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SPIN(9) GEOMETRY OF THE OCTONIONIC HOPF FIBRATION

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Abstract. We deal with Riemannian properties of the octonionic Hopf fibration $S^{15} \to S^8$, in terms of the structure given by its symmetry group Spin(9). In particular, we show that any vertical vector field has at least one zero, thus reproving the non-existence of S^1 subfibrations. We then discuss Spin(9)-structures from a conformal viewpoint and determine the structure of compact locally conformally parallel Spin(9)-manifolds. Eventually, we give a list of examples of locally conformally parallel Spin(9)-manifolds.

1. Introduction

There are some features that distinguish S^{15} among spheres of arbitrary dimension. For example, S^{15} is the only sphere that admits three homogeneous Einstein metrics (see [Zil82]), and the only one that appears as regular orbit in three coho-

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mogeneity one actions on projective spaces, namely of SU(8), Sp(4) and Spin(9) on $\mathbb{C}P^8$, $\mathbb{H}P^4$ and $\mathbb{O}P^2$ respectively (see [Kol02]). Moreover, according to a famous problem of vector fields on spheres, S^{15} is the lowest dimensional sphere with more than 7 linearly independent vector fields (cf. for example [Hus94]). Finally, it has been shown that the Killing superalgebra of S^{15} is isomorphic to the exceptional compact real Lie algebra $\mathfrak{e}_8$ (see [FO08]).

All of these features can somehow be traced back to the transitive action of the subgroup $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$ on the octonionic Hopf fibration $S^{15} \to S^8$. This latter has a quite exceptional character: it does not admit any S^1-subfibration (see [LV92]), and there is no Hopf fibration over the Cayley projective plane $\mathbb{O}P^2$, although its volume is the quotient of those of the spheres S^{23} and S^7, natural candidates to its possible total space and fiber (cf. [Ber72, page 8]).

The mentioned characterizations of S^{15} and the role of Spin(9) in 16-dimensional Riemannian geometry have been a first motivation for the present paper.

In this respect, a first result we get is the following:

Theorem A. *Any global vector field on S^{15} which is tangent to the fibers of the octonionic Hopf fibration $S^{15} \to S^8$ has at least one zero.*

Note that the non-existence of S^1-subfibrations follows (cf. results obtained in [LV92] and Corollary 4.1).

A second motivation for this paper is to complete a general scheme of description for metrics which are locally conformally parallel with respect to the G-structures that refer to Riemannian holonomies. We next recall this general scheme. We say that we have a *locally conformally parallel G-structure* on a manifold M if one has a Riemannian metric g on M, a covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of M, and for each $\alpha \in A$ a metric g_α defined on U_α which has holonomy contained in G such that the restriction of g to each U_α is conformal to g_α:

$$g|_{U_\alpha} = e^{f_\alpha} g_\alpha$$

for some smooth map f_α defined on U_α.

Some of the possible cases here are:

- $G = \mathrm{U}(n)$, where we have the *locally conformally Kähler metrics*;
- $G = \mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$, yielding the *locally conformally quaternion Kähler metrics*;
- $G = \mathrm{Spin}(9)$, which is the case we are dealing with.

In any of the cases above, one can show that for each overlapping U_α, U_β the functions f_α, f_β differ by a constant:

$$f_\alpha - f_\beta = c_{\alpha,\beta} \text{ on } U_\alpha \cap U_\beta.$$

This implies that $df_\alpha = df_\beta$ on $U_\alpha \cap U_\beta \neq \varnothing$, hence defining a global, closed 1-form, usually denoted by θ and called the *Lee form*. Its metric dual with respect to g is denoted by B:

$$B = \theta^\sharp$$

and is called the *Lee vector field.*

The case $G = \mathrm{U}(n)$ is extensively studied: see, for instance, [DO98].

Choosing G to be $\mathrm{Sp}(n)$ or $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$, we get close relations to 3-Sasakian geometry: see [OP97] or the surveys [BG99], [CP99]. Finally, locally conformally parallel G_2 and Spin(7)-structures have been studied in [IPP06], and they relate to nearly parallel SU(3) and G_2 geometries, respectively.

In the case we deal with in this paper, it is a classical result by D. Alekseevsky that holonomy Spin(9) is only possible on manifolds that are either flat or locally isometric to $\mathbb{O}P^2$ or to the hyperbolic Cayley plane $\mathbb{O}H^2$ (see [Ale68] and [BrGr72]). Still, weakened holonomy conditions have also been considered. In particular, the article [Fri01] points out how, exactly as in the frameworks of structure groups $\mathrm{U}(n)$ and G_2, one can obtain 16 classes of Spin(9)-structures.

One of these classes consists of structures *of vectorial type* (see [AF06] and [Fri01, p. 148]); we show that this class fits into the locally conformally parallel scheme above (see Remark 6.2).

Besides this Remark, our contribution to the completion of the above general scheme with the case $G = \mathrm{Spin}(9)$ consists in the following Theorems.

Theorem B. *Let M^{16} be a compact manifold equipped with a locally, non-globally, conformally parallel* Spin(9) *metric g. Then*:

(1) *The Riemannian universal covering $(\widetilde{M}, \widetilde{g})$ of M is conformally equivalent to the Euclidean space $\mathbb{R}^{16} \setminus \{0\}$, that is, the Riemannian cone over S^{15}, and M is finitely isometrically covered by $S^{15} \times \mathbb{R}$.*
(2) *M is equipped with a canonical 8-dimensional foliation.*
(3) *If all the leaves of $\mathcal{F}$ are compact, then M fibers over an orbifold $\mathcal{O}^8$ finitely covered by S^8 and all fibers are finitely covered by $S^7 \times S^1$.*

Theorem C. *Let (M, g) be a compact Riemannian manifold. Then (M, g) is locally, non-globally, conformally parallel* Spin(9) *if and only if the following three properties are satisfied*:

(1) *M is the total space of a fiber bundle*

$$M \xrightarrow{\pi} S^1_r$$

where π is a Riemannian submersion over the circle of a certain radius r.
(2) *The fibers of π are isometric to a 15-dimensional spherical space form S^{15}/K, where $K \subset \mathrm{Spin}(9)$.*
(3) *The structure group of π is contained in the normalizer $N_{\mathrm{Spin}(9)}(K)$ of K in* Spin(9) *(that is, the isometries of S^{15}/K induced by* Spin(9)*).*

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2. Preliminaries

Let $\mathbb{O}$ be the algebra of octonions. The multiplication of $x = h_1 + h_2 e$, $x' = h'_1 + h'_2 e \in \mathbb{O}$ is defined through the one in quaternions $\mathbb{H}$ by the Cayley-Dickson process:

$$xx' = (h_1 h'_1 - \overline{h}'_2 h_2) + (h_2 \overline{h}'_1 + h'_2 h_1)e, \qquad (2.1)$$

where $\overline{h'_1}, \overline{h'_2}$ are the conjugates of $h'_1, h'_2 \in \mathbb{H}$. The conjugation in $\mathbb{O}$ is defined by $\overline{x} = \overline{h_1} - h_2 e$ and relates with the non-commutativity in $\mathbb{O}$ by $\overline{xx'} = \overline{x}'\overline{x}$. The non-associativity of $\mathbb{O}$ gives rise to the associator

$$[x, x', x''] = (xx')x'' - x(x'x''),$$

that vanishes whenever two among x, x', x'' are equal or conjugate. For a survey on octonions and their applications in geometry, topology and mathematical physics, see [Bae02].

We recall in particular the decomposition of the real vector space $\mathbb{O}^2$ into its *octonionic lines*

$$l_m \stackrel{\text{def}}{=} \{(x, mx) \mid x \in \mathbb{O}\} \quad \text{or} \quad l_\infty \stackrel{\text{def}}{=} \{(0, x') \mid x' \in \mathbb{O}\},$$

that intersect each other only in $(0,0) \in \mathbb{O}^2$ (cf. Section 4). Here $m \in S^8 = \mathbb{O}P^1 = \mathbb{O} \cup \{\infty\}$ parametrizes the set of octonionic lines l, whose volume elements $\nu_l \in \Lambda^8 l$ allow one to define the following *canonical 8-form* on $\mathbb{O}^2 = \mathbb{R}^{16}$:

$$\Phi_{\mathrm{Spin}(9)} \stackrel{\text{def}}{=} \int_{\mathbb{O}P^1} p_l^* \nu_l \, dl \in \Lambda^8(\mathbb{R}^{16}),$$

where p_l denotes the orthogonal projection $\mathbb{O}^2 \to l$. This definition of $\Phi_{\mathrm{Spin}(9)}$ is due to M. Berger (cf. [Ber72]). The following statement motivates our choice of notation for the canonical 8-form:

Proposition 2.1 ([Cor92, p. 170, Prop. 1.4])**.** *The subgroup of* $\mathrm{GL}(16, \mathbb{R})$ *preserving* $\Phi_{\mathrm{Spin}(9)}$ *is the image of* $\mathrm{Spin}(9)$ *under its spin representation into* $\mathbb{R}^{16}$*.*

Therefore, one can look at Spin(9) as a subgroup of SO(16). Accordingly, Spin(9)-structures can be considered on 16-dimensional oriented Riemannian manifolds. The following definition collects different approaches that have been used (see [Cor92], [Fri01], [PP12]):

Definition 2.2. Let M be a 16-dimensional oriented Riemannian manifold. A Spin(9)-*structure* on M is the datum of any of the following equivalent alternatives.

