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The Mixture Transition Distribution approach to networks: Evidence from stock markets

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ABSTRACT

Networks can be built by using correlations between time series. The approach based on correlations has many advantages which are essentially related to its simplicity. Nevertheless, it is well known that time series may show strong dependence even if they are uncorrelated. In this paper, we will advance a multivariate Markov chain model based on the Mixture Transition Distribution (MTD) model to build networks between time series. The multivariate MTD is able to consider the dependence between time series and, at the same time, reduce the number of parameters to be estimated compared to the classical multivariate Markov chain. We show, by a numerical example, that the multivariate MTD outperforms the classical correlation approach. Moreover, using the same model, we build the network of the 30 constituents of the Dow Jones index showing the usefulness of the methodology in real problems in financial markets.

1. Introduction

In recent years, the study of complex networks captured the attention of scholars in several scientific fields [see, e.g.,1–6]. In particular, other authors considered the stock networks to study financial markets [see, e.g.,7–9] or economic networks [10]. Stock networks are a kind of network in which the stocks represent the nodes and the relationships among the stocks are the links. In the literature, the relationships are generally built from the stock return correlations, after the application of a filter to reduce the number of links. One of the first examples of stock networks was proposed by Mantegna [7] who analyzed the returns of constituents of the Dow Jones Industrial Average (DJIA) and Standard and Poor's 500 (S&P 500) indexes. Starting from a measure of distance derived from the returns correlations, the author built the network using the concept of minimum spanning tree (MST) which reduced the number of links to $n-1$, being n the number of stocks. The author found that the connected stocks are grouped according to the relative industry. Similarly, Onnela et al. [11] introduced the dynamic asset graphs building the network of 477 stocks traded at the New York Stock Exchange (NYSE). The networks are again based on the returns correlation, however, the authors decided to keep only the first $n-1$ closest nodes. The consequence is that the resulting network is not a tree but a graph, or several graphs not necessarily interconnected. A variation has been proposed by Tse et al. [8] who built the full network of US stocks, with 19 807 nodes. The authors considered two nodes connected if the correlation of their returns was above a certain threshold. More recent work by Guo et al. [12] proposed a method based on the maximum likelihood estimation to select a different threshold value for each stock. Moreover, Lyócsa et al. [13] proposed a different application of the minimum spanning tree. They employed the dynamic conditional correlation (DCC) approach to compute the correlations over time and build the network of the S&P 100 constituents. However, a comparison with the traditional moving window approach revealed that the latter is more robust and exhibits a higher industry cluster.

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Departing from the correlations, Billio et al. [14] exploited the concepts of principal component analysis and pairwise Granger-causality to build the network of hedge funds, mutual funds, insurance companies, banks, and broker/dealers. A different approach has been pursued by Yang et al. [15] who analyzed the cointegration relationships among 26 global stock market indices. Differently from the use of correlation, the advantage of these two methodologies is the possibility of building directed connections within the network. However, because both methods employ a statistical test to assess the causality or cointegration, they are not able to assign a weight to the network edges. On the contrary, Su et al. [16] combined both the Granger-causality and the cointegration test in a sliding window setting to build a directed-weighted stock network via meso-scale. Also, Diebold and Yılmaz [17] built a weighted and directed network of the major US financial institutions based on the vector autoregression (VAR) variance decomposition. On a similar path, Yang et al. [18] constructed a sovereign default network and used some measures of centrality to detect whether the network properties drive the currency risk premia. Finally, Chen et al. [19] proposed to build a three-layer network model based on correlation coefficient, grey relational analysis, and maximum information coefficient.

In our paper, we introduce a new methodology to build the stock network. We assume that the stock returns are described by a multivariate Markov chain modeled through the mixture transition distribution (MTD) initially proposed by Raftery [20] to model high-order Markov chain and extended to a multivariate setting by Ching et al. [21]. Several applications of the multivariate MTD approach to financial markets have been proposed in the literature, from the stock valuation to the price discovery and credit risk [see, e.g.,22–27]. In this context, we apply the multivariate MTD to obtain a matrix of connectedness among the stocks which can be employed to build the adjacency matrix of the network. We design a specific numerical example showing that the advantage of using such an approach is that it allows us to capture the dependencies among the stocks going beyond the simple linear correlations. An application to the 30 constituents of the Dow Jones index, demonstrates the potential of the proposed methodology for real-life applications and allows to compute the asymmetric dependence structure among various stocks, along with some common centrality measures, such as out- and in-degree centrality.

The paper is structured as follows. Section 2 presents the methodology, while Section 3 introduces the numerical example and some network measures. Following, Section 4 illustrates an application of the proposed methodology to the Dow Jones constituents. Finally, Section 5 concludes the manuscript.

2. The model

In this section, we present briefly the concept of the Markov chain and its multivariate extension to introduce the notation and the basic idea generating these stochastic models. A detailed theoretical treatment, with applications and examples, is available in many textbooks on the subject [see, e.g.,28–30].

2.1. The discrete-time Markov chain

A sequence of random variables $\{S_t\}_{t\in\mathbb{N}}$ taking values in the set $\mathcal{M} = \{1,\dots,m\}$ is called a *Markov Chain* when it satisfies the following *Markov Property*

$$\mathbb{P}(S_{t+1} = j|S_t = i, S_{t-1} = i_{t-1},\dots,S_0 = i_0) = \mathbb{P}(S_{t+1} = j|S_t = i). \tag{1}$$

When this condition is independent of the time t, then the process is called a *Homogeneous Markov Chain* (HMC), and the probability to move from state i to state j at any point in time can be expressed as

$$\mathbb{P}(S_{t+1} = j|S_t = i) = p_{ij}, \quad \forall t \in \mathbb{N}, \forall i, j \in \mathcal{M}. \tag{2}$$

All the possible combinations of changing from one state to another form the one-step *transition probability matrix* of the HMC:

$$\mathbf{P} = \begin{array}{cc} & S_{t+1} \\ S_t & \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix} \end{array}, \tag{3}$$

subject to $0 \le p_{ij} \le 1$, $\forall i, j \in \mathcal{M}$ and $\sum_{j=1}^{m} p_{ij} = 1$, $\forall i \in \mathcal{M}$.

Given the transition probability matrix $\mathbf{P}$ and the initial probability distribution $\mathbf{A}(0) := [A_1(0),\dots,A_m(0)]$, where $A_i(0) := \mathbb{P}\big(S_0 = i\big)$ and $i \in \mathcal{M}$, we can define the probability distribution at each time t as

$$\mathbf{A}(t) := [A_1(t),\dots,A_m(t)] \tag{4}$$

where $A_i(t) = \mathbb{P}(S_t = i)$, and compute it according to

$$\mathbf{A}(t) = \mathbf{A}(0)\mathbf{P}^t. \tag{5}$$

Thus, the probability distribution of the random variable S_t can be calculated by multiplying the initial probability distribution with the power t of the transition probability matrix $\mathbf{P}$.

