \{\alpha_i\}$ into $-(\Pi \setminus \{\alpha_i\})$ is well known. Moreover w_i permutes the positive roots on $\Phi \setminus \Phi(\Pi \setminus \{\alpha_i\})$. Hence, $ht(w_i(\theta)) = \sum m_j ht(w_i(\alpha_j)) = ht(w_i(\alpha_i)) - \sum_{j \neq i} m_j |ht(w_i(\alpha_j))| = ht(w_i(\alpha_i)) - (ht(\theta) - 1)$, and $ht(w_i(\theta)) > 0$. This implies that $ht(w_i(\alpha_i)) = ht(\theta)$ and $ht(w_i(\theta)) = 1$, whence $w_i(\alpha_i) = \theta$, and $w_i(\theta) = \alpha_i$. $\square$

For the general theory of affine root systems, we refer the reader to [13]. We briefly describe the (untwisted) affine root system $\widehat{\Phi}$ associated with Φ.

We extend $Span_{\mathbb{R}}\Phi$ to a $n+1$ dimensional real vector space $Span_{\mathbb{R}}\Phi \oplus \mathbb{R}\delta$ and set

$$\widehat{\Phi} = \Phi + \mathbb{Z}\delta := \{\alpha + k\delta \mid \alpha \in \Phi, \ k \in \mathbb{Z}\}.$$

Then $\widehat{\Phi}$ is an affine root system in $Span_{\mathbb{R}}\Phi \oplus \mathbb{R}\delta$ endowed with the positive semidefinite symmetric bilinear form that extends the scalar product of $Span_{\mathbb{R}}\Phi$ and has $\mathbb{R}\delta$ as its kernel.

If we take $\alpha_0 = -\theta + \delta$, then $\widehat{\Pi} := \{\alpha_0\} \cup \Pi$ is a root basis for $\widehat{\Phi}$. The set of positive roots of $\widehat{\Phi}$ with respect to $\widehat{\Pi}$ is $\widehat{\Phi}^+ := \Phi^+ \cup (\Phi + \mathbb{Z}^+\delta)$, where $\mathbb{Z}^+$ is the set of positive integers.

Let $\mathfrak{g}$ be a complex simple Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ such that Φ is the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. For each $\alpha \in \Phi$, let $\mathfrak{g}_\alpha$ be the root space of α. For every choice of a basis Π of Φ, we have the corresponding standard Borel subalgebra $\mathfrak{b}(\Pi) := \mathfrak{h} \oplus \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$. We let $\mathfrak{b} := \mathfrak{b}(\Pi)$ if no confusion arises. Being $\mathfrak{h}$-stable, any ideal $\mathfrak{i}$ of $\mathfrak{b}$ is compatible with the root space decomposition. Since, given $\alpha, \alpha' \in \Phi^+$, $[\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha'}]$ is equal to $\mathfrak{g}_{\alpha+\alpha'}$ if $\alpha + \alpha' \in \Phi$ and is trivial otherwise, if $\mathfrak{i} = \sum_{\alpha \in \Gamma} \mathfrak{g}_\alpha$ is an ideal of $\mathfrak{b}$, then $\Gamma \subseteq \Phi^+$ satisfies $(\Gamma + \Phi^+) \cap \Phi \subseteq \Gamma$, or, equivalently, Γ is a dual order ideal in the root poset. If we further require that $\mathfrak{i}$ be abelian, than Γ must satisfy also the abelian condition: $(\Gamma + \Gamma) \cap \Phi = \emptyset$. Indeed, all abelian ideals of $\mathfrak{b}$ are of this kind since they must be *ad*-nilpotent (i.e., included in $\sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$). By a *principal abelian ideal of* $\mathfrak{b}$, we mean an abelian ideal $\mathfrak{i}$ of the form $\mathfrak{i} = \sum_{\alpha \in \Gamma} \mathfrak{g}_\alpha$, where Γ, as a subposet of the root poset, has a minimum η (hence Γ is an interval since the highest root θ is the maximum). A principal abelian ideal is generated by any non-zero vector of the root space $\mathfrak{g}_\eta$.

3. Standard Parabolic Faces

In this section, we consider a set of distinguished faces of the root polytope P_Φ, and analyze their rich combinatorial structure.

A proper face of $\mathcal{P}_\Phi$ is, by definition, the intersection of $\mathcal{P}_\Phi$ with some affine hyperplane that does not split $\mathcal{P}_\Phi$. Moreover, any intersection of faces is a face. Recall that, in our notation, the highest root θ has m_i as i-th coordinate w.r.t. Π, i.e., $(\breve{\omega}_i, \theta) = m_i$. Hence, each hyperplane $(\breve{\omega}_i, -) = m_i$ contains a face, of some dimension, of $\mathcal{P}_\Phi$.

Definition 3.1. For each $I \subseteq [n]$ and $i \in [n]$, we set
$$F_I := \{x \in \mathcal{P}_\Phi \mid (\breve{\omega}_i, x) = m_i \; \forall i \in I\} \qquad \text{and} \qquad F_i := F_{\{i\}}.$$

Thus, the F_i are proper faces of $\mathcal{P}_\Phi$, all containing θ. We call them the *coordinate faces*. Moreover, $F_\emptyset = \mathcal{P}_\Phi$, and, for all $I \neq \emptyset$, F_I is a proper nonempty face, since $F_I = \cap_{i \in I} F_i$. We call the face F_I, for $I \neq \emptyset$, the *standard parabolic faces* of P_Φ. We also set
$$V_i := F_i \cap \Phi, \qquad \text{and} \qquad V_I := F_I \cap \Phi.$$
It is clear that F_I is the convex hull of V_I.

For any $I \subseteq [n]$, we set
$$\Pi_I := \{\alpha_i \in \Pi \mid i \in I\}.$$

Moreover, we denote by $\widehat{\Phi}_0(\widehat{\Pi} \setminus \Pi_I)$ the irreducible component of α_0 in $\widehat{\Phi}(\widehat{\Pi} \setminus \Pi_I)$, and set
$$\widehat{\Phi}_0^+(\widehat{\Pi} \setminus \Pi_I) := \widehat{\Phi}_0(\widehat{\Pi} \setminus \Pi_I) \cap \widehat{\Phi}^+, \qquad (\widehat{\Pi} \setminus \Pi_I)_0 := \widehat{\Phi}_0(\widehat{\Pi} \setminus \Pi_I) \cap \widehat{\Pi}.$$

The following result provides a first connection of the root polytope with the extended root system.

Lemma 3.1. *Let $I \subseteq [n]$, $I \neq \emptyset$, and consider the subset*

$$-V_I + \delta := \{-\alpha + \delta \mid \alpha \in V_I\}$$

of the affine root system $\widehat{\Phi}$. Then

$$-V_I + \delta = \widehat{\Phi}^+(\widehat{\Pi} \setminus \Pi_I) \setminus \Phi = \widehat{\Phi}_0^+(\widehat{\Pi} \setminus \Pi_I) \setminus \Phi.$$

Proof. It is clear that the coordinate c_0 is constantly 1 on $-\Phi^+ + \delta$. Since $\beta \in V_I$ if and only if $(\beta, \breve{\omega}_i) = (\theta, \breve{\omega}_i)$ for all $i \in I$, and $\alpha_0 = -\theta + \delta$, we obtain that $-V_I + \delta$ is contained in the standard parabolic subsystem $\widehat{\Phi}(\widehat{\Pi} \setminus \Pi_I)$ of $\widehat{\Phi}$. Indeed, since each root in $-V_I + \delta$ has α_0 in its support, $-V_I + \delta$ is contained in the set of the positive roots of the irreducible component of $\widehat{\Phi}(\widehat{\Pi} \setminus \Pi_I)$ that contains α_0.

Conversely, it is clear that each positive root in $\Phi_0(\widehat{\Pi} \setminus \Pi_I)$ that has α_0 in its support belongs to $-V_I + \delta$. $\qquad\square$

Proposition 3.2. *Let $I \subseteq [n]$, $I \neq \emptyset$. Then:*
  (1) *as a subposet of Φ^+, V_I has maximum and minimum, and both of them are long roots;*
  (2) *V_I is an abelian dual order ideal, hence the subspace $\mathfrak{i}_{V_I} := \sum_{\alpha \in V_I} \mathfrak{g}_\alpha$ is a principal abelian ideal of $\mathfrak{b}$.*

Proof. (1) It is clear that θ is the maximum of V_I.