(1) A rank 9 vector subbundle $V^9 \subset \mathrm{End}(TM)$, locally spanned by endomorphisms

$$\{\mathcal{I}_\alpha\}_{\alpha=1,\dots 9}$$

satisfying

$$\mathcal{I}_\alpha^2 = \mathrm{Id}, \qquad \mathcal{I}_\alpha^* = \mathcal{I}_\alpha, \quad \text{and} \quad \mathcal{I}_\alpha \mathcal{I}_\beta = -\mathcal{I}_\beta \mathcal{I}_\alpha \quad \text{for} \quad \alpha \neq \beta, \tag{2.2}$$

where $\mathcal{I}_\alpha^*$ denotes the adjoint of $\mathcal{I}_\alpha$.

(2) An 8-form $\Phi_{\mathrm{Spin}(9)} \in \Lambda^8(M)$ which can be locally written as in [PP12, Table B], for a certain orthonormal local coframe $\{e^1, \dots, e^{16}\}$.

(3) A reduction $\mathcal{R}$ of the principal bundle of orthonormal frames on M from SO(16) to Spin(9).

Remark 2.3. From any of the Definitions 2.2, it follows that admitting a Spin(9)-structure depends only on the conformal class of M.

We describe now the rank 9 vector bundle of endomorphisms when M is the model space $\mathbb{R}^{16}$. Here $\mathcal{I}_1, \ldots, \mathcal{I}_9$ can be chosen as generators of the Clifford algebra Cl(9), the endomorphisms' algebra of its 16-dimensional real representation $\Delta_9 = \mathbb{R}^{16} = \mathbb{O}^2$. Accordingly, unit vectors $v \in S^8 \subset \mathbb{R}^9$ can be viewed, via the Clifford multiplication, as symmetric endomorphisms $v : \Delta_9 \to \Delta_9$.

The explicit way to describe this action is by $v = u + r \in S^8$ ($u \in \mathbb{O}$, $r \in \mathbb{R}$, $u\overline{u} + r^2 = 1$), acting on pairs $(x, x') \in \mathbb{O}^2$ by

$$\begin{pmatrix} x \\ x' \end{pmatrix} \longrightarrow \begin{pmatrix} r & R_{\overline{u}} \\ R_u & -r \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}, \tag{2.3}$$

where $R_u, R_{\overline{u}}$ denote the right multiplications by $u, \overline{u}$, respectively (cf. [Har90, p. 288]).

A basis of the standard Spin(9)-structure on $\mathbb{O}^2 = \mathbb{R}^{16}$ can be written by looking at the action (2.3) and at the nine vectors

$$(0,1), (0,i), (0,j), (0,k), (0,e), (0,f), (0,g), (0,h) \quad \text{and} \quad (1,0) \in S^8 \subset \mathbb{O} \times \mathbb{R} = \mathbb{R}^9.$$

In this way, one gets the following symmetric endomorphisms:

$$\mathcal{I}_1 = \left(\begin{array}{c|c} 0 & \mathrm{Id} \\ \hline \mathrm{Id} & 0 \end{array}\right), \mathcal{I}_2 = \left(\begin{array}{c|c} 0 & -R_i \\ \hline R_i & 0 \end{array}\right), \ldots, \mathcal{I}_8 = \left(\begin{array}{c|c} 0 & -R_h \\ \hline R_h & 0 \end{array}\right), \mathcal{I}_9 = \left(\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline 0 & -\mathrm{Id} \end{array}\right),$$

where $R_i, \ldots, R_h$ are the right multiplications by the 7 unit octonions $i, \ldots, h$. The subgroup $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$ is then characterized as preserving the 9-dimensional vector space

$$V^9 \stackrel{\text{def}}{=\!=} \langle \mathcal{I}_1, \ldots, \mathcal{I}_9 \rangle \subset \mathrm{End}(\mathbb{R}^{16}). \tag{2.4}$$

3. The quaternionic Hopf fibration

It is useful to look at $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$ as the octonionic analogue of the quaternionic group $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1) \subset \mathrm{SO}(8)$. A simple aspect of the analogy is given by the symmetry groups of the Hopf fibrations $S^7 \to S^4$ and $S^{15} \to S^8$, that are $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1) \subset \mathrm{SO}(8)$ and $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$, respectively (see [GWZ86, pp. 183 and 190]).

In the symmetry group of the quaternionic Hopf fibration, the two factors Sp(2) and Sp(1) act on the basis S^4 on the left, and on the S^3 fibers on the right, respectively. This action is thus related with the reducibility of the Lie algebra $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ and with the associativity of quaternions. All of this fails for the octonionic Hopf fibration, due to the irreducibility of $\mathfrak{spin}(9)$ and to the non-associativity of octonions.

However, the approach to a Spin(9)-structure on a 16-dimensional manifold M through the vector bundle $V^9 \subset \mathrm{End}(TM)$ admits a strict analogy for $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$. The same formula (2.3) defines a similar action on the sphere S^4 in $\mathbb{H}^2$, and this can be viewed as defining a $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$-structure. An explicit description of a canonical basis $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_5$ of sections of $V^5 \subset \mathrm{End}(\mathbb{H}^2)$ is given by the choices $(r, u) = (0, 1), (0, i), (0, j), (0, k), (1, 0)$ in equation (2.3), where now $u, x, x' \in \mathbb{H}$, and thus $(r, u) \in S^4$ (cf. [PP12]).

The ten compositions $\mathcal{I}_\alpha \mathcal{I}_\beta$, for $\alpha < \beta$, yield complex structures on $\mathbb{R}^8 = \mathbb{H}^2$, and a basis of the Lie algebra $\mathfrak{sp}(2)$. In particular, the sum of squares of their Kähler forms $\omega_{\alpha\beta}$ gives (cf. [PP12, p. 329]):

$$\sum_{1 \leq \alpha < \beta \leq 5} \omega_{\alpha\beta}^2 = -2\Omega_L,$$

where Ω_L is the left quaternionic 4-form in $\mathbb{R}^8$, defined as usual by

$$\Omega_L \stackrel{\text{def}}{=} \omega_{L_i}^2 + \omega_{L_j}^2 + \omega_{L_k}^2,$$

in terms of the Kähler forms ω of the left multiplications L_i, L_j and L_k.

Thus, on a Riemannian manifold M^8, the datum of a $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$-structure can be given through two different approaches. One can simply fix the usual rank 3 vector subbundle Q^3 of skew-symmetric elements in $\mathrm{End}(TM)$, whose local generators can be denoted by I, J, K. In the model space $\mathbb{R}^8$, the subgroup of rotations commuting with the standard complex structures I, J, K is $\mathrm{Sp}(2) \subset \mathrm{SO}(8)$, and the second factor $\mathrm{Sp}(1)$ of the reduced structure group here works as the double covering of $\mathrm{SO}(3)$, allowing one to change the admissible hypercomplex structure.

Since both factors of $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ are double coverings of rotation groups—namely, of $\mathrm{SO}(5)$ and $\mathrm{SO}(3)$, respectively—one can reverse the roles of the two factors. Accordingly, one can follow a different approach to fix a $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ reduction of the structure group on a Riemannian M^8. This second approach is what can be called a *quaternionic Hopf structure* (cf. [PP12, p. 327]), and consists of a vector subbundle $V^5 \subset \mathrm{End}(TM)$ of symmetric elements, whose local bases of sections $\mathcal{I}_\alpha \in \Gamma(V^5)$ $(\alpha = 1, \ldots, 5)$ satisfy relations (2.2). On the model space $\mathbb{R}^8$, the subgroup of rotations commuting with the standard $\mathcal{I}_1, \ldots, \mathcal{I}_5$ is the diagonal $\mathrm{Sp}(1)$ subgroup of $\mathrm{SO}(8)$, and now it is the left factor of $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ that allows admissible five dimensional rotations in the choice of bases of sections in V^5. As already recalled, the quaternionic 4-form of $\mathbb{H}^2 \cong \mathbb{R}^8$ can be easily written according to both the mentioned approaches.