2.2. The multivariate Markov process

As demonstrated by Ching et al. [21], the previous model can be extended into a multivariate setting, with more than one time series expressing the dynamic of the system's components. Towards this end, we consider $\mathbf{S} = (S_t^{(\alpha)}, \forall \alpha \in \Gamma = \{1, 2, \dots, \gamma\})$, a multivariate sequence of random variables defined on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Each sequence $\{S_t^{(\alpha)}\}_{t\in\mathbb{N}}$ takes values in the same finite state space $\mathcal{M}$. For every series α, the probability of being in state j depends on the state $i_1, \dots, i_\gamma$ occupied by all the available series one time step before. The Markov Property in (1) can be extended as follows

$$\begin{aligned}&\mathbb{P}(S_{t+1}^{(\alpha)} = j | (S_t^{(1)} = i_t^{(1)}, S_{t-1}^{(1)} = i_{t-1}^{(1)}, \dots, S_0^{(1)} = i_0^{(1)}), \dots, \\ &\quad (S_t^{(\gamma)} = i_t^{(\gamma)}, S_{t-1}^{(\gamma)} = i_{t-1}^{(\gamma)}, \dots, S_0^{(\gamma)} = i_0^{(\gamma)})) \\ &= \mathbb{P}(S_{t+1}^{(\alpha)} = j | S_t^{(1)} = i_t^{(1)}, \dots, S_t^{(\gamma)} = i_t^{(\gamma)}).\end{aligned} \tag{6}$$

The Property in (6) shows that there are multiple dependencies between the series. Therefore, the transition probability matrix of the multivariate model must include each possible combination, m^γ, for the initial states, and every initial state must end in one of the possible final combinations. The result is $m^\gamma(m^\gamma - 1)$ total parameters to estimate for the multivariate Markov model, given that there are $m^\gamma - 1$ independent probabilities in each row. Such a configuration is not practical in a real-world application because the number of parameters will increase exponentially when the number of series and states increases.

Raftery [20] proposed the Mixture Transition Distribution model (MTD) to reduce the number of parameters to estimate for high order Markov chains, and Ching et al. [21] applied it to the multivariate Markov chains. A review of the MTD model and its application is available in [31]. Applying the MTD model the probability vector for series α at time $t+1$ becomes

$$\mathbf{A}^{(\alpha)}(t+1) = \sum_{\beta=1}^{\gamma} \mathbf{A}^{(\beta)}(t) \cdot \lambda_{\beta,\alpha} \cdot \mathbf{P}^{(\beta,\alpha)}, \tag{7}$$

where $\mathbf{A}^{\alpha}(t) := [A_1^{(\alpha)}, \dots, A_m^{(\alpha)}]$ and $A_i^{(\alpha)}(t) := \mathbb{P}(S_t^{(\alpha)} = i)$.

According to this condition, we can build γ^2 transitions probability matrices $\mathbf{P}^{(\beta,\alpha)} = (p_{ij}^{(\beta,\alpha)})_{i,j\in\mathcal{M}}$, each one containing the transition probabilities from state i in series β to state j in series α, with $\alpha, \beta \in \Gamma$,

$$\mathbf{P}^{(\beta,\alpha)} = \begin{array}{cc} & S_{t+1}^{(\alpha)} \\ S_t^{(\beta)} & \begin{bmatrix} p_{11}^{(\beta,\alpha)} & p_{12}^{(\beta,\alpha)} & \cdots & p_{1m}^{(\beta,\alpha)} \\ p_{21}^{(\beta,\alpha)} & p_{22}^{(\beta,\alpha)} & \cdots & p_{2m}^{(\beta,\alpha)} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1}^{(\beta,\alpha)} & p_{m2}^{(\beta,\alpha)} & \cdots & p_{mm}^{(\beta,\alpha)} \end{bmatrix} \end{array}. \tag{8}$$

The parameters $\lambda_{\beta,\alpha}$ are the scalar weights that combine all the series, and are subject to:

$$\sum_{\beta=1}^{\gamma} \lambda_{\beta,\alpha} = 1, \quad \lambda_{\beta,\alpha} \geq 0. \tag{9}$$

They provide a measure of the degree of dependence among the different components of the systems. Large values of $\lambda_{\beta,\alpha}$ imply a strong influence of the component β over the component α.

The MTD model permits to reduce the total number parameters to estimate from $m^\gamma(m^\gamma - 1)$ to $\gamma^2 m(m-1) + \gamma(\gamma - 1)$, the first addend being the number of $\mathbf{P}^{(\beta,\alpha)}$ parameters and the second the number of weights $\lambda_{\beta,\alpha}$.

If the series represents financial returns, the equation in (7) tells us that the probability for a return change in series α of being in a specific state (e.g., negative, positive, or null) is a linear combination of all γ series of weighted transition probabilities from each series initial states to the arrival state in series α. In other words, the $\lambda_{\beta,\alpha}$ weights indicate how much series β influences series α in changing the return of the latter.

In general, there are γ^2 values of $\lambda_{\beta,\alpha}$ subject to conditions (9) that can be organized in a matrix form,

$$\mathbf{\Lambda} = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & \dots & \gamma \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ \gamma \end{array} & \begin{bmatrix} \lambda_{1,1} & \lambda_{1,2} & \dots & \lambda_{1,\gamma} \\ \lambda_{2,1} & \lambda_{2,2} & \dots & \lambda_{2,\gamma} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{\gamma,1} & \lambda_{\gamma,2} & \dots & \lambda_{\gamma,\gamma} \end{bmatrix} \end{array}. \tag{10}$$

Each element of the matrix (10) measures the price change influence that a series has on other series, thus revealing the connectedness among the stocks. For example, element $\lambda_{1,1}$ is the portion of influence of series 1 on series 1, element $\lambda_{2,1}$ is the portion of influence of series 2 on series 1, and so on. Each column of the matrix contains the influence shares from all series to a specific series α, including the self-influence, and the sum of all column's elements is equal to one. It is worth noting that if there is no dependency between the price change series, the model still works but the matrix (10) becomes an identity matrix.

The transition probabilities in $\mathbf{P}^{\beta,\alpha}$ and the parameters $\lambda_{\beta,\alpha}$ can be estimated with a two stage process. First, we estimate the transition probabilities using the maximum likelihood estimator from [32]

$$\hat{p}_{ij}^{(\beta,\alpha)} = \frac{n_{ij}^{(\beta,\alpha)}}{\sum_{j=1}^{m} n_{ij}^{(\beta,\alpha)}},$$

where $n_{ij}^{(\beta,\alpha)}$ is the occurrences of transitions from state i in series β to state j in series α.

