ON QUATERNION ALGEBRAS OVER SOME EXTENSIONS OF QUADRATIC NUMBER FIELDS

VINCENZO ACCIARO, DIANA SAVIN, MOHAMMED TAOUS, AND ABDELKADER ZEKHNINI

ABSTRACT. Let p and q be two positive primes. Let ℓ be an odd positive prime integer and F a quadratic number field. Let K be an extension of F such that K is a dihedral extension of $\mathbb{Q}$ of degree ℓ over F or K is an abelian ℓ-extension unramified over F assuming ℓ divides the class number of F. In this paper, we obtain a complete characterization of division quaternion algebras $H_K(p,q)$ over K.

1. INTRODUCTION

Let F be a field with $char(F) \neq 2$ and let $a, b \in F \backslash \{0\}$. The generalized quaternion algebra $H_F(a,b)$ is the associative algebra generated over the field F by two elements i and j, subject to the relations $i^2 = a$, $j^2 = b$ and $ij = -ji$.

Quaternion algebras turn out to be central simple algebras of dimension 4 over F. A basis for $H_F(a,b)$ over F is given by $\{1, i, j, ij\}$.

It can be shown that every four dimensional central simple algebra over a field of characteristic $\neq 2$ is a quaternion algebra.

If $x = x_1 1 + x_2 i + x_3 j + x_4 k \in H_F(a,b)$, with $x_i \in F$, the conjugate $\overline{x}$ of x is defined as $\overline{x} = x_1 1 - x_2 i - x_3 j - x_4 k$, and the norm of x as $\boldsymbol{n}(x) = x\overline{x} = x_1^2 - a x_2^2 - b x_3^2 + ab x_4^2$.

If the equations $ax = b$, $ya = b$ have unique solutions for all $a, b \in A$, $a \neq 0$, then the algebra A is called *a division algebra*. If A is a finite-dimensional algebra, then A is a division algebra if and only if A has no zero divisors. In the case of generalized quaternion algebras there is a simple criterion that guarantees them to be division algebras: $H_F(a,b)$ is a division algebra if and only if there is a unique element of zero norm, namely $x = 0$.

Let L be an extension field of F, and let A be a central simple algebra over F. We recall that A is said to split over L, and L is called a splitting field for A, if $A \otimes_F L$ is isomorphic to a full matrix algebra over L.

Several results are known about the splitting behavior of quaternion algebras and symbol algebras over specific fields [4, 5, 7, 11, 12].

Explicit conditions which guarantee that a quaternion algebra splits over the field of rationals numbers, or else is a division algebra, were studied in [3].

In [1] we studied the splitting behavior of some quaternion algebras over quadratic fields. Then, in [2] we extended the previous results to the composite of quadratic number fields.

2000 *Mathematics Subject Classification.* Primary 11R11; 11R21 ; 11R32, 11R52; 11S15; Secondary 11R37, 11R29, 11A41, 11R04, 11F85.

Key words and phrases. Quaternion algebras, quadratic fields, dihedral extension, Hilbert class field .

Traditionally, in order to decide whether a quaternion algebra is a division algebra or it splits, one either looks for the primes which ramify in the algebra or otherwise one has to appeal to Hasse's norm theorem which allows to reduce the problem to computations of local Hilbert symbols [13, 14]. Very recently, Goldstein [8] tried to combine these two techniques.

In this paper, as we did in our previous papers [1, 2], we adopt the former approach, i.e we study the ramification of certain integral primes, and we obtain a nice characterization of quaternion division algebras $H_K(p, q)$ solely in terms of quadratic residues, assuming that p and q are positive primes and K is a dihedral extension of $\mathbb{Q}$ of prime degree over an imaginary quadratic field. The layout of the paper is the following. In Section 2 we state some preliminary results which we will need later. In Section 3 we apply these results to study quaternion algebras over dihedral extensions K of $\mathbb{Q}$ of prime degree l over an imaginary quadratic field.

2. Preliminary results

In this section we recall some basic results concerning quaternion algebras. Unless otherwise stated, when we say "prime integer" we mean "positive prime integer".