In Section 5, we will deal with locally conformally parallel $\mathrm{Spin}(9)$-structures. It will be useful to have in mind some known facts for their corresponding 8-dimensional analogues, Riemannian manifolds M^8 whose metric is locally conformally related to metrics with holonomy $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$. We rephrase here some of these facts in terms of the rank 5 vector bundle $V^5 \subset \mathrm{End}(TM)$.

As mentioned in the Introduction, a quaternion Hermitian manifold (M^8, g) is called *locally conformally quaternion Kähler* (or LCQK, briefly) if $g|_U = e^{f_U} g'_U$ with local quaternion Kähler metrics g'_U, defined over open neighborhoods U covering M. The *Lee form* θ, locally $\theta|_U = df_U$, allows one to characterize globally the LCQK condition (cf. [OP97, p. 643]):

$$d\Omega_L = \theta \wedge \Omega_L, \qquad d\theta = 0.$$

The Levi-Civita connections of local quaternion Kähler metrics g'_U glue together to the *Weyl connection* D, defined on tangent vector fields X, Y as

$$D_X Y = \nabla_X Y - \tfrac{1}{2}\{\theta(X)Y + \theta(Y)X - g(X,Y)B\},$$

where ∇ is the Levi-Civita connection of g and $B = \theta^\sharp$ is the *Lee vector field*. Then the LCQK condition can be viewed as an example of *Einstein–Weyl structure*, i.e. the datum of the conformal class $[g]$ of metrics together with the torsion-free connection D satisfying the Einstein condition and preserving both the conformal class $[g]$ and the vector bundle $V^5 \to M^8$, that is, satisfying $Dg = \theta \otimes g$ and $DV^5 \subset V^5$.

Abundant examples exist in the subclass of 8-dimensional compact locally conformally hyperkähler manifolds: for instance any product $\mathcal{S} \times S^1$ of a compact 3-Sasakian 7-dimensional manifold $\mathcal{S}$ with a circle, where the former can be chosen having any second Betti number $b_2(\mathcal{S})$ (see [BGMR98]).

However, LCQK metrics on compact M^8 are either globally conformally quaternion Kähler or locally conformally quaternion Kähler with the local quaternion Kähler metrics of vanishing scalar curvature ([OP97, p. 645]), so that the locally quaternion Kähler metrics g'_U are necessarily Ricci flat.

Note that this does not imply the existence of a global hypercomplex structure on M^8, even on the open neighborhood where the local hyperkähler metrics g'_U are defined.

In the following, we see how a locally conformally quaternion Kähler manifold M^8 can be described by looking at the vector bundle $V^5 \to M^8$, and by using the vector fields $\mathcal{I}_1 B, \dots, \mathcal{I}_5 B$ on M^8.

Lemma 3.1. *Let M^8 be a compact manifold equipped with a locally, non-globally,* LCQK *metric g. Let B be its Lee vector field and $V^5 \subset \mathrm{End}(TM)$ the vector bundle defining the* $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$*-structure, locally spanned by $\mathcal{I}_1, \dots, \mathcal{I}_5$. Then the local vector fields $\mathcal{I}_1 B, \dots, \mathcal{I}_5 B$ are orthonormal and B belongs to their 5-dimensional distribution VB. The orthogonal complement $(VB)^\perp$ is integrable.*

Proof. Consider on M the distribution $\mathcal{F}$ spanned by the Lee vector field B and its transformation under the (local) compatible almost complex structures. As already mentioned, the whole $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$-structure can be given either by a rank 3 vector subbundle Q^3 of skew-symmetric elements in $\mathrm{End}(TM)$ (whose local generators are compatible almost complex structures usually denoted by I, J, K), or by a vector subbundle $V^5 \subset \mathrm{End}(TM)$ of symmetric elements, whose local generators we denote here by $\mathcal{I}_1, \dots \mathcal{I}_5$.

To prove the statement, there are now two possibilities. The first one is to refer to the work [OP97], and to rephrase the integrability of $\mathcal{F}$, a consequence of the Frobenius Theorem in [OP97, p. 645], in terms of the vector bundle V^5. The geometric interplay between the foliation $\mathcal{F}$ and the distribution VB, locally spanned by the vector fields $\mathcal{I}_1 B, \dots, \mathcal{I}_5 B$, follows from a computation that can be performed in the model space $\mathbb{R}^8$. This gives rise to the situation described in the statement. The same computation shows that none of the $\mathcal{I}_\alpha B$ is in general perpendicular to B, and that the orthogonal complement $(VB)^\perp$ is locally spanned by IB, JB, KB.

A second way to prove the statement is by a straightforward computation. This will be essentially done later in the proof of Theorem B, and more precisely of its statement (2). Although this latter refers to the Spin(9) case, the same computations, if limited to the choices $1, \dots, 5$, prove the present statement. $\square$

The following Proposition gives now a more complete description of LCQK manifolds in terms of the vector bundle V^5. Again, its proof follows from results in [OP97] (see in particular Theorem 3.8 at page 649).

Proposition 3.2. *Let M^8 be a compact manifold equipped with a locally, non-globally,* LCQK *metric g, with the same notation as in Lemma* 3.1.

(1) *There exists a metric in the conformal class of g whose Lee form θ is parallel.*
(2) *On each integral manifold N^7 of $\ker(\theta)$, the distribution $(VB)^\perp$, orthogonal in M to VB, is integrable and its leaves are 3-dimensional spherical space forms. The distribution on M spanned by $(VB)^\perp$ and B is the 4-dimensional vertical foliation $\mathcal{F}$, whose leaves are* LCQK *(generally non-primary) Hopf surfaces.*
(3) *The leaf space $M/\mathcal{F}$, when a manifold or an orbifold, carries a projected positive self-dual Einstein metric.*

4. Spin(9) and the octonionic Hopf fibration

For any $(x, y) \in S^{15} \subset \mathbb{O}^2 = \mathbb{R}^{16}$, we denote by

$$B \stackrel{\text{def}}{=} (x, y) \stackrel{\text{def}}{=} (x_1, \dots, x_8, y_1, \dots, y_8)$$

the (outward) unit normal vector field of S^{15} in $\mathbb{R}^{16}$. Here and in the following, we are identifying the tangent spaces $T_{(x,y)}(\mathbb{R}^{16})$ with $\mathbb{R}^{16}$.

Through the involutions $\mathcal{I}_1, \dots, \mathcal{I}_9$ one gets then the following sections of $T(\mathbb{R}^{16})|_{S^{15}}$ of length one:

$$\begin{aligned}
\mathcal{I}_1 B &= (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8),\\
\mathcal{I}_2 B &= (y_2, -y_1, -y_4, y_3, -y_6, y_5, y_8, -y_7, -x_2, x_1, x_4, -x_3, x_6, -x_5, -x_8, x_7),\\
\mathcal{I}_3 B &= (y_3, y_4, -y_1, -y_2, -y_7, -y_8, y_5, y_6, -x_3, -x_4, x_1, x_2, x_7, x_8, -x_5, -x_6),\\
\mathcal{I}_4 B &= (y_4, -y_3, y_2, -y_1, -y_8, y_7, -y_6, y_5, -x_4, x_3, -x_2, x_1, x_8, -x_7, x_6, -x_5),\\
\mathcal{I}_5 B &= (y_5, y_6, y_7, y_8, -y_1, -y_2, -y_3, -y_4, -x_5, -x_6, -x_7, -x_8, x_1, x_2, x_3, x_4),\\
\mathcal{I}_6 B &= (y_6, -y_5, y_8, -y_7, y_2, -y_1, y_4, -y_3, -x_6, x_5, -x_8, x_7, -x_2, x_1, -x_4, x_3),\\
\mathcal{I}_7 B &= (y_7, -y_8, -y_5, y_6, y_3, -y_4, -y_1, y_2, -x_7, x_8, x_5, -x_6, -x_3, x_4, x_1, -x_2),\\
\mathcal{I}_8 B &= (y_8, y_7, -y_6, -y_5, y_4, y_3, -y_2, -y_1, -x_8, -x_7, x_6, x_5, -x_4, -x_3, x_2, x_1),\\
\mathcal{I}_9 B &= (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, -y_1, -y_2, -y_3, -y_4, -y_5, -y_6, -y_7, -y_8).
\end{aligned} \tag{4.1}$$