Then, the parameters $\lambda_{\beta,\alpha}$ can be estimated by maximizing the log-likelihood function of the multivariate model respecting constraints in (9) (see e.g., [31,33]),

$$log L(\boldsymbol{\Lambda}) = \sum_{i_1=1}^{m} \dots \sum_{i_\gamma=1}^{m} \sum_{j=1}^{m} n_{i_1,\dots,i_\gamma,j}\, log\left(\sum_{\beta=1}^{\gamma} \lambda_{\beta,\alpha} \hat{p}_{i_\beta,j}^{(\beta,\alpha)}\right), \tag{11}$$

where $n_{i_1,\dots,i_\gamma,j}$ is the observed number of sequences of the type $S_{t-1}^{(1)} = i_1, \dots, S_{t-1}^{(\gamma)} = i_\gamma, S_t^{(\alpha)} = j,$.

Finally, for each series α, the constrained numerical optimization is performed using the *Sequential Least SQuares Programming* (SLSQP) algorithm from [34]. Moreover, as the cited algorithm is applied to each series α, we select the initial values of the parameters $\lambda_{\beta,\alpha}$ with $\beta \in \Gamma$, from a uniform distribution, i.e.

$$\lambda_{\beta,\alpha} = \frac{1}{\gamma},\ \forall \beta = 1, 2, \dots, \gamma.$$

This decision is supported by the indifference principle based on the use of a non-informative prior distribution over the lambdas because there was no initial information about the values of these parameters.

Among the estimation methodologies, Ching et al. [21] proposed to minimize the distance from the stationary distribution. In the asymptotic situation, this approach may be useful for precise fitting; but, in the transient analysis, it may not be as effective.

3. The MTD network

As seen in the previous section, the multivariate MTD approach allows us to consider the dependence between time series by reducing the total number of parameters to be estimated. Thanks to this captured dependence, we are able to build a network of stocks. The weights matrix in (10) contains the influence of one series on another. The stronger the influence, the higher the value. In network terms, we can consider higher values of the weights as stronger connections between the stocks. Therefore, we can use the weights matrix as the adjacency matrix and construct the network accordingly. Moreover, contrary to the correlation network, the matrix is not symmetric. Thus, it permits to build weighted and directed networks.

It is well known that many systems show zero correlations even though there is a strong dependence between their components. Therefore, we are going to show, with the design of a numerical example, that if we are in the case where there exists a dependence between uncorrelated time series, the classical approach based on correlations gives meaningless results if employed to build a network. We design a hierarchical stochastic system of interacting components. In order to keep things as simple as possible, we consider simple dynamics of the components that are based on autoregressive processes of the first order.

Let us consider the first level of the hierarchy of the random system by considering two random processes $X^{(1)}(t)$ and $X^{(4)}(t)$ defined according to the next equations:

$$X^{(1)}(t) = \phi X^{(1)}(t-1) + \epsilon_t,\quad X^{(1)}(0) = 0, \tag{12}$$

$$X^{(4)}(t) = \tilde{\phi} X^{(4)}(t-1) - \epsilon_t,\quad X^{(4)}(0) = 0, \tag{13}$$

where $\epsilon_t \sim \mathcal{N}(0,1)$ and $|\phi| < 1$ and $|\tilde{\phi}| < 1$ guarantee the stationarity of the two processes as well as the finiteness of the variance.

Clearly, the two components are dependent on each other due to the exposition to the same noise process ϵ_t with different algebraic signs. From these components, we introduce four additional components of the second level in the hierarchy. Precisely, first, we introduce two random variables W_a and W_b independent of each other and on the first level processes at any time t. We assume they have Rademacher distributions given by:

$$W_a = \begin{cases} a & \text{with probability } 1/2 \\ -a & \text{with probability } 1/2, \end{cases} \tag{14}$$

$$W_b = \begin{cases} b & \text{with probability } 1/2 \\ -b & \text{with probability } 1/2, \end{cases} \tag{15}$$

with a and b real numbers.

Then, we introduce two additional components of the second level of hierarchy directly related to the component $X^{(1)}$:

$$\begin{aligned} X^{(2)}(t) &= W_a X^{(1)}(t), \\ X^{(3)}(t) &= -W_a X^{(1)}(t), \end{aligned} \tag{16}$$

and similarly, two additional components directly related to the variable $X^{(4)}$:

$$\begin{aligned} X^{(5)}(t) &= W_b X^{(4)}(t), \\ X^{(6)}(t) &= -W_b X^{(4)}(t). \end{aligned} \tag{17}$$

Table 1
Correlation coefficients of the numerical example.

	$X^{(1)}$	$X^{(2)}$	$X^{(3)}$	$X^{(4)}$	$X^{(5)}$	$X^{(6)}$
$X^{(1)}$	1	0	0	−0.934	0	0
$X^{(2)}$	0	1	−1	0	0	0
$X^{(3)}$	0	−1	1	0	0	0
$X^{(4)}$	−0.934	0	0	1	0	0
$X^{(5)}$	0	0	0	0	1	−1
$X^{(6)}$	0	0	0	0	−1	1

In summary, the designed system has six components: two dependent on each other on the first level and four components on the second level which are dependent on the specific root on the first order. Thus, by construction, both $X^{(2)}$ and $X^{(3)}$ are dependent on their root $X^{(1)}$ and both $X^{(5)}$ and $X^{(6)}$ are dependent on their own root $X^{(4)}$.

Since our objective is to compare the network approach we are proposing, the one based on MTD against the classical one based on correlations, we observe some particular features of the proposed random system expressed in terms of correlations.

First, we observe that even if the random components $X^{(1)}$ and $X^{(2)}$ are dependent, their correlation is zero. Indeed, for any time t we have that:

$$\begin{aligned}
&\mathbb{E}[X^{(1)}(t)\cdot X^{(2)}(t)] - \mathbb{E}[X^{(1)}(t)]\cdot\mathbb{E}[X^{(2)}(t)] = \mathbb{E}[X^{(1)}(t)\cdot W_a X^{(1)}(t)] - \mathbb{E}[X^{(1)}(t)]\cdot\mathbb{E}[W_a X^{(1)}(t)]\\
&= \mathbb{E}[W_a]\mathbb{E}[(X^{(1)}(t))^2] - \mathbb{E}[W_a](\mathbb{E}[X^{(1)}(t)])^2 = 0,
\end{aligned}$$

where the last equality is obtained from the independence of W_a on $X^{(1)}$ and the fact that W_a has zero expectation. The fact that the covariance function is zero immediately provides a zero correlation coefficient i.e. $\rho_{(X^{(1)},X^{(2)})}(t) = 0$. A symmetric argument gives the same result for the correlation structure between the random components $X^{(1)}$ and $X^{(3)}$, i.e. $\rho_{(X^{(1)},X^{(3)})}(t) = 0$. It is also simple to realize that $\rho_{(X^{(2)},X^{(3)})}(t) = -1$ because they are just one the opposite each other. Similarly, we obtain the correlations for the remaining series, as reported in Table 1.