Let $\widehat{\eta}_I$ be the highest root of the irreducible component $\widehat{\Phi}_0(\widehat{\Pi} \setminus \Pi_I)$, and set $\eta_I := -\widehat{\eta}_I + \delta$. By Lemma 3.1, we directly obtain that η_I is the minimum of V_I.

(2) The fact that V_I is an abelian dual order ideal follows by noting that the functional $(\breve{\omega}_i, -)$ cannot take values $> m_i$ on the roots. In fact, given $\alpha \in V_I$ and $\beta \in \Phi^+$, $\beta \geq \alpha$, then $m_i = (\breve{\omega}_i, \alpha) \leq (\breve{\omega}_i, \beta) \leq m_i$, for all $i \in I$: hence $\beta \in V_I$ and V_I is a dual order ideal. The abelianity follows by the fact that, for $\alpha, \alpha' \in V_I$, $\alpha + \alpha'$ cannot be a root since $(\breve{\omega}_i, \alpha + \alpha') = 2m_i$. Since V_I has a minimum, the abelian ideal of $\mathfrak{b}$ corresponding to V_I is principal. $\qquad\square$

Remark 3.1. For any subset Σ of Π, the root subsystem $\widehat{\Phi}(\{\alpha_0\} \cup \Sigma)$, through the natural projection of $Span_{\mathbb{R}} \widehat{\Phi}$ onto $Span_{\mathbb{R}} \Phi$, maps onto the root subsystem $\Phi(\{\theta\} \cup \Sigma)$ of Φ. If Σ is a proper subset of Π, this is a bijection and a root system isomorphism. In particular, $\{-\theta\} \cup \Sigma$ is a root basis for $\Phi(\{\theta\} \cup \Sigma)$. It is clear that, with respect to this basis, the coordinate relative to $-\theta$ is at most 1, for all roots in $\Phi(\{\theta\} \cup \Sigma)$. Therefore, if $-\eta$ is the highest root, by Proposition 2.4, $\{\eta\} \cup \Sigma$ is a root basis, too.

The set of positive roots with respect to this latter basis is $\Phi(\{\theta\} \cup \Sigma) \cap \Phi^+$, i.e., $\Phi^+(\{\theta\} \cup \Sigma)$, according to our notation.

It is clear, from Lemma 3.1, that the map $I \mapsto F_I$, that associates to the subset I of $[n]$ the corresponding standard parabolic face, is not injective, in general. In fact, this is an injective map only when Φ is of type A_1 or A_2. We determine explicitly, for each parabolic face F, the set of all I such that $F = F_I$.

For any $I \subseteq [n]$, we set

$$\overline{I} := \{k \mid \alpha_k \notin (\widehat{\Pi} \setminus \Pi_I)_0\}$$

and

$$\partial I := \{j \mid \alpha_j \in \Pi_I, \text{ and } \exists \beta \in (\widehat{\Pi} \setminus \Pi_I)_0 \text{ s. t. } \beta \not\perp \alpha_j\}.$$

We call $\overline{I}$ the *closure* and ∂I the *border* of I.

By definition

$$(\widehat{\Pi} \setminus \Pi_I)_0 = \widehat{\Pi} \setminus \Pi_{\overline{I}}.$$

The closure $\overline{I}$ and the border ∂I depend only on $(\widehat{\Pi} \setminus \Pi_I)_0$, hence $\overline{I} = \overline{\partial I}$ and $\partial I = \partial \overline{I}$.

If we denote by Γ the extended Dynkin diagram of Φ, and by $\Gamma(\Sigma)$, the subdiagram of Γ induced by Σ, for any $\Sigma \subseteq \widehat{\Pi}$, then $\Pi_{\overline{I}}$ is the set of all simple roots exterior to the connected subdiagram $\Gamma((\widehat{\Pi} \setminus \Pi_I)_0)$, while ∂I is the set of simple roots that are exterior and adjacent to $\Gamma((\widehat{\Pi} \setminus \Pi_I)_0)$. In this sense, ∂I is indeed the border of $\overline{I}$.

Remark 3.2. The map $I \mapsto \overline{I}$ has actually the properties of a topological closure operator on the power set of $[n]$. Indeed, it is clear from the definition that $I \subseteq \overline{I}$, and that $I = \overline{I}$ if and only if $\Gamma(\widehat{\Pi} \setminus \Pi_I)$ is connected. Hence we get $\overline{\overline{I}} = \overline{I}$ and $\overline{\emptyset} = \emptyset$; moreover, $\overline{I \cup J} = \overline{I} \cup \overline{J}$, for all $I, J \subseteq [n]$, since the Dynkin diagram of any finite system is a tree.

We illustrate the definition of $\overline{I}$ and ∂I in the following example.

Example 3.1. Let $I = \{5, 7\}$ in Φ of type B_9. Then we see that $\partial I = \{5\}$ and $\overline{I} = \{5, \ldots, 9\}$:
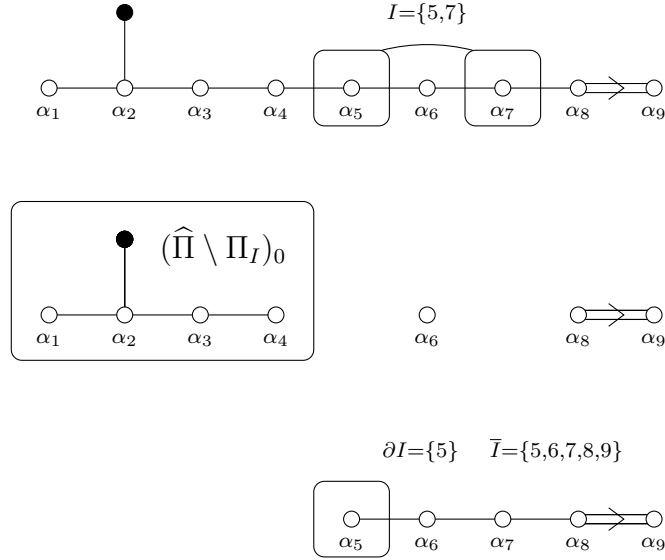
The following proposition follows directly from Lemma 3.1 and from the above definitions.

Proposition 3.3. *Fix $I \subseteq [n]$ and let $J \subseteq [n]$. Then $F_J = F_I$ if and only if*

$$\partial I \subseteq J \subseteq \overline{I}.$$

In particular, the standard parabolic faces of $\mathcal{P}_\Phi$ are in bijection with the connected subdiagrams of the extended Dynkin diagram of Φ that contain the affine node. This bijection is an isomorphism of posets with respect to the inclusions.

Remark 3.3. Let $I \subseteq [n]$, $I \neq \emptyset$.
(1) Consider V_I and $\widehat{\Phi}^+(\widehat{\Pi} \setminus \Pi_{\overline{I}}) \setminus \Phi$ as partial order subsets of the corresponding root posets with respect to the bases Π and $\widehat{\Pi} \setminus \Pi_{\overline{I}}$, respectively. The map from V_I to $\widehat{\Phi}^+(\widehat{\Pi} \setminus \Pi_{\overline{I}}) \setminus \Phi$ sending β to $-\beta + \delta$ (Lemma 3.1) is an anti-isomorphism of posets.
(2) Let η_I be the minimal root in V_I (Proposition 3.2). Then $-\eta_I + \delta$ is the highest root of $\widehat{\Pi} \setminus \Pi_{\overline{I}}$, hence has positive coefficient in all roots in $\widehat{\Pi} \setminus \Pi_{\overline{I}}$ and, as observed in the the proof of Lemma 3.1, the coefficient c_0 is 1. It follows that $\overline{I}$ can be characterized in term of η_I:

$$\overline{I} = \{i \in [n] \mid c_i(\eta_I) = m_i\}$$

(recall that, in our notation, $\theta = \sum m_i \alpha_i$).