As mentioned, $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$ is the group of symmetries of the octonionic Hopf fibration. This latter is defined by looking at the decomposition of $\mathbb{O}^2$ into the *octonionic lines*

$$l_m \stackrel{\text{def}}{=} \{(x, mx) \mid x \in \mathbb{O}\} \quad \text{or} \quad l_\infty \stackrel{\text{def}}{=} \{(0, y) \mid y \in \mathbb{O}\},$$

mentioned in Section 2. One has to be careful that the octonionic line through $(0,0)$ and $(x,y) \in \mathbb{O}^2$ *is not* $\{(xo, yo) \mid o \in \mathbb{O}\}$. This latter in fact is not even an octonionic line, the correct line being instead $l_{yx^{-1}} = \{(o, (yx^{-1})o \mid o \in \mathbb{O}\}$ if $x \neq 0$, and l_∞ if $x = 0$. In this way the fibration

$$\mathbb{O}^2 \setminus 0 \to S^8 = \{m \in \mathbb{O}\} \cup \{\infty\}$$

is obtained, with fibers $\mathbb{O} \setminus 0$, and the intersection with the unit sphere $S^{15} \subset \mathbb{O}^2$ provides the octonionic Hopf fibration

$$S^{15} \to S^8, \qquad \text{or as homogeneous fibration} \qquad \frac{\mathrm{Spin}(9)}{\mathrm{Spin}(7)} \xrightarrow{\frac{\mathrm{Spin}(8)}{\mathrm{Spin}(7)}} \frac{\mathrm{Spin}(9)}{\mathrm{Spin}(8)}.$$

Denote by VB the 9-dimensional span of $\mathcal{I}_1 B, \dots, \mathcal{I}_9 B$:

$$VB \stackrel{\text{def}}{=} \langle \mathcal{I}_1 B, \dots, \mathcal{I}_9 B \rangle,$$

and note that 9-planes of VB are generally not tangent to S^{15}.

Proof of Theorem A. First, note that VB is invariant under Spin(9): this is clear for the unit normal $B = (x,y)$, since $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$, and on the other hand the nine endomorphisms $\mathcal{I}_\alpha$ are rotating under the Spin(9) action inside their vector space $V^9 \subset \mathrm{End}(\mathbb{R}^{16})$.

Next, VB contains B. In fact:

$$B = \lambda_1 \mathcal{I}_1 B + \lambda_2 \mathcal{I}_2 B + \dots + \lambda_8 \mathcal{I}_8 B + \lambda_9 \mathcal{I}_9 B,$$

where the coefficients λ_α can be computed from 4.1 in terms of the inner products (here all the arrows denote vectors in $\mathbb{R}^8$)

$$\vec{x} = (x_1, \dots, x_8),\ \vec{y} = (y_1, \dots, y_8) \in \mathbb{R}^8$$

and of the right translations $R_i, \dots, R_h$ as follows:

$$\lambda_1 = 2\vec{x} \cdot \vec{y},\ \lambda_2 = -2\vec{x} \cdot \vec{R_i y},\ \dots,\ \lambda_8 = -2\vec{x} \cdot \vec{R_h y},\ \lambda_9 = |\vec{x}|^2 - |\vec{y}|^2.$$

In particular, at points with $\vec{x} = \vec{0}$, that is on the octonionic line l_∞, the vector fields $\mathcal{I}_1 B, \dots, \mathcal{I}_9 B$ are orthogonal to the unit sphere $S^7_\infty \subset l_\infty$. This latter is the fiber of the Hopf fibration $S^{15} \to S^8$ over the north pole $(0, \dots, 0, 1) \in S^8$, and the mentioned orthogonality of this fiber S^7 is immediate from 4.1 for $\mathcal{I}_1 B, \dots, \mathcal{I}_8 B$. Also, for these points, we have $\mathcal{I}_9 B = B$, so $\mathcal{I}_9 B$ is ortohogonal to S^7_∞. Now, the invariance under Spin(9) of the octonionic Hopf fibration shows that all its fibers are characterized as orthogonal in $\mathbb{R}^{16}$ to the vector fields $\mathcal{I}_1 B, \dots, \mathcal{I}_9 B$.

Now, assume that X is a vertical vector field of $S^{15} \to S^8$. By the previous characterization we have the following orthogonality relations in $\mathbb{R}^{16}$:

$$\langle X, \mathcal{I}_\alpha B \rangle = 0 \quad \text{for } \alpha = 1, \dots, 9,$$

and it follows that $\langle \mathcal{I}_\alpha X, B \rangle = 0$. But from the definition of a Spin(9)-structure we see that if $\alpha \neq \beta$, then $\langle \mathcal{I}_\alpha X, \mathcal{I}_\beta X \rangle = 0$. Thus, if X is a nowhere zero vertical vector field, we would obtain in this way 9 pairwise orthogonal vector fields $\mathcal{I}_1 X, \dots, \mathcal{I}_9 X$, all tangent to S^{15}. But S^{15} is known to admit at most 8 linearly independent vector fields by the classical Hurwitz–Radon–Adams result (see, for example, [Hus94] or [PP13]). Thus X cannot be vertical and nowhere zero, and Theorem A is proved. $\square$

One gets as a consequence the following alternative proof of a result in [LV92]:

Corollary 4.1. *The octonionic Hopf fibration $S^{15} \to S^8$ does not admit any S^1 subfibration.*

Proof. In fact, any S^1 subfibration would give rise to a real line subbundle $L \subset T_{\mathrm{vert}}(S^{15})$ of the vertical subbundle of $T(S^{15})$. Such line bundle L is necessarily trivial, due to the vanishing of its first Stiefel-Whitney class $w_1(L) \in H^1(S^{15}; \mathbb{Z}_2) = 0$. It follows that L would admit a nowhere zero section, thus a global vertical nowhere zero vector field. □

5. Locally conformally parallel Spin(9) manifolds

Definition 5.1. A Riemannian manifold (M^{16}, g) is *locally conformally parallel* Spin(9) (LCP, briefly) if over open neighbourhoods $\{U\}$ covering M the restriction $g|_U$ of the metric g is conformal to a (local) metric g'_U having holonomy contained in Spin(9). □

The conformality relations $g|_U = e^{f_U} g'_U$ give rise to a *Lee form* θ, locally defined as $\theta|_U \stackrel{\mathrm{def}}{=} df_U$. Next, recall that Spin(9) is characterized as the subgroup of $\mathrm{GL}(16, \mathbb{R})$ that preserves the 8-form $\Phi_{\mathrm{Spin}(9)}$ (cf. the already quoted [Cor92, p. 170]). Thus, a Spin(9)-structure on M^{16} is equivalent to the datum of a $\Phi \in \Lambda^8(M)$, which can be locally written as in [PP12, Table B] and, under the LCP hypothesis, on each U the metric g'_U defines a similar 8-form Φ'_U parallel with respect to the Levi-Civita connection of g'_U. It follows that the restriction of Φ to U satisfies

$$\Phi|_U = e^{4f_U} \Phi'_U,$$

henceforth one has

$$d\Phi = \theta \wedge \Phi.$$

Moreover, the Levi-Civita connections of the local parallel metrics g'_U glue together to the global Weyl connection D on M:

$$D_X Y = \nabla_X Y - \tfrac{1}{2}\{\theta(X)Y + \theta(Y)X - g(X,Y)B\},$$

where ∇ is the Levi-Civita connection of g. Recall that, since the metrics g'_U are assumed to have holonomy contained in Spin(9), they are Einstein metrics. Thus the conditions $DV \subset V$, $Dg = \theta \otimes g$, $d\theta = 0$ and g'_U Einstein, insure that the conformal class $[g]$ defines a closed Einstein–Weyl manifold $(M, [g], D)$.

We will next give the

Proof of Theorem B. First, recall that g defines a closed Einstein–Weyl structure on a compact manifold, with Lee form θ non-exact (but closed). Then the following properties hold (cf. [Gau95, p. 10, Thm. 3]): (a) its Weyl connection D is Ricci-flat; (b) one can choose, in the conformal class $[g]$, a metric g_0 (unique up to homotheties) such that its Lee form θ_0 is parallel with respect to the Levi-Civita connection ∇^{g_0} of g_0.

Thus, to prove our statement we can assume, without loss of generality, that the Lee form θ of g is parallel with respect to the Levi Civita connection ∇^g.