It is interesting now to assess the correlation structure between the random components $X^{(1)}$ and $X^{(4)}$ which form the first level of the hierarchical model. Let us start by computing the covariance function.

$$\begin{aligned}
Cov[X^{(1)}(t), X^{(4)}(t)] &= \mathbb{E}[(\phi X^{(1)}(t-1)+\epsilon_t)\cdot(\tilde{\phi}X^{(4)}(t-1)-\epsilon_t)]\\
&-\ \mathbb{E}[\phi X^{(1)}(t-1)+\epsilon_t]\cdot\mathbb{E}[\tilde{\phi}X^{(4)}(t-1)-\epsilon_t]\\
&= \phi\tilde{\phi}\mathbb{E}[X^{(1)}(t-1)X^{(4)}(t-1)] - \phi\mathbb{E}[X^{(1)}(t-1)]\mathbb{E}[\epsilon_t] + \tilde{\phi}\mathbb{E}[X^{(4)}(t-1)]\mathbb{E}[\epsilon_t] - \mathbb{E}[(\epsilon_t)^2]\\
&-\ \Big[\big(\mathbb{E}[\phi X^{(1)}(t-1)] + \mathbb{E}[\epsilon_t]\big)\cdot\big(\mathbb{E}[\tilde{\phi}X^{(4)}(t-1)] - \mathbb{E}[\epsilon_t]\big)\Big]\\
&= \phi\tilde{\phi}\cdot Cov[X^{(1)}(t-1), X^{(4)}(t-1)] - 1.
\end{aligned}$$

Thus, an iteration of the previous computation with respect to the time variable and the deterministic nature of $X^{(1)}(0)$ and $X^{(4)}(0)$ gives:

$$Cov[X^{(1)}(t), X^{(4)}(t)] = -\sum_{l=0}^{t-1}(\phi\cdot\tilde{\phi})^l. \tag{18}$$

Since our processes are stationary we have

$$Cov[X^{(1)}, X^{(4)}] := \lim_{t\to\infty} Cov[X^{(1)}(t), X^{(4)}(t)] = -\sum_{l\geq 0}(\phi\cdot\tilde{\phi})^l = -\frac{1}{1-(\phi\cdot\tilde{\phi})}. \tag{19}$$

Furthermore, we observe that the variances for the series $X^{(1)}$ and $X^{(4)}$ are

$$V(X^{(1)}(t)) = \sum_{r=0}^{t-1}(\phi^2)^r \quad\text{and}\quad V(X^{(4)}(t)) = \sum_{r=0}^{t-1}(\tilde{\phi}^2)^r,$$

and passing to the limit we get

$$V(X^{(1)}) := \lim_{t\to\infty} V(X^{(1)}(t)) = \frac{1}{1-\phi^2} \quad\text{and}\quad V(X^{(4)}) := \lim_{t\to\infty} V(X^{(4)}(t)) = \frac{1}{1-\tilde{\phi}^2}.$$

Finally, we get

$$\begin{aligned}
&\rho_{(X^{(1)},X^{(4)})}(t) = \frac{-\sum_{l=0}^{t-1}(\phi\cdot\tilde{\phi})^l}{\sqrt{\sum_{r=0}^{t-1}(\phi^2)^r}\cdot\sqrt{\sum_{r=0}^{t-1}(\tilde{\phi}^2)^r}},\\
&\rho_{(X^{(1)},X^{(4)})} := \lim_{t\to\infty}\rho_{(X^{(1)},X^{(4)})}(t) = \frac{-\frac{1}{1-(\phi\cdot\tilde{\phi})}}{\sqrt{\frac{1}{1-\phi^2}}\cdot\sqrt{\frac{1}{1-\tilde{\phi}^2}}}.
\end{aligned}\tag{20}$$

Table 2
Average of the correlation coefficients of the 1000 Monte Carlo simulations.

	$X^{(1)}$	$X^{(2)}$	$X^{(3)}$	$X^{(4)}$	$X^{(5)}$	$X^{(6)}$
$X^{(1)}$	1.000	−0.001	0.001	−0.935	0.002	−0.002
$X^{(2)}$	−0.001	1.000	−1.000	0.000	0.006	−0.006
$X^{(3)}$	0.001	−1.000	1.000	−0.000	−0.006	0.006
$X^{(4)}$	−0.935	0.000	−0.000	1.000	−0.002	0.002
$X^{(5)}$	0.002	0.006	−0.006	−0.002	1.000	−1.000
$X^{(6)}$	−0.002	−0.006	0.006	0.002	−1.000	1.000

Table 3
Average values of $\lambda_{\beta,\alpha}$ from the MTD estimation of 1000 Monte Carlo simulations.

	$X^{(1)}$	$X^{(2)}$	$X^{(3)}$	$X^{(4)}$	$X^{(5)}$	$X^{(6)}$
$X^{(1)}$	0.795	0.343	0.343	0.214	0.094	0.094
$X^{(2)}$	0.000	0.221	0.221	0.000	0.091	0.091
$X^{(3)}$	0.000	0.221	0.221	0.000	0.091	0.091
$X^{(4)}$	0.205	0.103	0.103	0.786	0.318	0.318
$X^{(5)}$	0.000	0.056	0.056	0.000	0.203	0.203
$X^{(6)}$	0.000	0.056	0.056	0.000	0.203	0.203

Given the proposed numerical example, we perform a Monte Carlo simulation fixing the values for the parameters of the models, such as $\phi = 0.9$, $\tilde{\phi} = 0.8$, and $a = 1$, and $b = 1$. Then, we simulate the following random system:

$$
\begin{aligned}
X^{(1)}(t) &= 0.9 \cdot X^{(1)}(t-1) + \epsilon_t \\
X^{(2)}(t) &= W_a \cdot X^{(1)}(t), \\
X^{(3)}(t) &= -W_a \cdot X^{(1)}(t) \\
X^{(4)}(t) &= 0.8 \cdot X^{(4)}(t-1) - \epsilon_t \\
X^{(5)}(t) &= W_b \cdot X^{(4)}(t), \\
X^{(6)}(t) &= -W_b \cdot X^{(4)}(t),
\end{aligned}
$$

which represents a particular vector autoregressive (VAR) model, designed in such a way that its six components show some uncorrelations while being dependent by construction.

All series in the proposed example assume values in a general state space, however, the application of the MTD model requires a discrete state space. Therefore, we apply a discretization of the series into a 3-state space. Specifically, as the processes are stationary, we set a central state which includes all values within a half standard deviation radius from zero, and two external states corresponding to the higher and lower values.

The averaged values of the correlation coefficient and MTD weights are reported in Tables 2 and 3, respectively. The values of the correlations computed on the Monte Carlo simulations are perfectly in line with the theoretical values given by previous computations. For example, a substitution of the values $\phi = 0.9$ and $\tilde{\phi} = 0.8$ inside Eq. (20) produces a value of $\rho_{(X^{(1)},X^{(4)})} = -0.934$. The simulation is performed using a Monte Carlo method with 1000 simulations of length 1000 from which we compute the values of $\lambda_{\beta,\alpha}$ as in (11).