Let K be a number field and let $\mathcal{O}_K$ be its ring of integers. If v is a place of K, let us denote by K_v the completion of K at v. We recall that a quaternion algebra $H_K(a, b)$ is said to ramify at a place v of K - or v is said to ramify in $H_K(a, b)$ - if the quaternion K_v-algebra $H_v = K_v \otimes H_K(a, b)$ is a division algebra. This happens exactly when the Hilbert symbol $(a, b)_v$ is equal to -1, i.e. when the equation $ax^2 + by^2 = 1$ has no solutions in K_v. We also recall that the reduced discriminant $D_{H_K(a,b)}$ of the quaternion algebra $H_K(a, b)$ is defined as the product of those prime ideals of the ring of integers $\mathcal{O}_K$ of K which ramify in $H_K(a, b)$. The following splitting criterion for a quaternion algebras is well known [3, Corollary 1.10]:

Proposition 2.1. *Let K be a number field. Then, the quaternion algebra $H_K(a, b)$ is split if and only if its discriminant $D_{H_K(a,b)}$ is equal to the ring of integers $\mathcal{O}_K$ of K.*

If $\mathcal{O}_K$ is a principal ideal domain, then we may identify the ideals of $\mathcal{O}_K$ with their generators, up to units. Thus, in a quaternion algebra H over $\mathbb{Q}$, the element D_H turns out to be an integer, and H is split if and only if $D_H = 1$.

The next proposition gives us a geometric interpretation of splitting [7, Proposition 1.3.2]:

Proposition 2.2. *Let K be a field. Then, the quaternion algebra $H_K(a, b)$ is split if and only if the conic $C(a, b) : ax^2 + by^2 = z^2$ has a rational point in K, i.e. there are $x_0, y_0, z_0 \in K$ such that $ax_0^2 + by_0^2 = z_0^2$.*

The next proposition due to Hasse relates the norm group of extensions of the base field to the splitting behavior of a quaternion algebra [7, Proposition 1.1.7]:

Proposition 2.3. *Let F be a field. Then, the quaternion algebra $H_F(a, b)$ is split if and only if a is the norm of an element of $F(\sqrt{b})$.*

For quaternion algebras it is true the following [7, Proposition 1.1.7]:

Proposition 2.4. *Let K be a field with char $K \neq 2$ and let $a, b \in K\backslash\{0\}$. Then the quaternion algebra $H_K(a, b)$ is either split or a division algebra.*

In particular, this tells us that a quaternion algebra $H_{\mathbb{Q}}(a,b)$ is a division algebra if and only if there is a prime p such that $p|D_{H_{\mathbb{Q}}(a,b)}$.

It is known [10] that if a prime integer p divides $D_{H(a,b)}$ then it must divide $2ab$, hence we may restrict our attention to these primes. In other words, in order to obtain a sufficient condition for a quaternion algebra $H_{\mathbb{Q}(\sqrt{d})}(p,q)$ to be a division algebra, it is important to study the ramification of the primes $2,p,q$ in the algebra $H_{\mathbb{Q}}(p,q)$. The following lemma from the classical book by Alsina [3, Lemma 1.21] gives us a hint:

Lemma 2.5. *Let p and q be two primes, and let $H_{\mathbb{Q}}(p,q)$ be a quaternion algebra of discriminant D_H.*

i. *if $p \equiv q \equiv 3 \pmod 4$ and $(\frac{q}{p}) \neq 1$, then $D_H = 2p$;*
ii. *if $q = 2$ and $p \equiv 3 \pmod 8$, then $D_H = pq = 2p$;*
iii. *if p or $q \equiv 1 \pmod 4$, with $p \neq q$ and $(\frac{p}{q}) = -1$, then $D_H = pq$.*

We recall that a small ramified $\mathbb{Q}$-algebra is a rational quaternion algebra having the discriminant equal to the product of two distinct prime numbers. The following necessary and sufficient explicit condition for a small ramified $\mathbb{Q}$-algebra $H_{\mathbb{Q}}(p,q)$ to be a division algebra over a quadratic field $\mathbb{Q}(\sqrt{d})$ was proved in [1]:

Proposition 2.6. *Let p and q be two distinct odd primes, with p or $q \equiv 1 \pmod 4$ and $(\frac{p}{q}) = -1$. Let $K = \mathbb{Q}(\sqrt{d})$ and let Δ_K be the discriminant of K. Then the quaternion algebra $H_{\mathbb{Q}(\sqrt{d})}(p,q)$ is a division algebra if and only if $(\frac{\Delta_K}{p}) = 1$ or $(\frac{\Delta_K}{q}) = 1$.*