Henceforth also its Lee field $B = \theta^\sharp$ is parallel and as a consequence, by the de Rham decomposition theorem, the universal covering $(\widetilde{M}, \widetilde{g})$ is reducible:

$$(\widetilde{M}, \widetilde{g}) = (\mathbb{R}, dt^2) \times (\widetilde{N}, g_{\widetilde{N}}), \quad \widetilde{N} \text{ complete and simply connected.}$$

With respect to this decomposition we have that the pull-back of θ is $\widetilde{\theta} = dt$. The diffeomorphism $\mathbb{R} \times \widetilde{N} \to \mathbb{R}^+ \times \widetilde{N}$ given by

$$(t, x) \mapsto (s = e^t, x)$$

shows that $(\widetilde{M}, \widetilde{g})$ is globally conformal, with conformality factor $1/s^2$, to the so-called *metric cone*

$$C(\widetilde{N}) = (\widetilde{M}, ds^2 + s^2 g_{\widetilde{N}}).$$

Using the classical D. Alekseevsky theorem ([Ale68, p. 98, Cor. 1]) we see that the Ricci-flatness of the local metrics (as mentioned, the consequence of their holonomy contained in Spin(9) and of the Theorem of Gauduchon on closed Weyl structures), insures their flatness, so that the cone $C(\widetilde{N})$ is flat. We can use then the relation between the curvature operator R of the warped product $C(\widetilde{N}) = \mathbb{R}^+ \times_{s^2} \widetilde{N}$ and the curvature operator $R^{\widetilde{N}}$ of its fiber $\widetilde{N}$:

$$0 = R_{VW}Z = R^{\widetilde{N}}_{VW}Z - \frac{4}{s^2}(g_{\widetilde{N}}(V, Z)W - g_{\widetilde{N}}(W, Z)V)$$

(see, for example, [O'N83, p. 210]) to recognize that $\widetilde{N}$, being complete, is the sphere S^{15}. All of this insures that the universal covering of M is conformally equivalent to the cone $C(S^{15})$ and, since the Lee vector field B is parallel, that M is locally isometric, up to homotheties, to $S^{15} \times \mathbb{R}$. This proves statement 1.

We now prove statement 2. Denote by Θ the codimension 1 foliation on M defined by the equation $\theta = 0$, with $\theta = B^\sharp$, and note that the parallelism of θ insures that Θ is a totally geodesic foliation, so that the Levi-Civita connection on any leaf $T = T^{15}$ is just the restriction of ∇^g. Next, consider the vector bundle $V = V^9 \subset \mathrm{End}(TM)$ given by the Spin(9)-structure, locally spanned by $\mathcal{I}_1, \dots, \mathcal{I}_9$, and the corresponding distribution

$$VB \stackrel{\text{def}}{=\!=} \langle \mathcal{I}_1 B, \dots, \mathcal{I}_9 B \rangle \subset T(M)$$

generated by the orthonormal vector fields $\mathcal{I}_1 B, \dots, \mathcal{I}_9 B$. Then VB contains the Lee vector field B, as seen in the proof of Theorem A.

We now show that the 8-dimensional distribution

$$\mathcal{F} \stackrel{\text{def}}{=\!=} \langle \mathcal{I}_1 B, \dots, \mathcal{I}_9 B \rangle^\perp \oplus \langle B \rangle = (VB)^\perp \oplus \langle B \rangle$$

is integrable.

First, let $X, Y \in (VB)^\perp$, so that $g(X, \mathcal{I}_\alpha B) = g(Y, \mathcal{I}_\alpha B) = 0$ for $\alpha = 1, \dots, 9$. Then, in terms of the Weyl connection D, we have

$$g([X, Y], \mathcal{I}_\alpha B) = g(\mathcal{I}_\alpha(D_X Y), B) - g(\mathcal{I}_\alpha(D_Y X), B).$$

Recall now that $DV \subset V$ gives rise to 1-forms $a_{\alpha\beta}$ such that $D\mathcal{I}_\alpha = \sum a_{\alpha\beta} \otimes \mathcal{I}_\beta$. It follows:

$$\begin{aligned} g(\mathcal{I}_\alpha(D_X Y), B) &= g(D_X(\mathcal{I}_\alpha Y), B) - g((D_X(\mathcal{I}_\alpha)Y), B) \\ &= g(D_X(\mathcal{I}_\alpha Y), B) - \sum a_{\alpha\beta}(X) g(\mathcal{I}_\beta Y, B), \end{aligned}$$

and since $g(\mathcal{I}_\beta Y, B) = 0$, we obtain

$$g(\mathcal{I}_\alpha(D_X Y), B) = g(D_X(\mathcal{I}_\alpha Y), B).$$

On the other hand, since $\nabla B = 0$ we have also $D_Y B = -\frac{1}{2}Y$ (cf. [DO98, p. 37]). Thus, by applying D to the identity $g(\mathcal{I}_\alpha X, B) = 0$, we obtain

$$0 = Y(g(\mathcal{I}_\alpha X, B)) = g(D_Y(\mathcal{I}_\alpha X), B) + g(\mathcal{I}_\alpha X, D_Y B) + \theta(Y) g(\mathcal{I}_\alpha X, B),$$

so that

$$g(D_Y(\mathcal{I}_\alpha X), B) = \tfrac{1}{2} g(\mathcal{I}_\alpha X, Y).$$

All of this gives

$$\begin{aligned} g([X, Y], \mathcal{I}_\alpha B) &= g(D_X(\mathcal{I}_\alpha Y), B) - g(D_Y(\mathcal{I}_\alpha X), B) \\ &= \tfrac{1}{2} g(\mathcal{I}_\alpha Y, X) - \tfrac{1}{2} g(\mathcal{I}_\alpha X, Y) = 0, \end{aligned}$$

so we get that $[X, Y] \in (VB)^\perp$.

Now, to obtain the integrability of $\mathcal{F}$, we must further check that for $X \in (VB)^\perp$ the bracket $[X, B] = D_X B - D_B X = -X - D_B X$ belongs to $\mathcal{F}$. In fact $D_B X = \nabla_B X - \frac{1}{2}X \in (VB)^\perp$, and $\nabla_B X \in (VB)^\perp$ is a consequence of $g(X, B) = g(X, \mathcal{I}_\alpha B) = 0$. This ends the proof of statement 2.

As for statement 3, one can use the same argument as in [OP97, Thm. 2.1] to show that $\mathcal{F}$ is a Riemannian totally geodesic foliation and that the leaf space, when a manifold or an orbifold, carries a metric of spherical space form type. □

Proof of Theorem C. Our arguments will follow basically the same ideas as in [OV03], and we first show that the locally conformally parallel Spin(9) condition implies on compact manifolds the structure described by properties (1), (2) and (3).

In fact, if (M, g) is compact and locally, non-globally, conformally parallel Spin(9), recall from the proof of Theorem B that its universal covering $(\widetilde{M}, \widetilde{g})$ is conformally equivalent to the metric cone $C(S^{15}) = \mathbb{R}^{16} \setminus 0$ with conformal factor $\frac{1}{s^2} = e^{-2t}$, that is, the cone metric g_{cone} is given by

$$g_{\mathrm{cone}} = e^{-2t}\widetilde{g}.$$

Any $\gamma \in \pi_1(M)$ can be thought as a map $\gamma : \widetilde{M} \to \widetilde{M}$ preserving $\widetilde{g}$, and we get:

$$\gamma^*(g_{\mathrm{cone}}) = \gamma^*(e^{-2t}\widetilde{g}) = (e^{-2t} \circ \gamma)\gamma^*(\widetilde{g}) = (e^{-2t} \circ \gamma)\widetilde{g} = (e^{-2t} \circ \gamma) e^{2t} g_{\mathrm{cone}},$$

showing that π_1 acts by conformal maps. Moreover, taking differentials of

$$\gamma^*(\Phi_{\mathrm{Spin}(9)}) = (e^{-2t} \circ \gamma) e^{2t} \Phi_{\mathrm{Spin}(9)}$$

and using $d\Phi_{\mathrm{Spin}(9)} = 0$, we see that $\pi_1(M)$ acts by homotheties.

Indeed, the homothety factor $\rho(\gamma)$ of $\gamma \in \pi_1(M)$ defines a homomorphism $\rho : \pi_1(M) \to \mathbb{R}^+$, whose image is a finitely generated subgroup of $\mathbb{R}^+$, thus isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{N}$. The locally conformal flatness of the metric allows one to apply the arguments used to prove [GOPP06, Cor. 4.7], and to see that the image of ρ is isomorphic to $\mathbb{Z}$.