When applying a correlation-based network, the results are unsatisfactory because it is not able to interpret the true relationships among the variables. On the contrary, the network built on the weights matrix $\boldsymbol{\Lambda}$ of the MTD model captures these nonlinear dependencies. In particular, the values $\lambda_{\beta,\alpha}$ clearly capture the dependence among the series. For example the series $X^{(1)}$ has an effect of magnitude 0.343 on series $X^{(2)}$ and $X^{(3)}$ but lower on $X^{(5)}$ and $X^{(6)}$ as they directly depend on $X^{(4)}$. A similar effect is noticeable from series $X^{(4)}$ to $X^{(5)}$ and $X^{(6)}$ and to $X^{(2)}$ to $X^{(3)}$. Moreover, the model is able to capture the interdependence between $X^{(2)}$ and $X^{(3)}$ with a value of $\lambda = 0.221$ and between $X^{(5)}$ and $X^{(6)}$ with $\lambda = 0.203$. Both values are lower than the direct dependence from $X^{(1)}$ and $X^{(4)}$ respectively. Finally, the model shows no dependence from $X^{(2)}$ or $X^{(3)}$ to $X^{(1)}$ and from $X^{(5)}$ or $X^{(6)}$ to $X^{(4)}$ as expected. However, there is a minimal dependence from $X^{(5)}$ and $X^{(6)}$ to $X^{(2)}$ and $X^{(3)}$ and vice-versa.

Fig. 1 shows two possible networks implied by the correlations built from the application of the minimum spanning tree as in Mantegna [7]. We point out that the trees reported in Fig. 1 are not the unique trees with the minimum weight, due to the symmetry of the numerical example. However, in all cases it is not possible to identify a direct dependency among the series. In particular, in the first network on the left of the picture, we do not capture the dependency between series $X^{(3)}$ from $X^{(1)}$, and $X^{(5)}$ from $X^{(4)}$. A similar situation is clear from the tree on the right, where we miss the dependency between $X^{(2)}$ from $X^{(1)}$, and $X^{(6)}$ from $X^{(4)}$. On the contrary, we can see the full dependence captured by the MTD approach using the weights matrix as an adjacency matrix in Fig. 2. It is clear from the picture that the MTD network represents a faithful reproduction of the real system because it captures the non-linear dependencies as well as the hierarchical structure characterizing the system.

As shown, the multivariate MTD model clearly outperforms the correlation approach. However, even though the proposed model is useful in reducing the number of parameters to estimate compared to the full multivariate Markov chain, it still comes with the cost of performing a maximum likelihood estimation to obtain the values of the adjacency matrix. Therefore, it is computationally more expensive compared to the correlation-based network, which simply requires computing correlations.

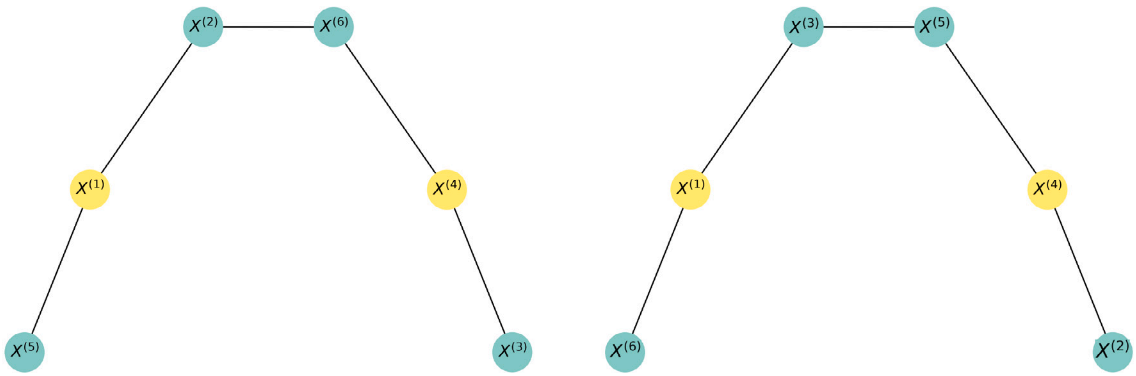

Fig. 1. Examples of correlation networks of the numerical example with the application of the Minimum Spanning Tree algorithm from [7].

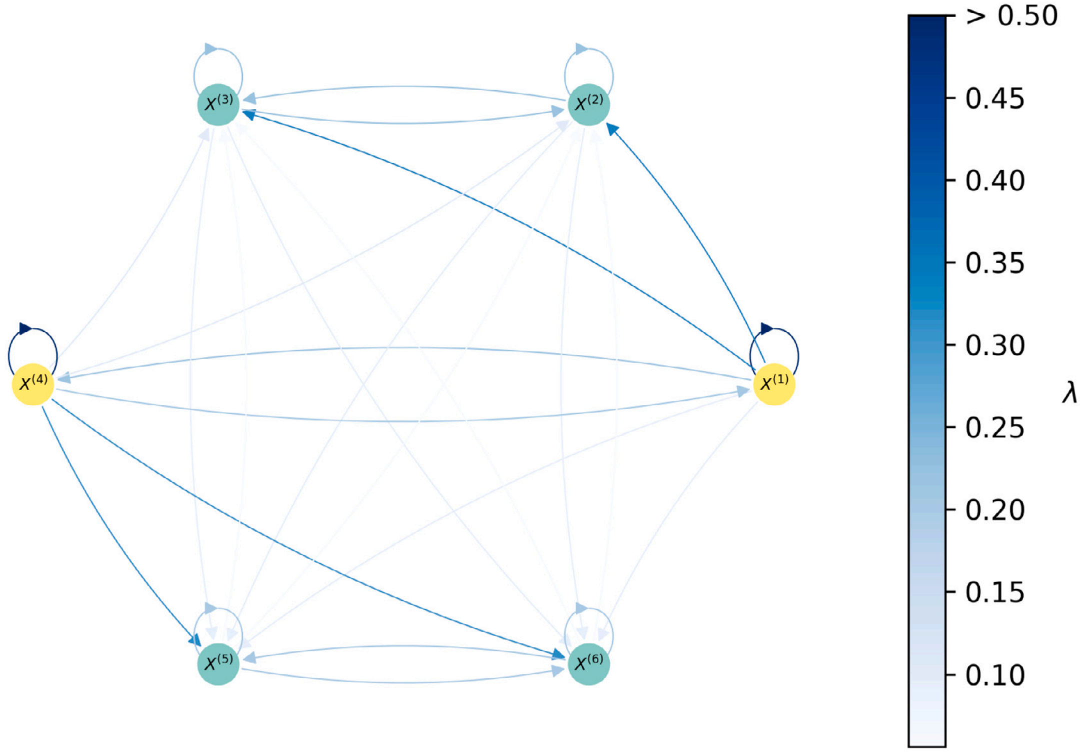

Fig. 2. MTD network of the numerical example.