When $q = 2$ and p is a prime such that $p \equiv 3 \pmod 8$, then, according to Lemma 2.5 the discriminant $D_{H_{\mathbb{Q}(p,q)}}$ is equal to $2p$, so $H_{\mathbb{Q}}(p,q)$ is a division algebra. The next proposition, which was proved in [1], shows what happens when we extend the field of scalars from $\mathbb{Q}$ to $\mathbb{Q}(\sqrt{d})$:

Proposition 2.7. *Let p be an odd prime, with $p \equiv 3 \pmod 8$. Let $K = \mathbb{Q}(\sqrt{d})$ and let Δ_K be the discriminant of K. Then $H_{\mathbb{Q}(\sqrt{d})}(p,2)$ is a division algebra if and only if $(\frac{\Delta_K}{p}) = 1$ or $d \equiv 1 \pmod 8$.*

When p and q are primes both congruent to 3 modulo 4 and $(\frac{q}{p}) \neq 1$ then, according to Lemma 2.5(i) the discriminant $D_{H_{\mathbb{Q}}(p,q)}$ is equal to $2p$, so $H_{\mathbb{Q}}(p,q)$ is a division algebra. The next proposition which also was proved in [1], tells us when the quaternion algebra $H_{\mathbb{Q}(\sqrt{d})}(p,q)$ is still a division algebra:

Proposition 2.8. *Let p and q be two odd prime integers, with $p \equiv q \equiv 3 \pmod 4$ and $(\frac{q}{p}) \neq 1$. Let $K = \mathbb{Q}(\sqrt{d})$ and let Δ_K be the discriminant of K. Then the quaternion algebra $H_{\mathbb{Q}(\sqrt{d})}(p,q)$ is a division algebra if and only if $(\frac{\Delta_K}{p}) = 1$ or $d \equiv 1 \pmod 8$.*

3. Main results

Let's ask ourselves now what happens when we consider a quaternion algebra over a Galois extension K of $\mathbb{Q}$, with nonabelian Galois group of order $2l$, where l is an odd prime integer. For this purpose, we recall the following result, which can be found as an exercise in [11, p. 77]:

Remark 3.1. *Let K/F be a finite extension of fields of odd degree, and let $a, b \in F\backslash\{0\}$. Then the quaternion algebra $H_K(a,b)$ splits if and only if $H_F(a,b)$ splits.*

Let us first consider the case $Gal(K/\mathbb{Q}) \cong S_3$, i.e. the dihedral group D_3. The following three propositions will deal with this case:

Proposition 3.2. *Let ϵ be a primitive third root of unity, and put $F = \mathbb{Q}(\epsilon)$. Let $\alpha \in K\backslash\{0,1\}$ be a cubicfree integer, put $K = F(\sqrt[3]{\alpha})$ and let p, q be two distinct odd prime integers such that $(\frac{p}{q}) = -1$ and p or $q \equiv 1 \pmod 4$. Then the quaternion algebra $H_K(p,q)$ is a division algebra if and only if $(\frac{-3}{p}) = 1$ or $(\frac{-3}{q}) = 1$.*

Proof. Clearly $F = \mathbb{Q}(\epsilon) = \mathbb{Q}\left(i\sqrt{3}\right)$ is an imaginary quadratic number field and $[K:F] = 3$. Moreover, $K/\mathbb{Q}$ is Galois and $\mathrm{Gal}(K/\mathbb{Q}) \cong S_3$. According to Remark 3.1 and Proposition 2.4, $H_K(p,q)$ is a division algebra if and only if $H_F(p,q)$ is a division algebra. By Proposition 2.6, this can happen if and only if $(\frac{-3}{p}) = 1$ or $(\frac{-3}{q}) = 1$. □

Proposition 3.3. *Let ϵ be a primitive third root of unity, and put $F = \mathbb{Q}(\epsilon)$. Let also $\alpha \in K\backslash\{0,1\}$ be a cubicfree integer, put $K = F(\sqrt[3]{\alpha})$ and let p be an odd prime integer such that $p \equiv 3 \pmod 8$. Then the quaternion algebra $H_K(p,2)$ is a division algebra if and only if $(\frac{-3}{p}) = 1$.*