Next, notice that $K \overset{\text{def}}{=\!=} \ker \rho$ consists of isometries of $C(S^{15})$ that leave the form $\Phi_{\mathrm{Spin}(9)}$ invariant, so that in particular $K \subset \mathrm{Spin}(9)$. Moreover, any isometry of $C(S^{15})$ induces the identity map on the $\mathbb{R}^+$-component (see again [GOPP06, Thm. 5.1]), and it leaves the fibers of the projection $C(S^{15}) \to \mathbb{R}^+$ invariant. Since S^{15} is compact and $\pi_1(M)$ acts properly discontinously and freely on $C(S^{15})$, K is finite and without fixed points on S^{15}. It follows:

$$C(S^{15})/K = C\left(S^{15}/K\right).$$

Consider now a homothety $\gamma \in \pi_1(M)$ such that $h \overset{\text{def}}{=\!=} \rho(\gamma) \in \mathbb{R}^+$ generates $\mathrm{Im}(\rho)$. Then γ is a homothety on $C(S^{15}/K)$, and

$$\gamma(s,x) = (h \cdot s, \psi(x)) \quad \text{for } x \in \frac{S^{15}}{K},\ s \in \mathbb{R}^+ \text{ and } \psi \in \mathrm{Isom}\left(\frac{S^{15}}{K}\right). \tag{5.1}$$

Thus, for any $n \in \mathbb{Z}$ we have:

$$\gamma^n(s,x) = (h^n \cdot s, \psi^n(x)). \tag{5.2}$$

Consider the projection $\mathrm{pr} : C(S^{15}/K) \to \mathbb{R}^+$ on the first factor of the cone. Then formula (5.2) shows that pr is equivariant with respect to the actions of $\langle\gamma\rangle = \mathbb{Z}$ on $C(S^{15}/K)$ and of $n \in \mathbb{Z}$ on $s \in \mathbb{R}^+$, given by $h^n \cdot s$. The induced map

$$M = \frac{C(S^{15}/K)}{\langle\gamma\rangle} \xrightarrow{\pi} \frac{\mathbb{R}^+}{\mathbb{Z}} = S^1 \tag{5.3}$$

is, up to rescaling the metric on S^1, the map in (1) in the statement. Then (2) follows. As for (3), observe that ψ in formula (5.1) comes from an element of $\mathrm{SO}(16)$ which preserves $\Phi_{\mathrm{Spin}(9)}$.

To show that (1), (2) and (3) are necessary conditions for M to be locally conformally parallel Spin(9), we use a topological argument. Assume that (M,g) is a compact Riemannian manifold satisfying (1), (2) and (3) in Theorem C, and consider two open sets U_1 and U_2 covering S^1. Then the definition of fiber bundle implies that M can be recovered by glueing together $U_1 \times (S^{15}/K)$ and $U_2 \times (S^{15}/K)$ by a transition function $\psi_\pi : S^{15}/K \to S^{15}/K$. This transition function depends on π, and is usually called the *clutching function* of the bundle. Moreover, (3) implies that $\psi \in N_{\mathrm{Spin}(9)}(K)$ is an isometry of S^{15}/K. Now choose $h \in \mathbb{R}^+$, and use ψ_π, h to define a homothety γ_π on $C(S^{15}/K)$ as in formula (5.1):

$$\gamma_\pi(s,x) \overset{\text{def}}{=\!=} (h \cdot s, \psi_\pi(x)) \quad \text{for } x \in \frac{S^{15}}{K} \text{ and } s \in \mathbb{R}^+.$$

Then, let M_π be the locally conformally parallel Spin(9) manifold

$$M_\pi \stackrel{\text{def}}{=\!=} \frac{C(S^{15}/K)}{\langle\gamma_\pi\rangle}.$$

Since we already proved the sufficiency of conditions in Theorem C, we know that M_π is itself a fiber bundle over S^1 with the same clutching map $\psi_\pi : S^{15}/K \to S^{15}/K$. Recall on the other hand that for any Lie group G, the equivalence classes of principal G bundles over S^n is in natural bijection with the homotopy group $\pi_{n-1}(G)$ ([Ste99, p. 99, Thm. 18.5]). Thus M and M_π are isomorphic as fiber bundles over S^1, and in particular they are isometric. $\square$

Remark 5.2. Using the Galoisian terminology described in [GOPP06, Sect. 2], the pair

$$\left(C\left(\frac{S^{15}}{K}\right), \langle\gamma\rangle\right)$$

is the minimal presentation of M.

Remark 5.3. The fibers of π in Theorem C inherit a 7-Sasakian structure (in the sense of [Dea08]) induced by the foliation $(VB)^\perp$ as in the proof of Theorem B. Indeed, this notion of 7-Sasakian structure on 15-dimensional spherical space forms seems to be the induced counterpart on the leaves of a canonical codimension one foliation on M^{16}. Note that, in accordance with [Dea08], such a 7-Sasakian structure does not involve global vertical vector fields, but only a vertical foliation, whose transverse structure we have here related with the Spin(9)-structure of M^{16}.

The following is a different way of stating Theorem C.

Corollary 5.4. *The set of isometry classes of locally, non-globally, conformally parallel* Spin(9) *manifolds is in bijective correspondence with the set of triples*

$$\{(r, K, c_K) \mid r \in \mathbb{R}^+, K \leq \mathrm{Spin}(9) \textit{ finite and free on } S^{15}, c_K \in \pi_0\left(N_{\mathrm{Spin}(9)}(K)\right)\},$$

where π_0 stands for the connected component functor.

Remark 5.5. We could also describe the map $\pi : M \to S^1$ in Theorem C as the Albanese map defined as follows. Fix any $x_0 \in M$. For any $x \in M$ and any path γ joining x_0 and x, define:

$$\alpha(x) \stackrel{\text{def}}{=\!=} \left(\int_\gamma \theta\right) \bmod G.$$

Here θ is the Lee form of M, and $G \subset \mathbb{R}$ is the additive subgroup of "periods of θ", generated by the integrals $\int_\sigma \theta$ over the generators σ of $H_1(M, \mathbb{Z})$.

6. Examples

As a consequence of Theorem B, the examples will be in the context of the flat Spin(9)-structure on $\mathbb{R}^{16}$. Recalling the threefold approach to Spin(9)-structures given by Definition 2.2, we refer to the data $V = V^9$, $\Phi_{\mathrm{Spin}(9)}$, $\mathcal{R}$ and Spin(9) as the *standard data*, and the standard inclusion $\mathrm{SO}(16) \subset \mathrm{GL}(16, \mathbb{R})$ can be viewed as equivalent to the choice of the standard basis $\{e_1, \dots, e_{16}\}$ of $\mathbb{R}^{16}$ as orthonormal. Thus, another way to describe the flat Spin(9)-structure on $\mathbb{R}^{16}$ is the *standard structure with respect to the standard basis* $\{e_1, \dots, e_{16}\}$.

Thus, if we choose a different basis $\mathcal{B}$ on $\mathbb{R}^{16}$ that we declare to be orthonormal in a suitable metric $g_{\mathcal{B}}$, we can talk about the *standard structure with respect to* $\mathcal{B}$. This means that we are choosing a different inclusion $i : \mathrm{SO}(16) \hookrightarrow \mathrm{GL}(16, \mathbb{R})$, but the structure is still standard in the sense that V, $\Phi_{\mathrm{Spin}(9)}$ and $\mathcal{R}$ are induced by the standard ones using the inclusion i.

Observe that this holds even if the inclusion i depends on the point $x \in \mathbb{R}^{16}$, that is if $\mathcal{B}$ is not a basis on the vector space $\mathbb{R}^{16}$, but a parallelization on the manifold $\mathbb{R}^{16}$. In the same way, on any parallelizable M^{16} with a fixed parallelization $\mathcal{B}$ one can speak of the *standard* Spin(9)*-structure on M associated with $\mathcal{B}$*, whose associated objects will be denoted by $V_{\mathcal{B}}$, $\Phi_{\mathcal{B}}$, $\mathcal{R}_{\mathcal{B}}$ and $g_{\mathcal{B}}$ (see [Par01b] and [Par01a] for details).

Example 6.1. On $\mathbb{R}^{16} \setminus 0$, consider the parallelization $\widetilde{\mathcal{B}} \stackrel{\text{def}}{=} \{|x|\partial_1, \dots, |x|\partial_{16}\}$ where $\partial_1, \dots, \partial_{16}$ denotes the derivatives with respect to the standard coordinates. Look at the map

$$p : \mathbb{R}^{16} \setminus 0 \longrightarrow S^{15} \times S^1, \quad p(x) \stackrel{\text{def}}{=} (x/|x|, \log|x| \mod 2\pi),$$

and observe that p projects $\widetilde{\mathcal{B}}$ to a parallelization $\mathcal{B} \stackrel{\text{def}}{=} p_*(\widetilde{\mathcal{B}})$ on $S^{15} \times S^1$ (see also [Bru92, Sect. 6 and 7] and [Par03]). Consider the standard Spin(9)-structure $g_{\mathcal{B}}$ on $S^{15} \times S^1$ associated with $\mathcal{B}$. Then $g_{\mathcal{B}}$ is locally conformally parallel, since p is a covering map, bundle-like by definition, so that $g_{\mathcal{B}}$ is locally given by $g_{\widetilde{\mathcal{B}}}$, that is to say, by $|x|^{-2}g$, where g is the flat metric on $\mathbb{R}^{16}$.