In the following Proposition we characterize the non empty closed subsets of $[n]$. For any $\eta \in \Phi^+$, we set

$$I(\eta) := \{i \in [n] \mid c_i(\eta) = m_i\}.$$

Proposition 3.4. *Let $I \subseteq [n]$, $I \neq \emptyset$. Then $I = \bar{I}$ if and only there exists $\eta \in \Phi^+$ such that $I = I(\eta)$.*

Proof. As seen in the above remark, if $I = \bar{I}$, then $I = I(\eta_I)$. Conversely, let $\eta \in \Phi^+$. Then $\eta \in F_{I(\eta)}$, but $\eta \notin F_J$ for all $J \supsetneq I(\eta)$, hence $I(\eta)$ is the maximum of $\{I \subseteq [n] \mid F_I = F_{I(\eta)}\}$. By Proposition 3.3, it follows that $I(\eta) = \overline{I(\eta)}$. $\square$

Remark 3.4. For all w in the affine Weyl group $\widehat{W}$ of Φ, let $N(w) := \{\alpha \in \widehat{\Phi}^+ \mid w^{-1}(\alpha) \in -\widehat{\Phi}^+\}$. It is is well known that $N(w)$ uniquely determines w and that, for all $v, w \in \widehat{W}$, $N(vw) = \big(N(v) \triangle v(N^{\pm}(w))\big) \cap \widehat{\Phi}^+$, where, for all $S \subseteq \widehat{\Phi}^+$, $S^{\pm}$ stands for $S \cup -S$, and $\triangle$ denotes the symmetric difference. From this last relation, we easily obtain that, given a standard parabolic face F_I,

$$-V_I + \delta = \widehat{\Phi}^+(\widehat{\Pi} \setminus \Pi_{\bar{I}}) \setminus \widehat{\Phi}^+(\Pi \setminus \Pi_{\bar{I}}) = N(w_{0\bar{I}}\widehat{w}_{0\bar{I}}),$$

where $\widehat{w}_{0\bar{I}}$ is the longest element in $\widehat{W}\langle\widehat{\Pi} \setminus \Pi_{\bar{I}}\rangle$ and $w_{0\bar{I}}$ is the longest element in $W\langle\Pi \setminus \Pi_{\bar{I}}\rangle$.

It is a general fact that, if V is an abelian dual order ideal of Φ^+, then there exists an element w_V in $\widehat{W}$ such that $-V + \delta = N(w_V)$. In fact, the correspondence $V \leftrightarrow w_V$ is a bijection between the abelian dual order ideals of Φ^+ and the set of all w in $\widehat{W}$ such that $N(w) \subseteq -\Phi^+ + \delta$ (Peterson, see [15]).

By Proposition 3.3, the set of the J giving the face F_I is an interval in the poset of the subsets of $[n]$ with minimum ∂I and maximum $\bar{I}$. We thus obtain, for every root system of rank n, a decomposition of the Boolean algebra of rank n as a disjoint union of intervals whose number is the number of standard parabolic faces $+1$.

Next we show that the maximum $\bar{I}$ yields the dimension and the number of roots in the standard parabolic face F_I. We let $E_{F_I} := Span_{\mathbb{R}}\{\alpha - \alpha' \mid \alpha, \alpha' \in F_I\}$ be the vector subspace underlying the smallest affine subspace containing F_I.

Proposition 3.5. *For all nonempty $I \subseteq [n]$,*

$$E_{F_I} = Span_{\mathbb{R}}\left((\widehat{\Pi} \setminus \Pi_I)_0 \setminus \{\alpha_0\}\right) = Span_{\mathbb{R}}(\Pi \setminus \Pi_{\bar{I}}). \tag{1}$$

Hence, for all $I \subseteq [n]$,

$$\dim F_I = rk\, \widehat{\Phi}_0(\widehat{\Pi} \setminus \Pi_I) - 1 = n - |\bar{I}|. \tag{2}$$

Moreover,

$$|V_I| = \frac{1}{2}\left(\left|\widehat{\Phi}(\widehat{\Pi} \setminus \Pi_I)\right| - \left|\Phi(\Pi \setminus \Pi_I)\right|\right) = \frac{1}{2}\left(\left|\widehat{\Phi}(\widehat{\Pi} \setminus \Pi_{\bar{I}})\right| - \left|\Phi(\Pi \setminus \Pi_{\bar{I}})\right|\right). \tag{3}$$

Proof. By Lemma 3.1, E_{F_I} is cointained in the subspace generated by the differences of the roots in $\widehat{\Phi}_0(\widehat{\Pi} \setminus \Pi_I) \setminus \Phi$. Set $\beta := \sum_{\alpha \in (\widehat{\Pi} \setminus \Pi_I)_0} \alpha$. Since $\widehat{\Phi}_0(\widehat{\Pi} \setminus \Pi_I)$ is irreducible, β is a root and there exists a chain $\beta_0 = \alpha_0, \beta_1, \ldots, \beta_s = \beta$ in $\widehat{\Phi}(\widehat{\Pi} \setminus \Pi_I) \setminus \Phi$, such that $\{\beta_0, \beta_i - \beta_{i-1} \mid i = 1, \ldots, s\} = (\widehat{\Pi} \setminus \Pi_I)_0$. This proves (1).

Equation (2) is obvious for $I = \emptyset$, and follows by Equation (1) for $I \neq \emptyset$, since $\dim F_I = \dim E_{F_I}$.

Equation (3) follows directly from Lemma 3.1 and from the definition of $\overline{I}$.   □

We recall that, for every nonempty $I \subseteq [n]$, $(V_I)_\ell$ and $(V_I)_s$ denote, respectively, the long and the short roots in the standard parabolic face F_I.

Lemma 3.6. *The parabolic subgroup* $W\langle \Pi \setminus \Pi_{\overline{I}} \rangle$ *acts transitively on* $(V_I)_\ell$, *and on* $(V_I)_s$.

Proof. Clearly, $\theta \in (V_I)_\ell$. We prove that any $\gamma \in (V_I)_\ell$ can be transformed into θ by some w in $W\langle \Pi \setminus \Pi_{\overline{I}} \rangle$. By contradiction, let γ be a counterexample of maximal height. Then there exists some $\alpha \in \Pi$ such that $(\gamma, \alpha) < 0$ since θ is the unique long root in the closure of the fundamental Weyl chamber, and thus $s_\alpha(\gamma) = \gamma + c\alpha$ with $c > 0$. By definition of V_I, we have that $\alpha \in \Pi \setminus \Pi_I$ and $\gamma + c\alpha \in V_I$. Indeed, by Proposition 3.3, $\alpha \in \Pi \setminus \Pi_{\overline{I}}$. By the maximality of $ht(\gamma)$, there exists $w \in W\langle \Pi \setminus \Pi_{\overline{I}} \rangle$ such that $\theta = w(\gamma + c\alpha) = ws_\alpha(\gamma)$: a contradiction, since $ws_\alpha \in W\langle \Pi \setminus \Pi_{\overline{I}} \rangle$.

Now assume $(V_I)_s \neq \emptyset$. Then the highest short root θ_s belongs to it, since V_I is a dual order ideal. Since θ_s is the unique dominant short root, we can prove that any other short root β in V_I can be transformed into θ_s by some w in $W\langle \Pi \setminus \Pi_{\overline{I}} \rangle$, with the same argument used for the long roots.   □

In the next Proposition we see that while the maximum $\overline{I}$ yields the dimension and the number of roots in the standard parabolic face F_I (Proposition 3.5), the minimum ∂I yields the stabilizer of F_I.

Proposition 3.7. *For each* $I \subseteq [n]$, *let* $Stab_W F_I = \{w \in W \mid w(F_I) = F_I\}$. *Then*

$$Stab_W F_I = W\langle \Pi \setminus \Pi_{\partial I} \rangle = W\langle \Pi \setminus \Pi_{\overline{I}} \rangle \times W\langle \Pi_{\overline{I}} \setminus \Pi_{\partial I} \rangle.$$

Moreover, $W\langle \Pi_{\overline{I}} \setminus \Pi_{\partial I} \rangle$ *fixes the face* F_I *pointwise, while the action of each* w *in* $W\langle \Pi \setminus \Pi_{\overline{I}} \rangle$ *on* F_I *is nontrivial, unless* $w = e$, *the identity of* W.

Proof. It is clear that $Stab_W F_I = Stab_W V_I = \{w \in W \mid w(V_I) = V_I\}$, therefore it is immediate that $W\langle \Pi \setminus \Pi_I \rangle \subseteq Stab_W F_I$. This should happen for all J such that $F_J = F_I$, in particular for ∂I, therefore $W\langle \Pi \setminus \Pi_{\partial I} \rangle \subseteq Stab_W F_I$.