Proof. The proof is obtained from the proof of Proposition 3.2 by replacing Proposition 2.6 with Proposition 2.7. □

Proposition 3.4. *Let ϵ be a primitive third root of unity, and put $F = \mathbb{Q}(\epsilon)$. Let $\alpha \in K\backslash\{0,1\}$ be a cubicfree integer, put $K = F(\sqrt[3]{\alpha})$ and let p and q be distinct odd prime integers with $(\frac{q}{p}) \neq 1$ and $p \equiv q \equiv 3$ (mod 4). Then the quaternion algebra $H_K(p,q)$ is a division algebra if and only if $(\frac{-3}{p}) = 1$.*

Proof. The proof is obtained from the proof of Proposition 3.2 after replacing Proposition 2.6 with Proposition 2.8. □

In what follows, let ℓ be an odd positive prime integer and F a quadratic number field. Let K be an extension of F defined as follows: K is a dihedral extension of $\mathbb{Q}$ of prime degree ℓ over F, or K is an abelian ℓ-extension unramified over F assuming ℓ divides the class number of F (so $[K:F] = \ell^n$ which is odd, with $n \in \mathbb{N}^*$). The existence of such a K containing F is guaranteed, in the second case, by class field theory, but the first one is a tipical problem in inverse Galois theory. The following result from [9, p. 352-353] guarantees us that such a K indeed exists (for the first case):

Theorem 3.5. *For any prime ℓ and any quadratic field $F = \mathbb{Q}(\sqrt{d})$ there exist infinitely many dihedral fields K of degree 2ℓ containing F.*

If the quaternion algebra $H_F(p,q)$ is a division algebra, we would like to know when $H_K(p,q)$ is still a division algebra. The following three propositions will allow us to achieve this task:

Proposition 3.6. *Let F be a quadratic field and Δ_F its discriminant. Let K be an extension of F defined as above. Let p and q be distinct odd prime integers, with $(\frac{p}{q}) = -1$, and p or $q \equiv 1 \pmod 4$. Then the quaternion algebra $H_K(p,q)$ is a division algebra if and only if $\left(\frac{\Delta_F}{p}\right) = 1$ or $\left(\frac{\Delta_F}{q}\right) = 1$.*

Proof. Note first that the degree $[K:F]$ is odd, since it is equal to ℓ or ℓ^n with $n \in \mathbb{N}^*$. According to Remark 3.1 and Proposition 2.4, $H_K(p,q)$ is a division algebra if and only if $H_F(p,q)$ is a division algebra. By Proposition 2.6, this can happen if and only if $\left(\frac{\Delta_F}{p}\right) = 1$ or $\left(\frac{\Delta_F}{q}\right) = 1$. □

Proposition 3.7. *Let $d \neq 1$ be a squarefree integer and let $F = \mathbb{Q}(\sqrt{d})$ be a quadratic number field and Δ_F its discriminant. Let K be an extension of F defined as above. Let $p \equiv 3 \pmod 8$ be an odd prime integer. Then the quaternion algebra $H_K(p,q)$ is a division algebra if and only if $\left(\frac{\Delta_F}{p}\right) = 1$ or $d \equiv 1 \pmod 8$.*

Proof. The proof is obtained from the proof of Proposition 3.6, after replacing Proposition 2.6 with Proposition 2.7. □

Proposition 3.8. *Let $d \neq 1$ be a squarefree integer and let $F = \mathbb{Q}(\sqrt{d})$ be a quadratic number field and Δ_F its discriminant. Let K be an extension of F defined as above. Let p and q be distinct odd prime integers with $(\frac{q}{p}) \neq 1$ and $p \equiv q \equiv 3 \pmod 4$. Then the quaternion algebra $H_K(p,q)$ is a division algebra if and only if $\left(\frac{\Delta_F}{p}\right) = 1$ or $d \equiv 1 \pmod 8$.*

Proof. The proof is obtained from the proof of Proposition 3.6 after replacing Proposition 2.6 with Proposition 2.8. □

We conclude our discussion with the main theorem of the paper:

Theorem 3.9. *Let $d \neq 1$ be a squarefree integer and ℓ an odd positive prime integer. Let $F = \mathbb{Q}(\sqrt{d})$ be a quadratic number field and Δ_F its discriminant. Let K be a dihedral extension of $\mathbb{Q}$ of prime degree ℓ over F or K is an abelian ℓ-extension unramified over F whenever ℓ divides the class number of F. Let p and q be two distinct odd prime integers. Then the quaternion algebra $H_K(p,q)$ is a division algebra if and only if one of the following conditions is verified:*

1. *p or $q \equiv 1 \pmod 4$, $(\frac{p}{q}) = -1$, and $\left(\frac{\Delta_F}{p}\right) = 1$ or $\left(\frac{\Delta_F}{q}\right) = 1$;*
2. *$p \equiv 3 \pmod 8$, and $\left(\frac{\Delta_F}{p}\right) = 1$ or $d \equiv 1 \pmod 8$;*
3. *$p \equiv q \equiv 3 \pmod 4$, $(\frac{q}{p}) \neq 1$, and $\left(\frac{\Delta_F}{p}\right) = 1$ or $d \equiv 1 \pmod 8$.*

References

[1] V. Acciaro, D. Savin, M. Taous and A. Zekhnini, On quaternion algebras over quadratic number fields, preprint, 2019.

[2] V. Acciaro, D. Savin, M. Taous and A. Zekhnini, On quaternion algebras over the composite of quadratic number fields, preprint, 2019.

[3] M. Alsina and P. Bayer, Quaternion Orders, Quadratic Forms and Shimura Curves, CRM Monograph Series, 22, American Mathematical Society, 2004.

[4] A. Chapman, D. J.Grynkiewiczb, E. Matzri, L. H.Rowen, U. Vishne, Kummer spaces in symbol algebras of prime degree, J. of Pure and Applied Algebra, vol. 220, issue 10 (2016): 3363-3371.

[5] A. Chapman, Symbol length of p-algebras of prime exponent, J. of Algebra and its Applications, vol. 16, No. 7 (2017), 1750136 (9 p.).

[6] T. Chinburg and E. Friedman, An embedding theorem for quaternion algebras, J. London Math. Soc. (2), 60(1):33–44, 1999.

[7] P. Gille, T. Szamuely, Central Simple Algebras and Galois Cohomology, Cambridge University Press, 2006.

[8] D. Goldstein and M. Schacher, Norms in central simple algebras, Pacific Journal of Mathematics, Vol. 292, No. 2, 2018, pp. 373-388.
[9] C.U. Jensen, N. Yui, Polynomials with D_p as Galois Group, Journal of Number Theory 15, (1982) pp. 347-375.
[10] D.R. Kohel, Quaternion algebras, available online at http://www.i2m.univ-amu.fr/perso/david.kohel/alg/doc/AlgQuat.pdf.
[11] T. Y. Lam, Introduction to Quadratic Forms over Fields, American Mathematical Society, 2004.
[12] L. Rowen, D. J. Saltman, Tensor products of division algebras and fields, Journal of Algebra, 394 (2013), pp. 296-309.
[13] J. P. Serre, Local fields, Graduate Texts in Mathematics 67, Springer, New York, 1979.
[14] S.V. Vostokov, Explicit formulas for the Hilbert symbol, Geometry and Topology Monographs, Vol.3: Invitation to higher local fields, Part I, section 8, pp 81-89.

Vincenzo Acciaro, Dipartimento di Economia, Università di Chieti–Pescara, Viale della Pineta, 4, 65127 Pescara, Italy
E-mail address: v.acciaro@unich.it

Diana Savin, Faculty of Mathematics and Computer Science, Ovidius University, Bd. Mamaia 124, 900527, Constanta, Romania
E-mail address: savin.diana@univ-ovidius.ro

Mohammed Taous, Department of Mathematics, Faculty of Sciences and Technology, Moulay Ismail University, Errachidia, Morocco.
E-mail address: taousm@hotmail.com

Abdelkader Zekhnini, Mohammed First University, Pluridisciplinary Faculty of Nador, Department Mathematics and Informatics, Nador, Morocco
E-mail address: zekha1@yahoo.fr