4. Application to financial markets

For the empirical application, we consider the thirty constituents of the Dow Jones Industrial Average index. Details of the companies included in the index are reported in Table 4.

We compute the 1-min interval log-returns of each series. Then, as the application of the model requires a discrete state space, we apply a discretization of the returns into three states. Specifically, as previously reported in the numerical example, we set a central state corresponding to the null return which includes all returns within a half standard deviation radius from zero, and two external states corresponding to the positive and negative returns. Finally, we perform the estimation of the lambda values of the MTD model on a weekly basis. The dates start from 1 August 2022 to 30 October 2022 for a total of 13 weeks and are sourced from the Thomson Reuters Tick History database.

We perform the estimation of the adjacency matrix for each week. As an example, the first week results are reported in Table 5. Respecting constraints in (9), all values are non-negative, and all columns sum to one. For example, in the first column, we observe the share of influence of AAPL on other series (i.e., 0.029 on CSCO, 0.076 on HON, 0.242 on KO, 0.303 on MCD, and 0.35 on

Table 4
Details of the 30 companies of the Dow Jones Industrial Average index.

id	Name	Market	Industry
MMM	3M Company	NYSE	Conglomerate
AXP	American Express	NYSE	Financial services
AMGN	Amgen	NASDAQ	Pharmaceutical industry
AAPL	Apple Inc.	NASDAQ	Information technology
BA	Boeing	NYSE	Aerospace and defense
CAT	Caterpillar Inc.	NYSE	Construction and Mining
CVX	Chevron Corporation	NYSE	Petroleum industry
CSCO	Cisco Systems	NASDAQ	Information technology
DOW	Dow Inc.	NYSE	Chemical industry
GS	Goldman Sachs	NYSE	Financial services
HON	Honeywell	NYSE	Conglomerate
IBM	IBM	NYSE	Information technology
INTC	Intel	NASDAQ	Information technology
JNJ	Johnson & Johnson	NYSE	Pharmaceutical industry
JPM	JPMorgan Chase	NYSE	Financial services
MCD	McDonald's	NYSE	Food industry
MRK	Merck & Co.	NYSE	Pharmaceutical industry
MSFT	Microsoft	NASDAQ	Information technology
NKE	Nike	NYSE	Apparel
PG	Procter & Gamble	NYSE	Fast-moving consumer goods
CRM	Salesforce	NYSE	Information technology
KO	The Coca-Cola Company	NYSE	Food industry
HD	The Home Depot	NYSE	Retailing
TRV	The Travelers Companies	NYSE	Financial services
DIS	The Walt Disney Company	NYSE	Broadcasting and entertainment
UNH	UnitedHealth Group	NYSE	Managed health care
VZ	Verizon	NYSE	Telecommunication
V	Visa Inc.	NYSE	Financial services
WBA	Walgreens Boots Alliance	NASDAQ	Retailing
WMT	Walmart	NYSE	Retailing

Table 5
Lambda matrix estimated on the 1-min returns of the Dow-Jones constituents. Week starting on 1 August 2022.

	AAPL	AMGN	AXP	BA	CAT	CRM	CSCO	CVX	DIS	DOW	GS	HD	HON	IBM	INTC	JNJ	JPM	KO	MCD	MMM	MRK	MSFT	NKE	PG	TRV	UNH	V	VZ	WBA	WMT
AAPL	–	0.106	–	–	–	0.072	–	–	0.245	–	–	0.234	–	–	–	–	–	0.058	–	–	–	–	0.008	–	–	–	–	–	–	–
AMGN	–	0.169	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	0.019	–	–	–	–	–	–	0.168	–	–	–
AXP	–	–	0.442	–	–	–	–	0.350	–	–	0.195	–	0.099	0.054	–	–	–	0.111	–	–	–	–	–	–	–	–	0.132	–	–	–
BA	–	–	–	0.297	–	–	0.063	–	–	–	–	0.110	0.230	–	0.082	–	0.041	0.132	–	–	–	–	–	0.134	0.323	–	–	–	–	–
CAT	–	–	–	0.023	0.361	–	–	0.015	–	0.123	–	–	–	0.036	–	–	–	0.118	–	–	–	0.409	–	0.148	0.010	0.077	–	0.307	0.390	–
CRM	–	–	–	–	–	–	–	–	–	0.187	0.041	0.011	–	–	–	–	–	–	–	–	–	–	0.046	–	–	–	–	–	–	–
CSCO	0.029	0.054	–	–	–	0.209	–	–	0.364	0.195	0.424	–	–	–	–	–	–	–	0.078	0.029	–	–	–	–	–	–	–	–	0.179	–
CVX	–	–	–	–	–	0.450	–	–	–	–	–	–	–	0.076	–	0.279	–	0.166	–	0.033	0.260	0.314	–	–	–	0.277	–	0.085	–	–
DIS	–	–	–	0.033	–	–	–	–	0.125	–	–	–	0.043	–	–	0.343	–	–	0.112	–	–	–	0.118	–	–	–	–	–	–	–
DOW	–	0.199	–	–	0.009	–	–	0.045	–	0.327	–	–	–	–	–	–	–	–	–	0.213	–	–	–	–	0.342	–	0.376	–	–	–
GS	–	–	–	–	–	–	–	–	–	–	0.023	–	–	–	–	–	–	–	–	–	0.437	–	–	–	–	–	–	–	–	–
HD	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	0.065	–	–	–	–	–	–	–	–	–	–	–
HON	0.076	0.103	–	–	–	–	0.222	–	–	–	–	–	0.424	–	–	0.190	0.069	–	–	–	–	0.222	0.067	–	–	–	–	–	–	–
IBM	–	–	–	–	–	–	–	–	–	0.092	–	0.023	–	–	–	–	–	–	–	0.109	0.024	–	0.033	–	–	–	–	–	–	–
INTC	–	–	–	–	–	–	–	–	–	–	–	0.482	0.204	–	0.108	–	–	–	0.384	0.361	–	–	0.029	0.116	–	–	0.143	–	–	0.132
JNJ	–	–	–	–	–	–	–	–	–	–	–	–	–	0.367	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
JPM	–	–	–	–	–	0.028	–	–	–	–	–	–	–	–	–	–	0.740	–	–	–	–	–	–	–	–	–	–	–	–	–
KO	0.242	–	0.139	–	–	–	–	–	–	–	–	–	–	–	–	–	–	0.337	–	0.134	–	–	–	–	0.252	–	–	–	–	–
MCD	0.303	–	–	–	–	–	–	0.084	–	–	–	–	–	–	–	–	–	0.003	0.009	–	–	–	–	–	–	–	–	–	–	0.205
MMM	–	–	–	–	–	0.060	–	–	–	–	0.118	–	–	–	0.256	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
MRK	–	0.347	–	–	–	–	–	–	–	–	–	–	–	–	–	–	0.150	–	–	–	–	–	–	–	–	–	–	–	–	–
MSFT	–	–	–	–	–	–	–	–	–	–	–	–	–	0.179	0.035	–	–	0.074	–	–	0.073	–	0.455	–	–	0.099	–	–	–	–
NKE	–	0.022	–	0.284	0.294	0.044	0.093	0.213	–	–	0.011	–	–	0.288	–	–	–	–	–	–	0.094	0.055	0.019	–	–	–	–	–	–	–
PG	–	–	0.226	–	–	–	0.177	0.293	–	0.076	–	0.140	–	–	–	–	–	–	0.099	–	–	–	–	0.602	–	–	–	0.163	0.379	0.141
TRV	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	0.053	–	–	–	0.102	–	–	–	–	–	–	–	–	–	–
UNH	–	–	0.032	0.363	0.279	0.053	0.445	–	0.067	–	–	–	–	–	–	–	–	–	0.035	–	0.046	–	–	–	0.073	0.515	–	0.098	–	–
V	–	–	–	–	–	–	–	–	–	–	–	–	–	–	0.218	–	–	–	0.218	–	–	–	–	–	–	0.024	–	0.105	–	–
VZ	0.350	–	0.163	–	0.057	0.084	–	–	–	–	–	–	–	–	0.301	0.135	–	0.001	–	–	–	–	0.225	–	–	0.008	–	0.242	–	–
WBA	–	–	–	–	–	–	–	–	0.199	–	0.188	–	–	–	–	–	–	–	–	–	0.066	–	–	–	–	–	0.181	–	0.052	–
WMT	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	0.522