As observed in Theorem B, the flat metric on $\mathbb{R}^{16} \setminus 0$ is the cone metric on $C(S^{15})$. The metric $g_{\widetilde{\mathcal{B}}}$ is instead the cylinder metric on the Riemannian universal covering of $S^{15} \times S^1$. $\square$

Remark 6.2. In [Fri01] and [AF06] the class of locally conformally parallel Spin(9)-structures has been identified and studied, under the name of "Spin(9)-structures of vectorial type" (cf. the following Definition 6.3). We outline now a proof that, *for* Spin(9)*-structures, vectorial type is equivalent to locally conformally parallel.*

Following [Fri01] and [AF06], one can look at the splitting of the Levi-Civita connection in the principal bundle of orthonormal frames on M:

$$\nabla = \nabla^* \oplus \theta$$

where ∇^* is the connection in the induced bundle of Spin(9)-frames and θ is its orthogonal complement. Thus, θ is a 1-form with values in the orthogonal complement $\mathfrak{m}$ defined by the splitting $\mathfrak{so}(16) = \mathfrak{spin}(9) \oplus \mathfrak{m}$ and, under canonical identifications, θ can be seen as a 1-form with values in $\Lambda^3(V)$.

Under the action of Spin(9), the space $\Lambda^1(M) \otimes \Lambda^3(V)$ decomposes as a direct sum of 4 irreducible components:

$$\Lambda^1(M) \otimes \Lambda^3(V) = P_0 \oplus P_1 \oplus P_2 \oplus P_3,$$

and, looking at all the possible direct sums, this yields 16 types of Spin(9)-structures. The component P_0 identifies with $\Lambda^1(M)$. Thus:

Definition 6.3. [AF06] A Spin(9)-structure is of *vectorial type* if θ lives in P_0.

Now, let (M,g) be a Riemannian manifold endowed with a Spin(9)-structure of vectorial type. Let θ be as above, and let Φ be its Spin(9)-invariant 8-form. Now, $\theta=0$ implies that the holonomy of M is contained in Spin(9) (cf. [Fri01, p. 21]).

From [AF06, p. 5] we know that the following relations hold:

$$d\Phi = \theta\wedge\Phi, \qquad d\theta = 0. \tag{6.1}$$

Let $(M,\widetilde{g})$ be the Riemannian universal cover of (M,g) and let $\widetilde{\Phi}$, $\widetilde{\theta}$ be the lifts of Φ, θ, respectively. Then relations (6.1) hold as well for $\widetilde{\Phi}$ and $\widetilde{\theta}$. Since $\widetilde{M}$ is simply connected, then $\widetilde{\theta}=df$, for some $f:\widetilde{M}\to\mathbb{R}$. Then, defining $g_0 \overset{\text{def}}{=} e^{-f}\widetilde{g}$ and $\Phi_0 \overset{\text{def}}{=} e^{-4f}\widetilde{\Phi}$, we have $d\Phi_0=0$, that is, the θ-factor of Φ_0 is zero. Hence g_0 has holonomy contained in Spin(9), and on the other hand it is locally conformal to g. Thus M can be covered by open subsets on which the metric is conformal to a metric with holonomy in Spin(9), which is Definition 5.1.

Remark 6.4. With the notations of Theorem C, the locally conformally parallel Spin(9)-structure on $S^{15}\times S^1$ defined in Example 6.1 is associated with $K=\{\mathrm{Id}\}\subset\mathrm{Spin}(9)$. Since $N_{\mathrm{Spin}(9)}(K)=\mathrm{Spin}(9)$ is connected, there is only one locally conformally parallel Spin(9)-structure on $S^{15}\times S^1$ (see Corollary 5.4). Thus, Example 6.1 is an alternative description of the Spin(9)-structure of vectorial type on $S^{15}\times S^1$ given in [Fri01, p. 136, Example 2], where the terminology "W_4 type" is used for vectorial type. See also [AF06].

Example 6.5. According to Theorems B and C, to give examples of compact locally conformally parallel Spin(9) manifolds, one has to look at finite subgroups of Spin(9) acting without fixed points on S^{15}. The classification of such finite subgroups is not an easy problem, and we limit ourselves to exhibit only some of them. They will show, however, that many finite quotients of S^{15} may appear as fibers in the map of Theorem C.

We describe in particular how S^{15} is acted on "diagonally" and without fixed points by a subgroup $\mathrm{Sp}(1)_\Delta\subset\mathrm{Spin}(9)$. Let $(x=h_1+h_2e, x'=h_1'+h_2'e)\in\mathbb{O}^2$ and define the following action of $q\in\mathrm{Sp}(1)$ on the *first* octonionic coordinate $x=h_1+h_2e\in\mathbb{O}$:

$$A_q:\ h_1+h_2e\longrightarrow h_1q+(\bar{q}h_2)e.$$

Due to the identity $\overline{q_1q_2}=\overline{q}_2\overline{q}_1$, this is a right action $A_q:\mathbb{O}\to\mathbb{O}$ for each $q\in\mathrm{Sp}(1)$. In the real components of x, A_q is represented by a matrix of SO(8), and indeed by a matrix in its diagonal $\mathrm{SO}(4)\times\mathrm{SO}(4)$ subgroup.

Recall now the Triality Principle for SO(8). In the formulation we need here it can be stated as follows (cf. [GWZ86, p. 192] or [DS01, pp. 143–145]).

Triality Principle. *Consider the triples $A,B,C\in\mathrm{SO}(8)$ such that for any $x,m\in\mathbb{O}$:*

$$C(m)A(x)=B(mx).$$

If any of A, B, C is given, then the other two exist and are unique up to changing sign for both of them.

Given $A \in \mathrm{SO}(8)$ we will call any of such matrices $\pm B, \pm C$ a *triality companion* of A.

Going back to the transformation $A_q \in \mathrm{SO}(8)$ defined by any $q \in \mathrm{Sp}(1)$, consider a pair $B_q, C_q \in \mathrm{SO}(8)$ of its triality companions. Thus, for any $x, m \in \mathbb{O}$:

$$C_q(m)A_q(x) = B_q(mx),$$

and define the following right action of $q \in \mathrm{Sp}(1)$ on $\mathbb{O}^2$:

$$R_q : (x = h_1 + h_2 e, x' = h'_1 + h'_2 e) \to (A_q x, B_q x').$$

Thus, R_q carries octonionic lines to octonionic lines, so that $R_q \in \mathrm{Spin}(9)$. In this way, a "diagonal" subgroup $\mathrm{Sp}(1)_\Delta \subset \mathrm{Spin}(9)$ is defined, and $\mathrm{Sp}(1)_\Delta$ is indeed a subgroup of the $\mathrm{Spin}(8) \subset \mathrm{Spin}(9)$ defined by triples $(A, B, C) \in \mathrm{SO}(8) \times \mathrm{SO}(8) \times \mathrm{SO}(8)$ obeying to the triality principle.

This action is without fixed points on S^{15}: from $A_q x = h_1 q + \overline{q} h_2 e = h_1 + h_2 e, q \neq 1$ follows $h_1 = h_2 = 0$, so that the only fixed points of R_q could be on the unit sphere S^7_∞ of the octonionic line l_∞, on which we are acting by the triality companion B_q. Now, if $x' \in S^7_\infty$ is a fixed point of B_q, so is $-x'$ and then B_q has to belong to a $\mathrm{SO}(7)$ subgroup of $\mathrm{SO}(8)$, rotating the equator of S^7_∞ with respect to the poles x' and $-x'$. But then the triple (A_q, B_q, C_q) belongs to a $\mathrm{Spin}(7)$ subgroup of $\mathrm{Spin}(8)$ and hence any of A_q, B_q, C_q has to belong to a $\mathrm{SO}(7) \subset \mathrm{SO}(8)$ (cf. [Mur89, page 194]). Recall on the other hand that any subgroup $\mathrm{SO}(7) \subset \mathrm{SO}(8)$, when acting on the sphere S^7, admits a fixed point and it is conjugate with the standard $\mathrm{SO}(7)$ (cf. [Var01, p. 168, Lemma 4]). This is a contradiction, since A_q has no fixed points.

We can now consider finite subgroups of $\mathrm{Sp}(1)_\Delta$. Recall that any finite subgroup of $\mathrm{Sp}(1)$ is isomorphic to either a cyclic group or to the binary dihedral, tetrahedral, octahedral, or icosahedral group (see, for instance, [Cox91, Sect. 6.5]). In the following, we associate a subgroup of $\mathrm{Sp}(1)_\Delta$ with every group in the list of abstract finite subgroups of $\mathrm{Sp}(1)$.