Now, assume by contradiction that $Stab_W F_I \setminus W\langle \Pi \setminus \Pi_{\partial I} \rangle \neq \emptyset$ and let w be an element of minimal length in $Stab_W F_I \setminus W\langle \Pi \setminus \Pi_{\partial I} \rangle$. Then $D_r(w) \subseteq \partial I$, therefore there exists $i \in \partial I$ such that $w(\alpha_i) \in -\Phi^+$. Let $\eta = \min V_I$ and $\alpha_i = \alpha$ for short. Then, by the proof of Proposition 3.2 and by definition of ∂I, $(\eta, \alpha) > 0$. It follows,

by Proposition 2.1, that $\eta - \alpha \in \Phi \cup \{0\}$: on the other hand, $\eta \neq \alpha$ since $\eta \in F_I$ while $\alpha \notin F_I$, because $w(\alpha) \notin F_I$ and w stabilizes F_I. By assumption, $w(\eta) \in V_I$, in particular, $w(\eta) \in \Phi^+$, therefore $w(\eta - \alpha) = w(\eta) - w(\alpha)$ is a root which is a sum of positive roots, one of which is in F_I. Hence $w(\eta - \alpha) \in F_I$, which is a contradiction since $\eta - \alpha \notin F_I$.

Since the diagram of $\Pi \setminus \Pi_{\partial I}$ is the disjoint union of the diagrams of $\Pi \setminus \Pi_{\overline{I}}$ and $\Pi_{\overline{I}} \setminus \Pi_{\partial I}$, $W\langle \Pi \setminus \Pi_{\partial I} \rangle$ is the direct product $W\langle \Pi \setminus \Pi_{\overline{I}} \rangle \times W\langle \Pi_{\overline{I}} \setminus \Pi_{\partial I} \rangle$. By Lemma 3.1, all elements in V_I are orthogonal to all roots in $\Pi_{\overline{I}} \setminus \Pi_{\partial I}$, therefore $W\langle \Pi_{\overline{I}} \setminus \Pi_{\partial I} \rangle$ fixes F_I pointwise. Finally, each $w \in W\langle \Pi \setminus \Pi_{\overline{I}} \rangle$ is nontrivial on F_I, unless $w = e$, since $W\langle \Pi \setminus \Pi_{\overline{I}} \rangle$ acts faithfully on $Span_{\mathbb{R}}(\Pi \setminus \Pi_{\overline{I}}) = E_{F_I}$. $\qquad\square$

Remark 3.5. It is easy to see that the root polytope $\mathcal{P}_\Phi$ is indeed the convex hull of the long roots. In fact, if Φ is not simply laced, since Φ is irreducible, any short root is contained in a rank 2 non-simply laced subsystem. And it is immediate to check that in such a subsystem a short root can be obtained as a convex linear combination of two long roots.

In particular we have that:

$$\mathcal{P}_{B_3} = \mathcal{P}_{A_3}, \quad \mathcal{P}_{B_n} = \mathcal{P}_{D_n} \text{ for } n \geq 4, \quad \mathcal{P}_{F_4} = \mathcal{P}_{D_4}, \quad \text{and} \quad \mathcal{P}_{G_2} = \mathcal{P}_{A_2}.$$

We explicitly notice that $\mathcal{P}_{C_n}$ is the cross-polytope for all $n \geq 2$ (octahedron for $n = 3$), and that $\mathcal{P}_{A_n}$ and $\mathcal{P}_{B_n} = \mathcal{P}_{D_n}$, for $n \geq 4$, are distinct n-dimensional generalizations of the cuboctahedron $\mathcal{P}_{A_3} = \mathcal{P}_{B_3}$ (see [8]).

Thus, the vertices of $\mathcal{P}_\Phi$ are long roots, and are clearly all the long roots, since W acts on $\mathcal{P}_\Phi$ and is transitive on the long roots.

In particular, the set of vertices in the standard parabolic face F_I is the subset of long roots of V_I. By Lemma 3.6, this set is the orbit of θ under the action of $W\langle \Pi \setminus \Pi_{\overline{I}} \rangle$. Since θ is dominant, its stabilizer in $W\langle \Pi \setminus \Pi_{\overline{I}} \rangle$ is the parabolic subgroup generated by the simple reflections through the roots in $(\Pi \setminus \Pi_{\overline{I}}) \cap \theta^\perp$. Therefore, we obtain the following corollary.

Corollary 3.8. *The number of vertices of the standard parabolic face F_I is $[W\langle \Pi \setminus \Pi_{\overline{I}} \rangle : W\langle (\Pi \setminus \Pi_{\overline{I}}) \cap \theta^\perp \rangle]$.*

Clearly, $(\Pi \setminus \Pi_{\overline{I}}) \cap \theta^\perp$ is the subset of the roots in $\Pi \setminus \Pi_{\overline{I}}$ that are not adjacent to α_0 in the extended Dynkin diagram.

The following proposition, based on Lemma 3.6, is a characterization of the roots of type $\min V_I$, for some nonempty $I \subseteq [n]$.

Proposition 3.9. *Let $\eta \in \Phi^+$. Then $\eta = \min V_{I(\eta)}$ if and only η is long and $(\eta, \alpha_i) \leq 0$ for all $i \in [n] \setminus I(\eta)$.*

In particular, the set of the standard parabolic faces of $\mathcal{P}_\Phi$ is in bijection with the subset $\{\eta \in \Phi_\ell^+ \mid (\eta, \alpha_i) \leq 0, \text{ for all } i \in [n] \setminus I(\eta)\}$ of Φ^+.

Proof. First, we assume that $\eta = \eta_{I(\eta)}$. Then $\eta \in \Phi_\ell$ by Proposition 3.2. By Lemma 3.1 and the proof of Proposition 3.2, $-\eta+\delta$ is the highest root of $\Phi\left(\{\alpha_0\} \cup (\Pi \setminus \Pi_I)\right)$, in particular $-\eta+\delta$ is dominant for this root system, which implies that $(\eta, \alpha_i) \leq 0$ for all $i \in \Pi \setminus \Pi_{I(\eta)}$.

Next, we assume $(\eta, \alpha_i) \leq 0$ for all $i \in [n] \setminus I(\eta)$ and that $\eta \in \Phi_\ell$. The first condition implies that $w(\eta) \geq \eta$ for all $w \in W\langle \Pi \setminus \Pi_{I(\eta)}\rangle$: this is easily seen by induction on the length of w. Indeed, let $w = s_{\alpha_{i_1}} \ldots s_{\alpha_{i_k}}$ be a reduced expression. Then $w(\eta) = s_{\alpha_{i_1}} \ldots s_{\alpha_{i_{k-1}}}(\eta + c\alpha_{i_k})$, with $c \geq 0$. By induction, $s_{\alpha_{i_1}} \ldots s_{\alpha_{i_{k-1}}}(\eta) \geq \eta$ and, by the properties of reduced expressions, $s_{\alpha_{i_1}} \ldots s_{\alpha_{i_{k-1}}}(\alpha_{i_k}) > 0$. The claim follows. Hence, η is a minimal element in its orbit under the action of $W\langle \Pi \setminus \Pi_{I(\eta)}\rangle$. Since η is long, by Lemma 3.6, this orbit is equal to $(V_{I(\eta)})_\ell$, and by Proposition 3.2 $\min V_{I(\eta)} = \min(V_{I(\eta)})_\ell$. It follows that $\eta = \min V_{I(\eta)}$. $\qquad\square$

4. Coordinate faces

Recall that we call coordinate faces the standard parabolic faces of type F_i, $i \in [n]$. It is clear, from the definition, that the standard parabolic facets of $\mathcal{P}_\Phi$ must be of this type, but not all coordinate faces are facets. Indeed, two coordinate faces can be one included in the other: the resulting partial order structure on the set of coordinate faces (or, equivalently by Proposition 4.1, on $[n]$) has a simple uniform description in terms of the Dynkin diagram (see Remark 4.1).

By the results in the previous section, we know that, for all $I \subseteq [n]$, the standard parabolic face F_I coincides with $F_{\partial I}$: ∂I is a small subset of $[n]$, being the set of roots adjacent and exterior to some irreducible subdiagram of the extended Dynkin diagram. In fact, it is easy to check that ∂I has at most 3 elements (more precisely, exactly 3 elements only in a special case occurring in type D and at most 2 elements otherwise). This means that every standard parabolic face F_I can be obtained as the intersection of 3 or (almost always) fewer coordinate faces. We dedicate this section to a deeper analysis of the coordinate faces.