VZ). On the rows, we can read the received share of influence from a single stock, e.g., APPL is influenced by AMGN (0.106), CRM (0.072), DIS (0.245), HD (0.234), KO (0.058), and NKE (0.008). As stated in Sections 2 and 3 the matrix is not symmetric as for the case of the correlations. Thus, we can build a two-way directed graph, which considers the influence of the price series on each other with different weights, e.g., 0.242 of APPL on KO versus 0.058 of KO on AAPL.

From the collected data, we compute the adjacency matrix for each week and then compute the respective network for a total of 13 networks. For each of them, we compute the out-degree and in-degree centrality associated with each node, i.e. the number of edges going out of and into a node respectively. A standardized out-degree is a simple measure of node centrality, generally called out-degree centrality.

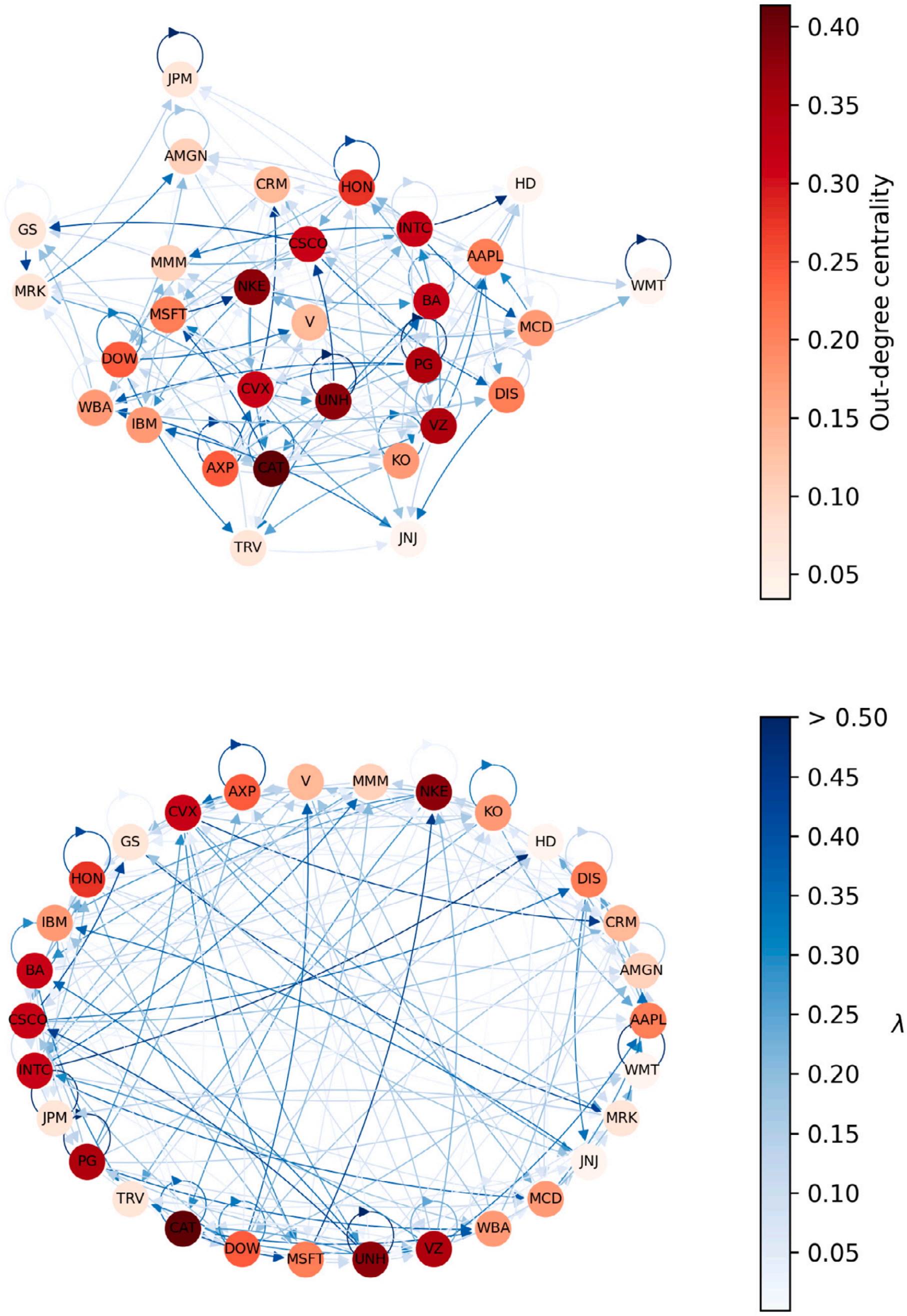

Fig. 3. MTD network of the Dow-Jones constituents represented by two different network layouts. Week starting on 1 August 2022.

The network associated with the first week adjacency matrix is reported in Fig. 3. To show different aspects of the stock network, we report two visualizations of the same network. The top graph is organized based on the standardized out-degree centrality. Nodes with higher connections (darker color) are more central and therefore positioned at the center of the network. On the contrary, the shape of the second graph at the bottom allows us to concentrate our attention on the connections and their relative weights. Darker links correspond to higher weights from the adjacency matrix, thus indicating a closer connection between the stocks.