- The cyclic group $\mathrm{C}_n = \langle a \mid a^n = 1 \rangle$, for $n \geq 1$. We can choose as generator R_a, where $a = e^{2\pi i/n}$.
- The binary dihedral group $\mathrm{D}_n = \langle a, b \mid a^{2n} = 1, b^2 = a^n, b^{-1}ab = a^{-1} \rangle$, for $n \geq 1$. Choose here as generators R_a, R_b with $a = e^{\pi i/n}$ and $b = j$.
- The binary tetrahedral group $\mathrm{T} = \langle a, b, c \mid a^2 = b^3 = c^3 = abc \rangle$. Choose now generators R_a, R_b, R_c with $b = (1 + i + j + k)/2$, $c = (1 + i + j - k)/2$ and $a = bc$.
- The binary octahedral group $\mathrm{O} = \langle a, b, c \mid a^2 = b^3 = c^4 = abc \rangle$. Choose generators R_a, R_b, R_c with $b = -(1 + i + j + k)/2$, $c = (1 + i)/\sqrt{2}$ and $a = bc$.
- The binary icosahedral group $\mathrm{I} = \langle a, b, c \mid a^2 = b^3 = c^5 = abc \rangle$. Let $\varphi \stackrel{\mathrm{def}}{=} (1 + \sqrt{5})/2$ be the golden ratio. Choose generators R_a, R_b, R_c where now $b = (1 + i + j + k)/2$, $c = (\varphi + \varphi^{-1} i + j)/2$ and $a = bc$.

Since any finite subgroup of Sp(1) is conjugate to one in the previous list, this classifies all locally conformally parallel Spin(9) manifolds such that $K = \ker \rho$ in Theorem C is contained in $\mathrm{Sp}(1)_\Delta$.

Remark 6.6. The Lee vector field on a locally conformally parallel Spin(9) manifold M is never vanishing (see proof of Theorem B). By [Fri01, Prop. 1] this means that M admits a $\mathrm{Spin}(7)_\Delta$-structure (in the sense of [Fri01]). Thus, the classification of isometry types of M reduces to the finding of finite subgroups of $\mathrm{Spin}(7)_\Delta \subset \mathrm{Spin}(9)$ acting without fixed points on S^{15}.

References

[AF06] I. Agricola, T. Friedrich, *Geometric structures of vectorial type*, J. Geom. Phys. **56** (2006), no. 12, 2403–2414.

[Ale68] Д. В. Алексеевский, *Римановы пространства с необычными группами голономии*, Функц. анализ и его прил. **2** (1968), вып. 2, 1–10. Engl. transl.: D. V. Alekseevskii, *Riemannian spaces with exceptional holonomy groups*, Func. Anal. Appl. **2** (1968), no. 2, 97–105.

[Bae02] J. C. Baez, *The octonions*, Bull. Amer. Math. Soc. (N.S.) **39** (2002), no. 2, 145–205.

[Ber72] M. Berger, *Du côté de chez Pu*, Ann. Sci. École Norm. Sup. (4) **5** (1972), 1–44.

[BrGr72] R. B. Brown, A. Gray, *Riemannian manifolds with holonomy group* Spin(9), in: *Differential Geometry, in Honor of K. Yano*, Kinokuniya, Tokyo, 1972, pp. 41–59.

[BG99] C. Boyer, K. Galicki, 3-*Sasakian manifolds*, in: *Surveys in Differential Geometry: Essays on Einstein Manifolds*, Int. Press, Boston, MA, 1999, pp. 123–184.

[BGMR98] C. P. Boyer, K. Galicki, B. M. Mann, E. G. Rees, *Compact* 3-*Sasakian* 7-*manifolds with arbitrary second Betti number*, Invent. Math. **131** (1998), no. 2, 321–344.

[Bru92] M. Bruni, *Sulla parallelizzazione esplicita dei prodotti di sfere*, Rend. di Mat., serie VII **12** (1992), 405–423.

[Cor92] K. Corlette, *Archimedean superrigidity and hyperbolic geometry*, Ann. Math. (2) **135** (1992), no. 1, 165–182.

[Cox91] H. S. M. Coxeter, *Regular Complex Polytopes*, 2nd ed., Cambridge University Press, Cambridge, 1991.

[CP99] D. M. J. Calderbank, H. Pedersen, *Einstein–Weyl geometry*, in: *Surveys in Differential Geometry: Essays on Einstein Manifolds*, Int. Press, Boston, MA, 1999, pp. 387–423.

[Dea08] O. Dearricott, *n-Sasakian manifolds*, Tohoku Math. J. (2) **60** (2008), no. 3, 329–347.

[DO98] S. Dragomir, L. Ornea, *Locally Conformal Kähler Geometry*, Progress in Math., Vol. 155, Birkhäuser, Boston, MA, 1998.

[DS01] R. De Sapio, *On* Spin(8) *and triality: A topological approach*, Expo. Math. **19** (2001), no. 2, 143–161.

[FO08] J. Figueroa-O'Farrill, *A geometric construction of the exceptional Lie algebras* F_4 *and* E_8, Comm. Math. Phys. **283** (2008), no. 3, 663–674.

[Fri01] T. Friedrich, *Weak* Spin(9)*-structures on* 16*-dimensional Riemannian manifolds*, Asian J. Math. **5** (2001), no. 1, 129–160.

[Gau95] P. Gauduchon, *Structures de Weyl–Einstein, espaces de twisteurs et variétés de type* $S^1 \times S^3$, J. Reine Angew. Math. **469** (1995), 1–50.

[GOPP06] R. Gini, L. Ornea, M. Parton, P. Piccinni, *Reduction of Vaisman structures in complex and quaternionic geometry*, J. Geom. Phys. **56** (2006), no. 12, 2501–2522.

[GWZ86] H. Gluck, F. Warner, W. Ziller, *The geometry of the Hopf fibrations*, Enseign. Math. (2) **32** (1986), no. 3–4, 173–198.

[Har90] F. R. Harvey, *Spinors and Calibrations*, Perspectives in Mathematics, Vol. 9, Academic Press, Boston, MA, 1990.

[Hus94] D. Husemoller, *Fibre Bundles*, 3rd ed., Graduate Texts in Mathematics, Vol. 20, Springer-Verlag, New York, 1994.

[IPP06] S. Ivanov, M. Parton, P. Piccinni, *Locally conformal parallel* G_2 *and* Spin(7) *manifolds*, Math. Res. Lett. **13** (2006), no. 2–3, 167–177.

[Kol02] A. Kollross, *A classification of hyperpolar and cohomogeneity one actions*, Trans. Amer. Math. Soc. **354** (2002), no. 2, 571–612.

[LV92] B. Loo, A. Verjovsky, *The Hopf fibration over* S^8 *admits no* S^1*-subfibration*, Topology **31** (1992), no. 2, 239–254.

[Mur89] S. Murakami, *Exceptional simple Lie groups and related topics in recent differential geometry*, in: *Differential Geometry and Topology* (Tianjin 1986–87), Lecture Notes in Mathematics, Vol. 1369, Springer, Berlin, 1989, pp. 183–221.

[O'N83] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Pure and Applied Mathematics, Vol. 103, Academic Press, New York, 1983.

[OP97] L. Ornea, P. Piccinni, *Locally conformal Kähler structures in quaternionic geometry*, Trans. Am. Math. Soc. **349** (1997), no. 2, 641–655.

[OV03] L. Ornea, M. Verbitsky, *Structure theorem for compact Vaisman manifolds*, Math. Res. Lett. **10** (2003), no. 5–6, 799–805.

[Par01a] M. Parton, *Hermitian and special structures on products of spheres*, PhD thesis, Dipartimento di Matematica, Università di Pisa, 2001, `www.sci.unich.it/~parton/rice/main.pdf`.

[Par01b] M. Parton, *Old and new structures on products of spheres*, in: *Global Differential Geometry: the Mathematical Legacy of Alfred Gray*, Bilbao, 2000, American Math. Soc., Providence, RI, 2001, pp. 406–410.

[Par03] M. Parton, *Explicit parallelizations on products of spheres and Calabi–Eckmann structures*, Rend. Ist. Mat. Univ. Trieste **35** (2003), no. 1–2, 61–67.

[PP12] M. Parton, P. Piccinni, Spin(9) *and almost complex structures on* 16*-dimensional manifolds*, Ann. Global Anal. Geom. **41** (2012), no. 3, 321–345.

[PP13] M. Parton, P. Piccinni, *Spheres with more than* 7 *vector fields: All the fault of* Spin (9), Linear Algebra Appl. **438** (2013), no. 3, 1113–1131.

[Ste99] N. Steenrod, *The Topology of Fibre Bundles*, 1st paperback ed., Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1999.

[Var01] V. S. Varadarajan, Spin(7)-*subgroups of* $SO(8)$ *and* Spin(8), Expo. Math. **19** (2001), no. 2, 163–177.

[Zil82] W. Ziller, *Homogeneous Einstein metrics on spheres and projective spaces*, Math. Ann. **259** (1982), no. 3, 351–358.