Proposition 4.1. *All coordinate faces are distinct.*

Proof. The proof follows by Lemma 3.1 since, for all $k, h \in [n]$, the connected components of α_0 in the Dynkin diagrams of $\widehat{\Pi} \setminus \{\alpha_k\}$ and $\widehat{\Pi} \setminus \{\alpha_h\}$ cannot coincide if $k \neq h$. $\qquad\square$

Proposition 4.1 follows also by the following argument. Let η_i be the minimal root in the face F_i, whose existence is established by Proposition 3.2. Then the only simple root that can have positive scalar product with η_i is α_i. Indeed, if α_j, $j \neq i$, were a simple root having positive scalar product with η_i, then $\eta_i - \alpha_j$ would be a root in F_i by Proposition 2.1, which is impossible since η_i is the minimum. On the other hand, η_i is a positive root and hence cannot have non-positive scalar

product with all simple roots. Thus $\eta_h \neq \eta_k$ if $h \neq k$, and the faces F_h and F_k cannot coincide.

Another property of the coordinate face F_i, implying Proposition 4.1, is that the barycenter of V_i is parallel to the i-th fundamental coweight $\breve{\omega}_i$. This follows by the following general lemma, which states that every α-string is centered on a vector orthogonal to α.

Lemma 4.2. *Let α and β be non-proportional roots in Φ, and let α-(β) be the α-string through β. Set $\mu := \sum_{\gamma \in \alpha\text{-}(\beta)} \gamma$. Then*

$$(\mu, \alpha) = 0.$$

Proof. We may suppose that β is the origin of its α-string. Then, by Proposition 2.2, α-$(\beta) = \left\{ \beta + j\alpha \mid j = 0, 1, \ldots, -\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \right\}$. The middle vector $\beta - \frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ is orthogonal to α. $\square$

Proposition 4.3. *The barycenter of the roots in the i-th coordinate face F_i is parallel to the i-th fundamental coweight:*

$$\sum_{\alpha \in V_i} \alpha = \frac{m_i |V_i|}{(\breve{\omega}_i, \breve{\omega}_i)} \, \breve{\omega}_i.$$

Proof. By the definition of V_i, if $\alpha \in V_i$ and $\alpha \pm \alpha_j \in \Phi$ for a certain $j \neq i$, then $\alpha \pm \alpha_j \in V_i$. Hence V_i is a union of α_j-string, for all $j \neq i$. By Lemma 4.2, $(\sum_{\alpha \in V_i} \alpha, \alpha_j) = 0$, for all $j \neq i$. Hence $\sum_{\alpha \in V_i} \alpha$ is a multiple of $\breve{\omega}_i$. Since $(\alpha, \breve{\omega}_i) = m_i$ for all $\alpha \in V_i$, we get the assertion. $\square$

Notice that, in the proof of Proposition 4.3, we use the fact that the set of roots V_i in the coordinate face F_i is union of α_j-string, for all $j \neq i$. Consequently, the set of roots V_I in the standard parabolic face F_I is a union of α_j-string, for all $j \notin I$. We obtain the following result on the barycenter of the roots in a standard parabolic face.

Proposition 4.4. *Let $I \subseteq [n]$. The barycenter of the set of roots in the standard parabolic face F_I is in the cone generated by the coweights $\breve{\omega}_i$, $i \in \partial I$.*

Proof. By Proposition 3.3, we may clearly assume that $I = \partial I$. Since V_I is union of α_j-string, for all $j \notin I$, the barycenter $\sum_{\alpha \in V_I} \alpha$ of F_I is orthogonal to α_j, for all $j \notin I$, by Lemma 4.2. Hence it is in the span of the coweights $\breve{\omega}_i$, $i \in I$. Moreover, by Proposition 2.1, $(\alpha, \alpha_i) \geq 0$ for all $\alpha \in V_I$ and $i \in I$, since $\alpha + \alpha_i$ cannot be a root. Hence the barycenter has nonnegative scalar product with all α_i, $i \in I$, and we have the assertion. $\square$

The following result provides several conditions equivalent to the fact that the coordinate face F_i is a facet of the root polytope $\mathcal{P}_\Phi$.

Theorem 4.5. *Let $i \in [n]$. The following are equivalent.*

(1) *The coordinate face F_i is a facet of $\mathcal{P}_\Phi$.*

(2) *The standard parabolic subsystem $\widehat{\Phi}(\widehat{\Pi} \setminus \{\alpha_i\})$ is irreducible.*

(3) *The minimal root η_i of V_i satisfies $(\eta_i, \breve{\omega}_j) \neq m_j$, for all $j \neq i$.*

(4) *$\overline{\{i\}} = \{i\}$.*

(5) *For all $j \neq i$, there exists $\alpha \in V_i$ such that $(\alpha, \breve{\omega}_j) \neq m_j$.*

(6) *The set V_i contains an α_j-string which is non-trivial (i.e. of cardinality > 1), for all $j \neq i$.*

(7) *F_i is maximal among the coordinate faces $\{F_j \mid j = 1, \ldots, n\}$.*

Proof. (4) is equivalent to (1) by Proposition 3.5, to (2) by definition, and to (3) by Remark 3.3.

We prove $3 \Rightarrow 5 \Rightarrow 6 \Rightarrow 1$ and then $1 \Rightarrow 7 \Rightarrow 6$.

The assertion $3 \Rightarrow 5$ is trivial. Fix $j \neq i$ and suppose there exists $\alpha \in V_i$ such that $(\alpha, \breve{\omega}_j) \neq m_j$. Take a chain $\alpha = \gamma_0 \lhd \gamma_1 \lhd \cdots \lhd \gamma_t = \theta$ in the root poset. Since $(\alpha, \breve{\omega}_j) \neq m_j$ and the root poset is ranked, there exists r such that $\gamma_{r+1} = \gamma_r + \alpha_j$. Hence the α_j-string through γ_r is non-trivial and we have $5 \Rightarrow 6$. Clearly $6 \Rightarrow 1$ since the difference of two consecutive roots in an α_j-string is α_j.

It is trivial that $1 \Rightarrow 7$. Let us prove $7 \Rightarrow 6$ by contradiction. So assume that there exists $j \neq i$ such that all α_j-strings in V_i are trivial. For every $\alpha \in V_i$, the difference of two consecutive roots in any chain in the root poset from α to θ cannot be α_j. This means that $(\alpha, \breve{\omega}_j) = m_j$ for all $\alpha \in V_i$. Hence $V_j \supseteq V_i$. Since all the coordinate faces F_k are distinct by Proposition 4.1, F_i is not maximal. $\qquad\square$

Note that, if $m_i = 1$, then α_i coincides with the minimal root η_i of V_i, and hence F_i is a facet. So, for example, in type A, all coordinate faces F_i are facets. On the other hand, in B_n ($n \geq 3$), C_n ($n \geq 2$), and D_n ($n \geq 4$) there are, respectively, only 2, 1, and 3 coordinate facets: F_1 and F_n in B_n, F_n in C_n, and F_1, F_{n-1} and F_n in D_n.

Remark 4.1. By the results in the previous section, the partial order on the set of the coordinate faces can be characterized as follows. Let $i, j \in [n]$. Then the following are equivalent:

(1) $F_i \subseteq F_j$,

(2) $\overline{\{i\}} \supseteq \overline{\{j\}}$,

(3) in the affine Dynkin diagram of Φ, every minimal path from α_j to α_0 contains α_i.

The Hasse diagrams of the posets of the coordinate faces under inclusion are depicted in Tables 1 and 2 for all types. The numerations of the simple roots follow those in [3].

5. UNIFORM DESCRIPTION OF $\mathcal{P}_\Phi$

Clearly, the Weyl group W acts on the set of the faces of $\mathcal{P}_\Phi$. We say that a face F is a *parabolic face* if it is transformed into a standard parabolic face by an element in W.