In Tables 6 and 7 we report the weekly evolution of both the out-degree and in-degree centrality associated with each node. The bold values represent the price series with the highest out-degree and in-degree for each week.

Table 6
Weekly out-degrees of the Dow-Jones constituents network built with the MTD approach.

Stocks	2022–08–01	2022-08-08	2022–08–15	2022-08-22	2022–08–29	2022-09-05	2022–09–12	2022-09-19	2022–09–26	2022-10-03	2022–10–10	2022-10-17	2022–10–24
AAPL	6	2	1	7	1	0	1	1	0	0	1	0	8
AMGN	3	4	4	2	1	4	4	11	6	7	3	15	3
AXP	7	9	15	3	8	2	4	5	8	3	**14**	3	3
BA	9	7	2	5	7	9	4	11	6	1	**14**	5	12
CAT	**12**	15	8	**17**	8	14	3	5	5	10	11	**18**	**25**
CRM	4	14	4	11	4	**16**	10	14	10	6	4	6	12
CSCO	9	4	6	0	2	5	1	1	0	2	2	2	3
CVX	9	6	2	12	0	6	**13**	3	10	**19**	5	4	3
DIS	6	6	6	5	4	8	10	5	6	3	9	3	4
DOW	7	8	11	6	5	7	12	4	1	8	9	9	4
GS	2	10	8	11	1	8	8	4	2	10	3	6	10
HD	1	7	9	5	13	12	1	3	3	7	3	1	14
HON	8	3	6	0	9	4	4	5	1	5	5	4	9
IBM	5	1	3	1	5	2	0	1	0	7	5	4	3
INTC	9	6	1	16	2	0	1	1	5	0	5	2	5
JNJ	1	2	6	2	6	5	**13**	17	3	16	12	9	1
JPM	2	2	0	4	4	1	6	4	2	2	3	3	3
KO	5	0	2	3	2	10	0	4	2	2	1	3	2
MCD	5	6	4	8	6	6	9	3	3	5	10	2	1
MMM	3	1	2	2	7	1	4	0	**17**	4	8	0	2
MRK	2	11	3	3	7	5	6	**22**	2	2	2	5	0
MSFT	6	4	7	**17**	4	5	6	12	8	4	1	9	7
NKE	11	9	1	2	2	0	2	13	4	3	9	11	6
PG	10	2	2	6	1	2	10	1	2	7	7	10	3
TRV	2	3	4	1	**16**	1	7	1	5	1	8	14	11
UNH	11	6	5	7	14	13	4	2	8	9	3	7	5
V	4	2	8	7	2	1	5	13	10	5	7	2	12
VZ	10	**17**	3	3	6	2	10	2	1	4	3	11	3
WBA	5	3	4	4	6	8	3	6	1	3	7	4	0
WMT	1	6	**17**	1	3	9	7	7	9	8	10	3	1

Table 7
Weekly in-degrees of the Dow-Jones constituents network built with the MTD approach.

Stocks	2022–08–01	2022-08-08	2022–08–15	2022-08-22	2022–08–29	2022-09-05	2022–09–12	2022-09-19	2022–09–26	2022-10-03	2022–10–10	2022-10-17	2022–10–24
AAPL	5	5	5	6	6	5	6	5	6	5	5	7	7
AMGN	7	5	4	6	5	5	8	4	4	4	6	4	7
AXP	5	5	3	6	6	6	6	6	3	4	6	5	4
BA	5	7	**8**	6	5	**8**	4	9	6	6	3	6	5
CAT	5	6	4	6	4	5	6	7	5	7	6	4	6
CRM	8	7	7	5	6	6	7	5	3	4	5	6	6
CSCO	5	5	5	5	**7**	4	6	5	7	5	6	7	6
CVX	6	7	7	7	6	5	4	6	4	6	6	5	7
DIS	5	5	5	4	5	6	4	**10**	5	6	8	6	6
DOW	6	**8**	4	6	6	5	5	5	**8**	5	8	8	5
GS	7	5	6	7	**7**	7	5	7	3	6	7	7	6
HD	6	6	4	4	6	7	4	6	6	5	4	7	5
HON	5	5	6	4	6	5	4	6	4	**8**	8	5	4
IBM	6	**8**	5	6	4	6	6	6	5	7	6	4	9
INTC	6	**8**	4	**10**	5	5	5	7	4	5	6	7	5
JNJ	5	**8**	5	5	5	**8**	6	6	3	6	5	6	8
JPM	4	6	5	6	6	5	6	5	6	5	6	5	7
KO	**9**	5	5	6	3	4	7	7	3	3	4	5	6
MCD	8	4	4	4	4	5	6	4	5	6	7	7	4
MMM	8	6	4	6	3	4	5	7	7	7	7	7	4
MRK	7	4	4	5	5	4	5	7	7	6	6	4	5
MSFT	4	4	7	5	4	7	4	5	5	4	8	5	7
NKE	**9**	4	6	4	**7**	6	5	6	6	6	5	6	**10**
PG	4	6	7	5	4	5	4	4	4	5	4	5	5
TRV	5	6	4	5	**7**	**8**	6	5	3	5	5	6	6
UNH	6	6	**8**	8	5	6	4	7	4	5	7	4	4
V	5	6	7	4	3	5	7	5	3	6	8	**9**	4
VZ	6	6	2	8	6	6	8	5	3	4	7	7	6
WBA	4	5	3	5	6	3	**9**	7	4	**8**	6	5	5
WMT	4	**8**	6	7	4	5	6	7	4	4	**9**	6	6

The centrality measures show a high variability of the network over time with some of the stocks passing from a central position to a more peripheral one. However, some of the stocks maintain a stronger position in terms of out-degree centrality, e.g., CAT, CRM, CVX, or DOW, while others keep a lower profile, e.g., AAPL, IBM, or HON.

5. Conclusion

We proposed a new methodology to build networks. Our approach is based on a multivariate Markov chain built on the Mixture Transition Distribution model. Using a numerical example, we demonstrated that, contrary to the correlation approach, our model is able to capture the dependence among the stocks. Thus, it allows us to build meaningful networks. In the application section, we showed the network of the 30 Dow Jones constituents and its dynamics over time along with some measures of centrality.

Future research might explore the other characteristics of the network, for example, other measures of centrality or distance between the nodes. Moreover, even though the proposed model outperforms the classical correlation approach, it presents some limitations. More specifically, it appears to be computationally more expensive because it requires performing a numerical maximum likelihood estimation.

CRediT authorship contribution statement

Guglielmo D'Amico: Conceptualization, Methodology, Writing. **Riccardo De Blasis:** Conceptualization, Data curation, Software, Validation, Writing. **Filippo Petroni:** Conceptualization, Writing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The authors do not have permission to share data.

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