TABLE 1. Hasse diagrams of the coordinate faces set for the classical Weyl groups
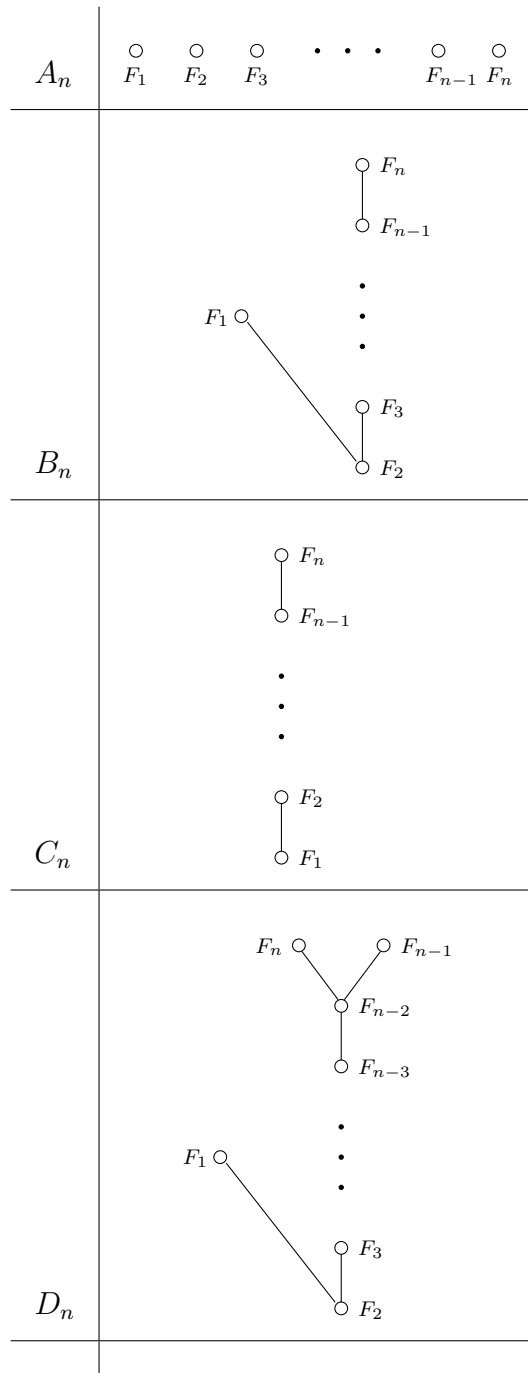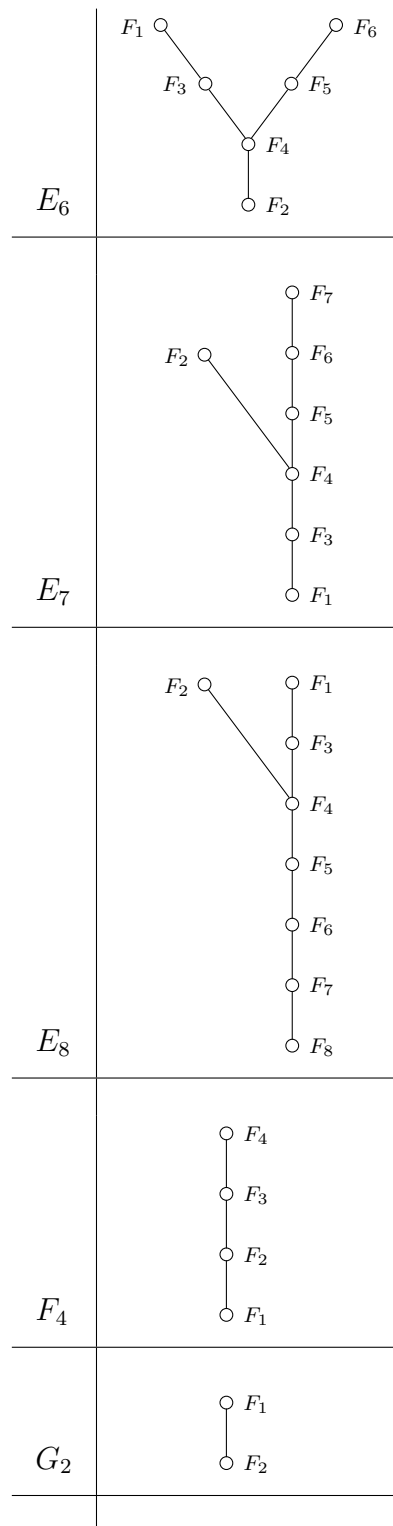
A_n | F_1 F_2 F_3 $\cdots$ F_{n-1} F_n

B_n |
F_n
F_{n-1}
$\vdots$
F_1 — F_2
F_3

C_n |
F_n
F_{n-1}
$\vdots$
F_2
F_1

D_n |
F_n F_{n-1}
F_{n-2}
F_{n-3}
$\vdots$
F_1 — F_2
F_3

TABLE 2. Hasse diagrams of the coordinate faces set for the exceptional Weyl groups

Proposition 5.1. *Two distinct standard parabolic faces cannot be transformed into one another by elements in the Weyl group W.*

Proof. The barycenter of every standard parabolic face is in the closure of the fundamental Weyl chamber, by Proposition 4.4. Since the closure of the fundamental chamber is a fundamental domain for the action of W, the barycenters of two standard parabolic faces in the same W-orbit must coincide. Since distinct faces have distinct barycenters, we get the assertion. $\square$

Given an arbitrary face F of $\mathcal{P}_\Phi$, we let $V_F := \Phi \cap F$ be the set of the roots in F (so, for all $I \subseteq [n]$, $V_{F_I} = V_I$), and $E_F := Span_\mathbb{R}\{\alpha - \alpha' \mid \alpha, \alpha' \in F\}$ be the vector subspace underlying the smallest affine subspace containing F.

The first main result of this section, Theorem 5.2, is that for any face F, V_F cannot be partitioned into two nonempty mutually orthogonal sets. We remark that this is false if we consider the orbit of θ instead of the set of all roots, as we can see in C_2.

Theorem 5.2. *Let F be a face of $\mathcal{P}_\Phi$. The set V_F of the roots in F is not the union of two non-trivial orthogonal subsets.*

Proof. By contradiction, suppose $V_F = V_1 \cup V_2$, where V_1 and V_2 are non-empty sets such that every root in V_1 is orthogonal to every root in V_2. We first prove that, under this assumption, the following holds:

(1) *there is no $\alpha \in \Phi$ that is simultaneously not orthogonal to V_1 and V_2.*

Assume, by contradiction, that there exist $\alpha \in \Phi$, $\beta_i \in V_i$, $i = 1, 2$, such that $(\alpha, \beta_i) \neq 0$, $i = 1, 2$. We may assume $(\alpha, \beta_1) < 0$, hence $\alpha + \beta_1 \in \Phi$. Let f be a linear functional such that $F = \{x \in \mathcal{P}_\Phi \mid f(x) = m_f\}$. We may assume $m_f > 0$. By the symmetry of $\mathcal{P}_\Phi$, f takes values between $-m_f$ and m_f on Φ. If $f(\alpha) = 0$, then $\alpha + \beta_1 \in F$, and since $\alpha + \beta_1 \not\perp \beta_2$, we obtain that $\alpha + \beta_1 \in V_2$. It follows that $\alpha + \beta_1 \perp \beta_1$, hence $s_{\beta_1}(\alpha) = s_{\beta_1}(\alpha + \beta_1 - \beta_1) = \alpha + \beta_1 - s_{\beta_1}(\beta_1) = \alpha + 2\beta_1$. But $f(\alpha + 2\beta_1) = 2m_f$: a contradiction. Thus, $f(\alpha) \neq 0$. Since α and $\alpha + \beta_1$ are roots, we have that $-m_f \leq f(\alpha) < 0$. This implies that $f(\beta - \alpha) > m_f$, for all $\beta \in V_F$, hence that $\beta - \alpha$ cannot be a root for all $\beta \in V_F$. Therefore, $(\alpha, \beta_2) < 0$, and hence $\alpha + \beta_1 + \beta_2$ is a root. This forces $f(\alpha) = -m_f$, hence $-\alpha \in V_F = V_1 \cup V_2$: a contradiction, since $-\alpha$ is not orthogonal to V_1 nor to V_2. Thus (1) is proven.

Now, let $\Phi_F := \Phi \cap Span_\mathbb{R} V_F$, $\Phi_1 := \Phi \cap Span_\mathbb{R} V_1$, and $\Phi_2 := \Phi \cap Span_\mathbb{R} V_2$. We prove that $\Phi_F = \Phi_1 \cup \Phi_2$. Suppose $\alpha \in \Phi_F \setminus (\Phi_1 \cup \Phi_2)$. Being $Span_\mathbb{R} V_F = Span_\mathbb{R} V_1 + Span_\mathbb{R} V_2$, $Span_\mathbb{R} V_1$ is the orthogonal complement of $Span_\mathbb{R} V_2$ in $Span_\mathbb{R} V_F$, and vice-versa. Hence, there exists $\beta_i \in V_i$ such that $(\alpha, \beta_i) \neq 0$, for $i = 1, 2$: this is can not happen, by (1), hence $\Phi_F = \Phi_1 \cup \Phi_2$.

Finally, we are going to prove that we can find $\alpha \in \Phi$ which is simultaneously not orthogonal to V_1 and V_2: this will conclude the proof, since contradicts (1). By

the previous step, and Proposition 2.3, we can find three subsets $\widetilde{\Pi}, \widetilde{\Pi}_1, \widetilde{\Pi}_2 \subseteq \Phi$, which are respectively bases of Φ, Φ_1 and Φ_2, and such that $\widetilde{\Pi}_1, \widetilde{\Pi}_2 \subseteq \widetilde{\Pi}$. Consider any pair of roots $\gamma_i \in \Pi_i$, $i = 1, 2$. Since Φ is irreducible, there is a simple path L connecting γ_1 and γ_2 on the Dynkin graph of $\widetilde{\Pi}$. Since $\widetilde{\Pi}_1$ and $\widetilde{\Pi}_2$ are mutually orthogonal, L contains at least one root other than its extremal points γ_1 and γ_2. Let $\alpha := \sum_{\gamma \in L \setminus \{\gamma_1, \gamma_2\}} \gamma$: then $\alpha \in \Phi$, and $(\alpha, \gamma_i) < 0$, for $i = 1, 2$. Since, for $i = 1, 2$, $\gamma_i \in Span_{\mathbb{R}} V_i$, α cannot be orthogonal to all the roots in V_1, nor to all the roots in V_2. $\qquad \square$

This result implies, as a direct corollary, that the intersection of the linear span of any face with the root system Φ is an irreducible parabolic subsystem (Corollary 5.3). A second direct consequence (Corollary 5.4) is that, for any face F, the vector space E_F is generated by roots, hence that E_F can be transformed by some w in W, into $\cap_{i \in I} \breve{\omega}_i^{\perp}$, for some $I \subseteq [n]$. This last fact holds for the polytope of the orbit of any weight ([18]).

Corollary 5.3. *Let F be any face of $\mathcal{P}_{\Phi}$. Then $Span_{\mathbb{R}} V_F \cap \Phi$ is an irreducible parabolic subsystem of Φ.*

Corollary 5.4. *Let F be any face of $\mathcal{P}_{\Phi}$. Then the subspace E_F is spanned by roots in Φ.*

Proof. Let Γ_F be the graph having V_F as vertex set and where $\{\beta, \beta'\} \subseteq V_F$ is an edge if and only if $(\beta, \beta') \neq 0$. If $(\beta, \beta') \neq 0$, $\beta \neq \beta'$, then $(\beta, \beta') > 0$ since the sum of two roots in a face cannot be a root, and hence $\beta - \beta'$ is a root. By Theorem 5.2, Γ_F is connected; this implies that E_F is spanned by roots: $Span_{\mathbb{R}}(\Phi \cap E_F) = E_F$. $\quad \square$

By Corollary 5.4, together with Lemma 3.1, we obtain a direct proof that all faces are parabolic, that is, that any face of $\mathcal{P}_{\Phi}$ is the transformed, by some element of W, of a standard parabolic face F_I. As like as Corollary 5.4, this result can be obtained (through our results in the previous sections) as a special case of Vinberg's ones on the polytope of the orbit of a weight. However, we prefer to give here a self-contained and well detailed proof.

Remark 5.1. Recall that we denote by w_0 the longest element of W. It is well known that $w_0(\Pi) = -\Pi$. In particular, for all $\beta \in \Phi$, $ht(\beta) = ht(-w_0(\beta))$ and hence $w_0(\theta) = -\theta$. It follows that, if $w_0(\alpha_i) = -\alpha_{i'}$, then $m_i = m_{i'}$. Moreover, since $w_0(\Pi \setminus \{\alpha_i\}) = -\Pi \setminus \{w_0(\alpha_i)\}$, the W-invariance of the scalar product implies that, if $w_0(\alpha_i) = -\alpha_{i'}$, then $w_0(\breve{\omega}_i) = -\breve{\omega}_{i'}$.

Theorem 5.5. *All faces of $\mathcal{P}_{\Phi}$ are parabolic.*

Proof. Let F be a face of dimension $\dim F = n - p$, with $1 \leq p \leq n$. We prove the claim by induction on p. If $p = 1$, then by Corollary 5.4 and Proposition 2.3, we get

that $\Phi \cap E_F$ is a parabolic root susbsystem of Φ of rank $n-1$. It follows that there exist $w \in W$ and $i \in [n]$ such that $w(E_F) = \breve{\omega}_i^\perp$. Therefore, there exists $a \in \mathbb{R}$ such that, for all $\beta \in w(F)$, $(\beta, \breve{\omega}_i) = a$. This forces $a = \pm m_i$. If $a = m_i$ we obtain that $w(F) = F_i$. Otherwise, by Remark 5.1, $w_0 w(F) = F_{i'}$, where $\alpha_{i'} = -w_0(\alpha_i)$.

Now, we assume $n \geq p > 1$ and let $\breve{\omega}_I^\perp = \cap_{i \in I} \breve{\omega}_i^\perp$, for short. Let $\tilde{F}$ be any face such that $F \subseteq \tilde{F}$ and $\dim \tilde{F} = n - p + 1$. By induction, we may assume that $\tilde{F}$ is standard parabolic, say $\tilde{F} = F_I$, with $I \subsetneq [n]$, $\bar{I} = I$, and hence that $E_{\tilde{F}} = \breve{\omega}_I^\perp$, by Proposition 3.5. It follows that $E_F \cap \Phi$ is contained in $\Phi(\Pi \setminus \Pi_I)$, and is a parabolic subsystem of it, of codimension 1. Therefore, there exist $w \in W\langle \Pi \setminus \Pi_I \rangle$ and $i \in [n] \setminus I$ such that $w(E_F) = \breve{\omega}_i^\perp \cap E_{\tilde{F}}$. Since, by Proposition 3.7, $w(F_I) = F_I$, this implies that $w(F) = F_I \cap \{x \mid (x, \breve{\omega}_i) = a\}$, for some $a \in \mathbb{R}$.

Now, let $\eta = \min V_I$ and $l_i = c_i(\eta)$. Since $\theta \in F_I$, we obtain that either $a = m_i$ or $a = l_i$. In the first case we are done, since then $w(F) = w(F_{I \cup \{i\}})$. Then, we assume that $a = l_i$. In this case, $\eta \in w(F)$, and $w(F) \subseteq \eta + \breve{\omega}_{I \cup \{i\}}^\perp$. If v is the longest element in $W\langle \Pi \setminus \Pi_I \rangle$, then, by Proposition 2.4 and Remark 3.3, applied to the root system $\Phi(-(\Pi \setminus \Pi_I) \cup \{\theta\})$, with the root basis $-(\Pi \setminus \Pi_I) \cup \{\theta\}$, we obtain that $v(\eta) = \theta$, and $v(\breve{\omega}_{I \cup \{i\}}^\perp) = \breve{\omega}_{I \cup \{j\}}^\perp$, for some j in $[n] \setminus I$. It follows that $vw(F) = vw(F_{I \cup \{j\}})$, hence F is parabolic. $\square$

In particular, the orbits of the facets of $\mathcal{P}_\Phi$ are in bijection with the simple roots that do not disconnect the extended Dynkin diagram, when removed. In Table 3, we list explicitly, type by type, the simple roots corresponding to the standard parabolic facets. In the pictures, the black node corresponds to the affine root, the crossed node and its label denotes the simple root α_i to be removed and its index i.

We can now give a half-space representation of $\mathcal{P}_\Phi$ and a more explicit expression for the f-polynomial of $\mathcal{P}_\Phi$.

Corollary 5.6. *The polytope $\mathcal{P}_\Phi$ is the intersection of the half-spaces $\{x \mid (x, w\breve{\omega}_i) \leq m_i, \forall w \in W, \forall i \in [n]\}$. A minimal half-space representation is obtained considering only the $i \in [n]$ such that $\Phi(\widehat{\Pi} \setminus \{\alpha_i\})$ is irreducible.*

Since the stabilizer of $\breve{\omega}_i$ is the parabolic subgroup $W\langle \Pi \setminus \{\alpha_i\} \rangle$, in Corollary 5.6 we may let w run over any complete set of left coset representatives of W modulo $W\langle \Pi \setminus \{\alpha_i\} \rangle$. For example, we may consider the set of the minimal coset representatives $\{w \in W \mid D_r(w) \subseteq \{i\}\}$.

Corollary 5.7. *For $\Gamma \subseteq \Pi$, let $\widehat{\Gamma} := \Gamma \cup \{\alpha_0\}$, and set*

$$\mathcal{I} := \{\Gamma \subseteq \Pi \mid \widehat{\Phi}(\widehat{\Gamma}) \text{ is irreducible}\}.$$

The f-polynomial of $\mathcal{P}_\Phi$ is

$$\sum_{\Gamma \in \mathcal{I}} [W : W\langle \Gamma^* \rangle] t^{|\Gamma|},$$

TABLE 3. Subdiagrams corresponding to the standard parabolic facets

where $W\langle\Gamma^*\rangle$ is the subgroup of the Weyl group generated by the reflections with respect to the roots in $\Gamma^* := \Gamma \cup (\widehat{\Gamma}^\perp \cap \Pi)$.

Proof. It follows by Proposition 3.5 , Proposition 3.7, and Theorem 5.5. $\qquad\square$

We end this section with the following elegant statement that can be obtained as an immediate consequence of Theorems 5.5 and Proposition 3.3 and is the $\lambda = \theta$ case of Proposition 3.2 of [18].

Theorem 5.8. *The orbits of the faces of $\mathcal{P}_\Phi$ under the action of W are in bijection with the connected subdiagrams of the extended Dynkin diagram that contain the*

affine node. Equivalently, the orbits of the faces are in bijection with the standard parabolic irreducible root subsystems of $\widehat{\Phi}$ that are not included in Φ.

6. FACES OF LOW DIMENSION

We know that the short roots are convex linear combinations of the long ones (Remark 3.5). Here we first see that if a short root lies on some face F, then it is a convex combination of two long roots in F. Moreover, we prove that if some short root lies on some face, then the faces of lowest dimension containing the short roots form a single orbit of W, and we determine the standard parabolic face in it. Then, we give a very peculiar property of the 1-skeleton of $\mathcal{P}_\Phi$, whose edges are made up of roots.

Lemma 6.1. *Let F be a face of $\mathcal{P}_\Phi$. Then F contains at least one long root. Moreover, if F contains some short root, then the ratio between the squared lengths of the long and the short roots is 2, and each short root in F is the halfsum of two long roots in F.*

Proof. We may assume F standard parabolic, say $F = F_I$, with $I \subseteq [n]$. The first assertion is clear, since θ belongs to any standard parabolic face. If F contains also some short root, then, by Theorem 5.2, there exists two non-orthogonal roots of different lengths, say β and β', in F, with $\frac{(\beta,\beta)}{(\beta',\beta')} = r > 1$. Then $s_{\beta'}(\beta) = \beta - r\beta'$ is a root and, since $\beta, \beta' \in V_I$, $c_i(s_{\beta'}(\beta)) = m_i - rm_i$ for all $i \in I$. This implies that $r = 2$, $c_i(s_{\beta'}(\beta)) = -m_i$, and $-s_{\beta'}(\beta) = 2\beta' - \beta \in F$. Therefore $\beta' = \frac{1}{2}(\beta - s_{\beta'}(\beta))$, a convex combination of the long roots β and $-s_{\beta'}(\beta)$, both lying in F_I. □

The following proposition tells how far the short roots are from being vertices. Recall that θ_s denotes the highest short root of Φ and $I(\theta_s) = \{i \in [n] \mid c_i(\theta_s) = m_i\}$.

Proposition 6.2. *Let Φ be not simply laced. Then the minimal dimension of the faces of $\mathcal{P}_\Phi$ containing a short root is $n - |I(\theta_s)|$. Moreover, the faces of dimension $n - |I(\theta_s)|$ containing a short root form a single W-orbit.*

Proof. Any standard parabolic face F containing a short root contains θ_s, since V_F is a dual order ideal in the root poset and θ_s is greater than any other short root. By Theorem 5.5, it is enough to find the dimension of the minimal standard parabolic face of $\mathcal{P}_\Phi$ containing θ_s. This is clearly $F_{I(\theta_s)}$, and its dimension is $n - |I(\theta_s)|$ by Propositions 3.4 and 3.5. The last statement follows since the action of W is transitive on the set of short roots and the intersection of two distinct faces of dimension $n - |I(\theta_s)|$ does not contain any short root. □

Suppose that Φ be not simply laced. By Lemma 6.1, we already know that, if the ratio between the squared lengths of the long and the short roots is 3, then the short roots lie in the inner part of $\mathcal{P}_\Phi$, and hence $I(\theta_s)$ must be empty. On the

other hand, if the ratio between the squared lengths of the long and the short roots is 2, by a case-by-case argument we see that $I \neq \emptyset$, and hence the short roots are always on the border of $\mathcal{P}_\Phi$. In particular, for types B_n and F_4, the minimal faces containing short roots are the facets, while, for type C_n, the short roots lie on the edges.

We end this section with a direct description of the 1-skeleton of $\mathcal{P}_\Phi$, whose edges are, in fact, made up of roots.

Corollary 6.3. *Let F be a face of $\mathcal{P}_\Phi$ of dimension 1. Then the roots in F form a string with 2 or 3 roots: if γ and γ' are the vertices of F, then either*

   (1) *γ and γ' are not perpendicular, there are no other roots in F, and $\gamma - \gamma'$ is a root, or*
   (2) *γ and γ' are perpendicular, there is only a third root γ'' in F (γ'' short), and $\gamma - \gamma'' = \gamma'' - \gamma'$ is a root (so that $\gamma - \gamma'$ is twice a root).*

Proof. Let $\gamma_0, \gamma_1, \ldots, \gamma_r$ be the roots in F. Since F has dimension 1, we may assume that $\gamma_i = \gamma_0 + k_i\beta$, $k_i \in \mathbb{R}$, for all $i \in [r]$, with $1 = k_1 < k_2 < \cdots < k_r$ and $\mathbb{R}\beta = E_F$. Recall that two roots in a face cannot have negative scalar product. Using Theorem 5.2, we easily show that $(\gamma_i, \gamma_{i+1}) > 0$, for all $i \in [0, r-1]$. Then $\beta \in \Phi$ and $\gamma_i - \gamma_{i-1}$ is both a root and a positive multiple of β, for all $i \in [r]$, by Proposition 2.1. Then $\{\gamma_0, \gamma_1, \ldots, \gamma_r\}$ is the β-string through γ_0. It is well known that the strings can have only 1,2,3, or 4 roots. In this case, it clearly does not have 1 root and it cannot have 4 roots since in a string of 4 roots the product of the first with the forth one is negative. Hence we have done, cause the string with 3 roots are necessarily of the type in the statement (see [3], Ch. VI, §1, n. 4). $\square$

Any face F of $\mathcal{P}_\Phi$ is a facet of the root polytope $\mathcal{P}_{\Phi \cap Span_\mathbb{R}F} \subseteq Span_\mathbb{R}F$. Since the root system $\Phi \cap Span_\mathbb{R}F$ is irreducible by Theorem 5.2, all possible types of faces of dimension k already occur in $(k+1)$-dimensional polytopes of the type studied in this work. In particular, Corollary 6.3 can also be proved by an immediate case-by-case verification of the 2-dimensional root polytopes.

Note that the standard parabolic edges of $\mathcal{P}_\Phi$ are made up of those roots α_k which are adjacent to α_0 in the affine Dynkin diagram of Φ (Proposition 3.5 and Corollary 6.3). Then, by Theorem 5.5, the 1-skeleton of $\mathcal{P}_\Phi$ is made up of the long or the short roots, depending on the length of the simple roots adjacent to α_0. In particular, in type C, the short roots occur, and in all the other cases the long ones.

7. Acknowledgements

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